POLYNOMIALS
Polynomials are sums of terms that look like $ax^n$, where $a$ is a real number called a coefficient, and $n$ is a whole number. We like to express the terms in order of descending powers, and the highest power is the degree of the polynomial. If a term doesn’t have a variable, it’s the constant term.

1. Consider the polynomial, $9 - 5x^2 - 3x^5 + 6x$.
   a. Write the polynomial in descending powers.

   b. How many terms are there?

   c. What is the constant term?

   d. What is the coefficient of $x^2$?

   e. What is the degree of the polynomial?

When polynomials are multiplied, each polynomial is called a factor. Remember – polynomial terms are added or subtracted, but factors are multiplied.

To multiply polynomials, we use the distributive property.

CHECK IT OUT!
We use the distributive property to multiply polynomials.

$$a(b + c) = ab + ac$$
When multiplying polynomials, we often need to combine products of variables raised to different powers. Think about how we multiply factors:

\[ 3x^2 \cdot 4x^5 = (3 \cdot 4)x^2 \cdot x^5 = 12(xx)(xxxxx) = 12x^7. \]

Note that the order of the quantities is changed so that the 3 and 4 are together, as are the \( x \) values.

2. Multiply the following polynomials.
   a. \( 5(2x - 7) = \) __________
   b. \( -3(4 - 6x) = \) __________
   c. \( 2x(3 - 5x) = \) __________
   d. \( -4x^2(5 - 3x^3) = \) __________
   e. \( 3x^3(2x - 4x^2 + 3) = \) __________

Sometimes we need to reverse the multiplication process with polynomials. When we do this, we reverse the distributive property. Reversing the distributive process is called factoring out the greatest common factor.

**CHECK IT OUT!**

We use the *distributive property* to factor away the greatest common factor.

\[ ab + ac = a(b + c) \]

For polynomials, the greatest common factor’s coefficient is the largest factor of the other coefficients, and its variable is raised to the lowest power present in the original terms.

To factor an expression like \( 21x^3 - 14x^5 \), think of the largest factor of 21 and 14, and factor it out. Also, factor away the smallest power of \( x \).

\[ 21x^3 - 14x^5 = 7x^3 \cdot 3 - 7x^3 \cdot 2x^2 = 7x^3(3 - 2x^2) \]

3. Factor away the greatest common factor.
   a. \( 12x^2 - 8x^6 = \) __________
SECTION 1.4

ALGEBRAIC SIMPLIFICATION

\[ b. \quad 35y^9 + 28y^4 = \quad \underline{\quad} \]
\[ c. \quad 22x^4 + 44x^2 - 33x = \quad \underline{\quad} \]
\[ d. \quad 19z^3 - 38z + 57z^2 = \quad \underline{\quad} \]

COMBINING LIKE TERMS

When two or more terms differ only by their coefficients (the numbers that are multiplied with the terms), we call them like terms. Like terms can be combined through factoring.

\[ 5xy^2 + 13xy^2 = (5 + 13)xy^2 \]
\[ = 18xy^2 \]

4. Add the polynomials by combining like terms.

a. \( 7x^3 + 9x^3 = \)

b. \( 8y^5 - 14y^5 = \)

c. \( (3x - 9) + (4x^2 + 6x) = \)

d. \( (2x^2 - 4x + 1) - (3x^2 + 2x - 6) = \)

REDUCING FRACTIONS WITH VARIABLES

Reducing fractions puts them in a simpler form. We reduce fractions using the idea that when a factor is common to a numerator and denominator, we can cancel it in both. But the key word here is factor. We can only cancel a common factor if both the numerator and denominator are factored. For example, if we factor the expression,

\[ \frac{6x^2 + 3x}{10x + 5} \]

we see a common factor that can be cancelled.

\[ \frac{6x^2 + 3x}{10x + 5} = \frac{3x(2x + 1)}{5(2x + 1)} \]
\[ = \frac{3x}{5} \]

But this can only happen after the expression is factored.
SECTION 1.4
ALGEBRAIC SIMPLIFICATION

YOUR TURN!

5. Reduce the following expressions, if possible.
   
   a. \( \frac{9x^3 - 6x^2}{12x - 8} = \)
   
   b. \( \frac{21x + 28x^2}{27x + 36x^2} = \)

   c. \( \frac{2x + 5}{2x + 7} = \)

   d. \( \frac{x^2 + x}{x^2 + 2x} = \)

PROPERTIES OF EXPONENTS

When multiplying the polynomials above you probably noticed one of the first important properties of exponents. This applies when multiplying exponential expressions of the same base.

6. Simplify this:
   \( x^4 \cdot x^6 = \) 

7. This rule can be summarized in a general setting as below. Write as a single base \((b)\) raised to a power.
   \( b^m \cdot b^n = \)

8. A similar rule applies when dividing exponential expressions of the same base.
   \( \frac{x^7}{x^3} = \)

9. In a general setting, we can state another principal for dividing exponentials of the same base. Rewrite as a single base, raised to a power.
   \( \frac{b^m}{b^n} = \)

10. Sometimes we raise an expression to a power, then raise the result to another power. How would you evaluate this? Write your answer as a single power of \(x\).
    \( (x^3)^2 = (xxx)^2 = (xxx)(xxx) = \)

11. Generalize this idea into a new rule. Write as the base raised to a single exponent.
    \( (b^m)^n = \)
12. How would you simplify the following?

\[
\frac{z^3}{z^2} = \underline{\hphantom{0}}
\]

13. There are actually two ways to think of question 10. You can simplify it by applying the property in question 9, yielding a negative exponent. You can also simplify it by cancelling powers of \(z\) to give a positive exponent. Write the result in both ways described here.

14. Think about your answer to question 13, and use this to make a statement about how to rewrite an exponential with a negative exponent.

\[
b^{-n} = \underline{\hphantom{0}}
\]

15. Think one more time about your rule in question 9.

\(a.\) Use this to rewrite the following as a power of \(x\).

\[
\frac{x^3}{x^3} = \underline{\hphantom{0}}
\]

\(b.\) But what is the value of any number divided by itself as you see above?

\(c.\) Does the argument work if \(x = 0\)? Explain your answer.

\(d.\) Give the principal for raising a number to the zero power.

\[
b^0 = \underline{\hphantom{0}}
\]

This rule is only valid if: \(\underline{\hphantom{0}}\).
Two more rules of exponents are as follows:

\[(ab)^n = a^n \cdot b^n,\]

and

\[\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.\]

16. Use the latest properties of exponents to evaluate the following.
   
   a. \((7x^2)^3 = \)
   
   b. \((9y^3)^4 = \)
   
   c. \(\left(\frac{5x^3}{2y^4}\right)^2 = \)
   
   d. \(\left(\frac{2a}{9b^2}\right)^3 = \)

17. Summarize the properties of exponents below.

   I. \(b^m \cdot b^n = \)

   II. \((b^m)^n = \)

   III. \(\frac{b^m}{b^n} = \)

   IV. \(b^{-n} = \)

   V. \(b^0 = \)

   VI. \((ab)^n = \)

   VII. \(\left(\frac{a}{b}\right)^n = \)
The properties of exponents can be used to simplify complicated fractions. Consider the following.

\[
\frac{(x^3y^2)^4z^5}{x^4(y^3z)^3} = \frac{(x^3)^4(y^2)^4z^5}{x^4(y^3)^3z^3} = \frac{x^{12}y^8z^5}{x^4y^9z^3} = \frac{x^{12-4}z^{5-3}}{y^{9-8}} = \frac{x^8z^2}{y}
\]

18. Simplify the following expressions.

a. \( \frac{a^2b^4c}{ab^6c^6} = \)

b. \( \frac{q^2(r^3)^2}{(q^2)^2s^3} = \)

c. \( \frac{(xy^2z^3)^2}{x^4(yz)^3} = \)

**FRACTIONAL EXPONENTS AND RADICALS**

Let's take a moment to remember what an \( n \)th root is.

\[ \sqrt[n]{b} = \text{the number that we raise to the } n \text{th power to get } b. \]

So \( \sqrt[3]{8} = 2 \) (because \( 2^3 = 8 \)), and \( \sqrt[3]{25} = \sqrt[5]{25} = 5 \) (since \( 5^2 = 25 \)).

Remember, when we raise an exponential expression to a power, the exponents multiply: \( (b^m)^n \). This fact holds the secret behind the meaning of **fractional exponents**. Note the following:

\[
(b^{1/n})^n = b^1 = b.
\]
In words, this tells us that $b^{1/n}$ is the number that we raise to the $n$th power to get $b$. But this is exactly the definition of an $n$th root. Thus, $b^{1/n} = \sqrt[n]{b}$. If we raise each side of this little equation to the power of $m$, we see that $b^{m/n} = \sqrt[n]{b^m}$.

**Check It Out!**

The first properties of radicals link them to exponents.

I. $b^{1/n} = \sqrt[n]{b}$

II. $b^{m/n} = \sqrt[n]{b^m}$

III. The remaining properties are direct consequences of corresponding properties of exponents.

IV. $\sqrt[n]{a b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

V. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

VI. $\frac{a}{b \sqrt{c}} = \frac{a \sqrt{c}}{bc}$ (This property is called *rationalizing the denominator*.)

Note that property two allows us to evaluate combinations of radicals and exponents. For example, to evaluate $\sqrt[3]{b^{18}}$, we use property II.

$$
\sqrt[3]{b^{18}} = b^{\frac{18}{3}} = b^6
$$

More complicated expressions require using several of these properties.

$$
\sqrt[2]{\frac{98x^6y}{2x^2y^3}} = \sqrt[2]{\frac{49x^4}{y^2}} = \frac{\sqrt[2]{49\sqrt{x^4}}}{\sqrt[2]{y^2}} = \frac{7x^2}{y^2} = \frac{7x^2}{y}
$$
19. Simplify the following expressions with radicals.
   a. $\sqrt[3]{b^9} =$
   b. $8^{1/3} =$
   c. $16^{3/4} =$
   d. $\sqrt[6]{\frac{63x^4y^6}{28x^2y^{10}}} =$
   e. $\sqrt[4]{\frac{5a^9b^3}{45a^3b}}$

Property VI above teaches us how to rationalize denominators. Sometimes we don’t like radicals (which usually result in irrational numbers) in denominators. One reason for this is that it is harder to divide a nice number by a messy number than it is to divide a messy number by a nice number.

For instance, $\frac{3}{\sqrt{2}}$ is hard to compute (try by replacing $\sqrt{2}$ with 1.4142, then using long division to compute it), but if we rationalize the denominator, the result is easier.

\[
\frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{3\sqrt{2}}{2}
\]

20. Rationalize the following denominators.
   a. $\frac{2}{\sqrt{5}} =$
   b. $\frac{6}{\sqrt{4}} =$
   c. $\frac{12}{5\sqrt{6}} =$

   (Hint for c: just multiply by the radical!)