INTRODUCTION
In this lesson, we learn about formulas for quickly solving linear systems of equations. The method is called Cramer’s rule. Cramer’s rule uses a mathematical computation that is known as a determinant. Determinants are used to help determine the properties of square arrays of numbers.

The simplest determinant is performed on an array of two rows and two columns.

\[
\begin{vmatrix}
2 & 3 \\
-4 & 1
\end{vmatrix}
\]

When we surround the array with vertical bars (similar to absolute value, but in this context the meaning is entirely different) as you see above, we are describing a computation. The computation is simple – multiply the values on the downward diagonal, then the upward diagonal, and subtract these results.

\[
\begin{vmatrix}
2 & 3 \\
-4 & 1
\end{vmatrix} = 2 \cdot 1 - (-4) \cdot 3 \\
= 2 + 12 \\
= 14
\]

Notice the double-negative in the second term of the determinant. Unfortunately, these arise frequently, so care must be taken to avoid mistakes.

CHECK IT OUT!
A determinant with two rows and columns is computed as follows.

\[
\begin{vmatrix}
a & b \\
d & e
\end{vmatrix} = ae - bd
\]

YOUR TURN!
1. Compute the determinants below.
   a. \[
   \begin{vmatrix}
   -3 & 2 \\
   4 & 1
   \end{vmatrix} = 
   \]
   b. \[
   \begin{vmatrix}
   2 & -5 \\
   3 & 1
   \end{vmatrix} = 
   \]
   c. \[
   \begin{vmatrix}
   4 & -6 \\
   -3 & 5
   \end{vmatrix} = 
   \]
   d. \[
   \begin{vmatrix}
   2.6 & 1.9 \\
   -3.8 & 0.2
   \end{vmatrix} = 
   \]
Moving On!

Watch how we solve a linear system of two equations and two unknowns.

\[
\begin{align*}
ax + by &= c \\
dx + ey &= f
\end{align*}
\]

We’ll use the method of substitution. If we divide the second equation by \( e \) we get

\[
\frac{d}{e}x + y = \frac{f}{e}
\]

This allows us to solve for \( y \). The common denominator allows terms to be combined.

\[
y = \frac{f - dx}{e}.
\]

If we substitute this into the first equation, we get

\[
ax + b \left( \frac{f - dx}{e} \right) = c.
\]

Distributing and clearing fractions gives the following

\[
aex + bf - bdx = ce.
\]

Combining terms containing \( x \) and moving the remaining term to the other side gives

\[
(ax - bd)x = ce - bf.
\]

Division gives the value of \( x \).

\[
x = \frac{ce - bf}{ae - bd}.
\]

The pattern in the numerator and denominator is the same sort of pattern that occurs when we compute a determinant. We have already said that

\[
\left| \begin{array}{cc} a & b \\ d & e \end{array} \right| = ae - bd.
\]

And this is exactly the denominator in the solution for \( x \) above. This is \( D \), the determinant of the array of \( x \) and \( y \) coefficients.

\[
\left| \begin{array}{cc} a & b \\ d & e \end{array} \right| = \frac{ce - bf}{ae - bd}.
\]

It is easy to see that the numerator for \( x \) is the determinant,

\[
\left| \begin{array}{cc} c & b \\ f & e \end{array} \right| = ce - bf.
\]

This determinant is called \( D_x \).
Dx is constructed by replacing the x coefficients with the constants, c and f. The y coefficients stay in place.

\[
\begin{align*}
\begin{cases}
ax + by &= c \\
dx + ey &= f
\end{cases}
\end{align*}
\]

\[
D_x = \begin{vmatrix}
c & b \\
f & e
\end{vmatrix}
\]

If you’re following carefully, you understand that \(x = \frac{D_x}{D}\).

Another determinant called \(D_y\) is constructed by replacing the y coefficients with c and f. The x coefficients return to their original positions.

\[
\begin{align*}
\begin{cases}
ax + by &= c \\
dx + ey &= f
\end{cases}
\end{align*}
\]

\[
D_y = \begin{vmatrix}
a & c \\
d & f
\end{vmatrix}
\]

As you might guess, \(D_y\) is used to determine the value of y. These formulas for x and y are Cramer's rule.

### Cramer's Rule

The solution of the system of equations,

\[
\begin{align*}
\begin{cases}
ax + by &= c \\
dx + ey &= f
\end{cases}
\end{align*}
\]

is

\[
(x, y) = \left( \frac{D_x}{D}, \frac{D_y}{D} \right),
\]

where

\[
D = \begin{vmatrix}
a & b \\
d & e
\end{vmatrix}, \quad D_x = \begin{vmatrix}
c & b \\
f & e
\end{vmatrix} \quad \text{and} \quad D_y = \begin{vmatrix}
a & c \\
d & f
\end{vmatrix}.
\]

### Moving On!

Let’s solve the system,

\[
\begin{align*}
\begin{cases}
2x + 3y &= 5 \\
-3x - 13y &= 1
\end{cases}
\end{align*}
\]

The first determinant is \(D\).

\[
D = \begin{vmatrix}
2 & 3 \\
-3 & -13
\end{vmatrix} = 2(-13) - 3(-3) = -26 + 9 = -17
\]
Next we compute \( D_x \).

\[
D_x = \begin{vmatrix} 5 & 3 \\ 1 & -13 \end{vmatrix} = 5(-13) - 3 \cdot 1 = -65 - 3 = -68
\]

The last determinant is \( D_y \).

\[
D_y = \begin{vmatrix} 2 & 5 \\ -3 & 1 \end{vmatrix} = 2 \cdot 1 - 5(-3) = 2 + 15 = 17
\]

The solution is computed using these.

\[
(x, y) = \left( \frac{D_x}{D}, \frac{D_y}{D} \right) = \left( \frac{-68}{17}, \frac{17}{17} \right) = (4, -1).
\]

**YOUR TURN!**

2. Use Cramer’s rule to solve the systems of equations below.

   a. \[
   \begin{cases} 
   2x - 3y = 5 \\
   x + 2y = 13 
   \end{cases}
   \]

   b. \[
   \begin{cases} 
   4x + 3y = 7 \\
   -3x - 2y = -4 
   \end{cases}
   \]

   c. \[
   \begin{cases} 
   2x + 7y = 11 \\
   3x - 4y = 2 
   \end{cases}
   \]