SECTION 5.3
MODELING WITH RATIONAL FUNCTIONS

Name: __________________________ Date: ______________

INTRODUCTION

Rational expressions are ratios of polynomials. Rational functions are represented with formulas involving rational expressions. The formulas below represent rational functions.

\[ y = \frac{2x + 5}{5x - 20} \quad y = 5x - \frac{x^2}{x + 1} \quad y = \frac{x}{x^2 - 1} \quad y = 2x + 3 \]

The last example above is a rational function because it can still be represented as a ratio, \( y = \frac{2x+3}{1} \).

In this lesson we graph simple rational functions. Before we do, we need to think about the **domain** of such functions, because the domain influences the appearance of the graph.

CHECK IT OUT!

To find the domain of a rational function, set any denominator that has variables **not equal to zero**, and solve.

For example, the domain of the function, \( y = \frac{3}{6-2x} \), is found by setting \( 6 - 2x \neq 0 \) and solving.

\[
\begin{align*}
6 - 2x & \neq 0 \\
-6 & \\
-2x & \neq -6 \\
-2 & \neq -6 \\
\frac{-2x}{-2} & \neq \frac{-6}{-2} \\
x & \neq 3
\end{align*}
\]

The domain consists of all values of \( x \) where \( x \neq 3 \).

YOUR TURN!

1. Give the domain for each rational function below.
   
   a. \( y = \frac{2x+5}{5x-20} \)
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b. \( y = 5x - \frac{x^2}{x+1} \)

c. \( y = \frac{x}{x^2-1} \)

Moving On!

One reason that we find the domain of a function is that it helps us graph the function.

2. We found that the domain for the function, \( y = \frac{3}{6-2x} \), is all values of \( x \) where \( x \neq 3 \). When we graph the function, we choose values on either side of 3, even very near to 3, but not equal to 3.

a. Fill in the \( y \)-values that correspond to the \( x \)-values for this function in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = \frac{3}{6-2x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td></td>
</tr>
<tr>
<td>3.2</td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

b. Think about the domain for the function, \( y = \frac{3}{6-2x} \). If we draw a vertical line where \( x = 3 \), can the graph of this function pass through this line? Explain your answer.
c. The graph of the function, \( y = \frac{3}{6-2x} \), cannot pass through the vertical line where \( x = 3 \). Sketch a dashed vertical line at \( x = 3 \) below.

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{X} & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
\hline
\text{Y} & -8 & -6 & -4 & -2 & 2 & 4 & 6 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{X} & 5 & 6 & 7 & 8 \\
\hline
\text{Y} & 2 & 4 & 6 & 8 \\
\hline
\end{array}
\]

d. Plot the points that you calculated for this graph above. Sketch the curve, remembering that it cannot cross the vertical line.

Notice that the graph of \( y = \frac{3}{6-2x} \) gets very close to the vertical line at \( x = 3 \). If we graph more points we see that the graph approaches the line ever more closely. When a function’s graph approaches a line like this we call the line an asymptote. The line, \( x = 3 \), is a vertical asymptote for this graph.

e. What happens to the \( y \)-coordinates on the graph as \( x \) approaches the vertical asymptote? Think about the left and right sides of the asymptote as you give your answer.

f. The graph has another asymptote, but it is not vertical. What line is this?

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**CHECK IT OUT!**

The graph of a rational function usually has a vertical asymptote at a point that is not in the function’s domain.
3. Let’s graph a simpler function: \( y = \frac{2}{x^2} \).

   a. What is the domain for this function?

   b. How does this affect the graph of the function?

   c. Fill in the table for points on the graph of \( y = \frac{2}{x^2} \).

   \[
   \begin{array}{c|c}
   x       & y = \frac{2}{x^2} \\
   \hline
   -4      & \text{---} \\
   -2      & \text{---} \\
   -1      & \text{---} \\
   -0.5    & \text{---} \\
   0.5     & \text{---} \\
   1       & \text{---} \\
   2       & \text{---} \\
   4       & \text{---} \\
   \end{array}
   \]

   d. Graph the vertical asymptote for this function, and plot the points computed on the graph below.
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e. What happens to the $y$-coordinates on the graph as $x$ approaches the vertical asymptote? Think about the left and right sides of the asymptote as you give your answer.

f. How does the behavior of $y$-coordinates differ from the graph in question 2? Can you explain this difference?

Moving On!

It is difficult to give applications of rational functions and equations that are not too complicated. There are a few, however, that we explore next.

Work Rate Problems

Work rate problems involve rational equations. They arise when multiple sources contribute to the work of getting some job done. To solve the problem we simply remember that the rate at which work is completed is equal to the sum of the rates for each contributing source.

\[ \text{Sum of Individual Rates} = \text{Combined Rate}. \]

Beyond this, all we need to remember is how to compute a rate.

\[ \text{Rate} = \frac{\text{Work Done}}{\text{Time}} \]

Suppose that Pietro can type 205 words in 4 minutes, and that Pietro and Juana can type 60 words per minute combined. In how many minutes can Juana type 1000 words?

We begin by letting $t$ represent the number of minutes needed for Juana to type 1000 words. Next we remember that

\[ \text{Sum of Individual Rates} = \text{Combined Rate}. \]

This gives the equation below.

\[
\frac{\text{Pietro's Rate}}{4 \text{ min}} + \frac{\text{Juana's Rate}}{t \text{ min}} = \frac{60 \text{ words}}{1 \text{ min}}
\]
If we rewrite this equation without units, it looks like this.

\[
\frac{205}{4} + \frac{1000}{t} = 60
\]

4. To solve this, we multiply by \(4t\) to clear denominators.

   a. Solve the equation above for \(t\).

   b. Explain what your answer represents.

5. Suppose that I can mow my lawn in 2 hours, but my son Ryan can mow the same lawn in 3 hours. If we work together, how long would it take for us to mow the lawn (assuming I have two lawnmowers and nobody gets hurt)?
6. Another important application of rational functions is inverse proportionalities. X-ray intensity, $I$, is inversely proportion to the square of distance, $D$, to the x-ray machine. This means that $I = \frac{k}{D^2}$, for some constant, $k$.

   a. At a distance of 25m, the intensity for a given machine is $I = 620$ R (Roentgens). What is the value of $k$, the constant of proportionality, for this machine?

   b. Give the equation for the proportionality using the value of $k$ determined above.

   c. Use the proportionality above to give the radiation intensity at a distance of 10m.