SECTION 6.2
ALGEBRAIC ANALYSIS OF RADICAL EXPRESSIONS

INTRODUCTION
In this lesson we review the algebraic processes that arise when working with radicals. We begin with several issues regarding the simplification of radical expressions.

When simplifying a square root, we rewrite the radical so that there are no perfect squares inside. With cube roots, there should be no perfect cubes inside. For example, let’s simplify the radical, \( \sqrt{98y^3} \) (assuming \( y > 0 \)).

\[
\sqrt{98y^3} = \sqrt{49y^2 \cdot 2y} \\
= \sqrt{49} \cdot \sqrt{y^2} \cdot \sqrt{2y} \\
= 7y\sqrt{2y}
\]

First we factor away the perfect squares that are in the radical.
Next, we apply the rule that \( \sqrt{ab} = \sqrt{a} \sqrt{b} \).
We finish by evaluating the radicals. Note that \( \sqrt{y^2} = y \).

YOUR TURN!

1. Simplify the following radicals. Assume all variables are positive.
   a. \( \sqrt{20} \)
   b. \( \sqrt{x^2} \)
   c. \( \sqrt{24y^3} \)
   d. \( \sqrt{200m^2n^4} \)

MOVING ON!

The last problem above brings back a property of exponents that relates exponents and radicals.

\[
\sqrt[n]{b^m} = b^{\frac{m}{n}}
\]

This allows us to handle radicals like this: \( \sqrt[4]{x^{12}} = x^{\frac{12}{4}} = x^3 \). Even worse, we can do problems like this:

\[
\sqrt[4]{y^{14}} = \sqrt[4]{y^{12} \cdot y^2} \\
= y^{\frac{12}{4}} \cdot y^{\frac{2}{4}} \\
= y^3 \cdot y^{\frac{1}{2}} \\
= y^3 \cdot \sqrt{y}.
\]
SECTION 6.2

ALGEBRAIC ANALYSIS OF RADICAL EXPRESSIONS

Note that we used the fact that $y^{\frac{1}{2}} = \sqrt{y}$. More generally, we know that $y^{\frac{1}{n}} = \sqrt[n]{y}$.

YOUR TURN!

2. Simplify the following radicals. Assume all variables are positive.
   a. $\sqrt[3]{50y^7}$
   b. $\sqrt[3]{72a^4b^5}$
   c. $\frac{2}{3}\sqrt[3]{52x^{11}}$
   d. $\frac{4}{5}\sqrt[5]{48m^{10}}$

MOVING ON!

When two radicals are exactly the same, they are like terms, and they can be added or subtracted by factoring. We can’t initially add $6\sqrt{200a^5} + 8a\sqrt{98a^3}$, but if we simplify first, the terms combine.

$$6\sqrt{200a^5} + 8a\sqrt{98a^3} = 6\sqrt{100a^4 \cdot 2a} + 8a\sqrt{49a^2 \cdot 2a}$$
$$= 6 \cdot 10a^2\sqrt{2a} + 8a \cdot 7a\sqrt{2a}$$
$$= 60a^2\sqrt{2a} + 56a^2\sqrt{2a}$$
$$= (60 + 56)a^2\sqrt{2a}$$
$$= 116a^2\sqrt{2a}$$

YOUR TURN!

3. Simplify, then combine like radicals. Assume all variables are positive.
   a. $5\sqrt{2x} - 7\sqrt{2x}$
   b. $7\sqrt{75a} - 17\sqrt{48a}$
   c. $8x\sqrt{2x} + 7\sqrt{18x^3}$
   d. $-2r^2\sqrt{128r^7} - 7r^3\sqrt{54r^{10}}$
**SECTION 6.2**

**ALGEBRAIC ANALYSIS OF RADICAL EXPRESSIONS**

**MOVING ON!**

In the following problems, the radicals multiply using the property: \( \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab} \). In each case, the result can be simplified.

\[
4\sqrt{35m} \cdot \sqrt{15m} = 4\sqrt{35 \cdot 15m^2} = 4\sqrt{7 \cdot 5 \cdot 5 \cdot 3m^2} = 4\sqrt{25m^2 \cdot 21} = 4 \cdot 5m\sqrt{21} = 20m\sqrt{21}
\]

First we multiply the radicals.

Instead of multiplying 35·15, we factor: 7·5·5·3 = (7·3)(5·5) = 21·25.

We separate the perfect square factors.

Apply the radical to the perfect squares.

Simplify.

**YOUR TURN!**

4. Multiply, then simplify. Assume all variables are positive.
   a. \( 8\sqrt{15\sqrt{6}} \)
   
   b. \( 8\sqrt{28x\sqrt{14x}} \)
   
   c. \( 12\sqrt{14r\sqrt{21r^5}} \)

**MOVING ON!**

Fractions with whole numbers are **rational numbers**. Rational numbers have decimal expansions that either terminate or repeat forever. For instance, \( 2/11 = 0.18181818... \), and \( 2/5 = 0.4 \). When we take the square root of a number that is not a perfect square, the result is an **irrational number**. The decimal expansion of an irrational number extends forever in a non-repeating pattern. For example,

\[
\sqrt{2} = 1.414213562373095048801688724209698078569671875376948073176679737990732...
\]

is an irrational number.

Simplified radicals, being irrational numbers, are hard to work with when they are in the denominators of fractions.
For this reason, we usually *rationalize denominators* by multiplying the numerator and denominator of the fraction by the radical.

\[
\frac{5x}{2\sqrt{3y}} = \frac{5x}{2\sqrt{3y}} \cdot \frac{\sqrt{3y}}{\sqrt{3y}} = \frac{5x\sqrt{3y}}{2\sqrt{9y^2}} = \frac{5x\sqrt{3y}}{2 \cdot 3y} = \frac{5x\sqrt{3y}}{6y}
\]

**YOUR TURN!**

5. Rationalize the denominators of the fractions below.
   a. \(\frac{3}{\sqrt{2}}\)
   b. \(\frac{5}{4\sqrt{50}}\)
   c. \(\frac{2x}{3\sqrt{2y}}\)
   d. \(\frac{6a}{5\sqrt{27b}}\)

**MOVING ON!**

Sometimes an entire fraction is inside of a radical. When this happens, we apply the rule that \(\sqrt[n]{a} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}\), then rationalize the denominator.

**YOUR TURN!**

6. Rationalize the denominators of the fractions below.
   a. \(\frac{\sqrt{2}}{\sqrt{3}}\)
b. \( \sqrt[5]{\frac{5x}{2y}} \)

MOVING ON!
The main goal of this lesson is to solve radical equations. When we solve an equation with a radical like a square root, we use the normal process to isolate the square root, then square both sides. Finish by isolating the variable. Squaring both sides of an equation can result in invalid answers, so check your answer.

7. Solve the following equations.
   a. \(-2\sqrt{6y} = -12\)
   b. \(6 = 18 - 3\sqrt{2x + 4}\)
   c. \(-2\sqrt{2 - 7t} + 3 = 3 + 4t\)
   d. \(\frac{4}{\sqrt{5y}} - \frac{6}{5\sqrt{20y}} = \frac{17}{25}\) (Hint: simplify radicals first, then clear fractions).