Solutions to Section 1.1

True-False Review:

1. FALSE. A derivative must involve some derivative of the function \( y = f(x) \), not necessarily the first derivative.

3. TRUE. If we define positive velocity to be oriented downward, then
\[
\frac{dv}{dt} = g,
\]
where \( g \) is the acceleration due to gravity.

5. FALSE. The restoring force is directed in the direction opposite to the displacement from the equilibrium position.

7. FALSE. The temperature of the object is given by \( T(t) = T_m + ce^{-kt} \), where \( T_m \) is the temperature of the medium, and \( c \) and \( k \) are constants. Since \( e^{-kt} \neq 0 \), we see that \( T(t) \neq T_m \) for all times \( t \). The temperature of the object approaches the temperature of the surrounding medium, but never equals it.

9. FALSE. The slopes of the two curves are negative reciprocals of each other.

11. FALSE. The family of orthogonal trajectories for a family of circles centered at the origin is the family of lines passing through the origin.

Problems:

1. \( \frac{d^2 y}{dt^2} = g \implies \frac{dv}{dt} = gt + c_1 \implies y(t) = \frac{gt^2}{2} + c_1 t + c_2. \) Now impose the initial conditions. \( y(0) = 0 \implies c_2 = 0, \frac{dy}{dt}(0) = c_1 = 0. \) Hence, the solution to the initial-value problem is: \( y(t) = \frac{gt^2}{2}. \) The object hits the ground at time, \( t_0, \) when \( y(t_0) = 100. \) Hence \( 100 = \frac{gt_0^2}{2}, \) so that \( t_0 = \sqrt{\frac{200}{g}} \approx 4.52 \) s, where we have taken \( g = 9.8 \) ms\(^{-2}.\)

3. From \( \frac{d^2 y}{dt^2} = g, \) we integrate twice to obtain the general equations for the velocity and the position of the rocket, respectively: \( \frac{dy}{dt} = gt + c \) and \( y(t) = \frac{1}{2}gt^2 + ct + d, \) where \( c, d \) are constants of integration. Setting \( y = 0 \) to be at ground level, we know that \( y(0) = 0. \) Thus, \( d = 0. \)

(a) The rocket reaches maximum height at the moment when \( y'(t) = 0. \) That is, \( gt + c = 0. \) Therefore, the time that the rocket achieves its maximum height is \( t = -\frac{c}{g}. \) At this time, \( y(t) = -90 \) (the negative sign accounts for the fact that the positive direction is chosen to be downward). Hence,
\[
-90 = y \left( -\frac{c}{g} \right) = \frac{1}{2}g \left( -\frac{c}{g} \right)^2 + c \left( -\frac{c}{g} \right) = \frac{c^2}{2g} - \frac{c^2}{g} = -\frac{c^2}{2g}.
\]
Solving this for \( c, \) we find that \( c = \pm \sqrt{180g}. \) However, since \( c \) represents the initial velocity of the rocket, and the initial velocity is negative (relative to the fact that the positive direction is downward), we choose \( c = -\sqrt{180g} \approx -42.02 \) ms\(^{-1}. \) and thus the initial speed at which the rocket must be launched for optimal viewing is approximately 42.02 ms\(^{-1}. \)

(b) The time that the rocket reaches its maximum height is \( t = -\frac{c}{g} \approx -\frac{-42.02}{9.81} = 4.28 \) s.
5. If \( y(t) \) denotes the displacement of the object from its initial position at time \( t \), the motion of the object can be described by the initial-value problem

\[
\frac{d^2y}{dt^2} = g, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = -2.
\]

We first integrate this differential equation:

\[
\frac{d^2y}{dt^2} = g \implies \frac{dy}{dt} = gt + c_1 \implies y(t) = \frac{gt^2}{2} + c_1 t + c_2.
\]

Now impose the initial conditions. \( y(0) = 0 \implies c_2 = 0 \). \( \frac{dy}{dt}(0) = -2 \implies c_1 = -2 \). Hence the solution to the initial-value problem is \( y(t) = \frac{gt^2}{2} - 2t \). We are given that \( y(10) = h \). Consequently, \( h = \frac{g(10)^2}{2} - 2 \cdot 10 \implies h = 10(5g - 2) = 470 \text{ m} \) where we have taken \( g = 9.8 \text{ ms}^{-2} \).

7. \( y(t) = A \cos(\omega t - \phi) \implies \frac{dy}{dt} = -A\omega \sin(\omega t - \phi) \implies \frac{d^2y}{dt^2} = -A\omega^2 \cos(\omega t - \phi) \). Hence, \( \frac{d^2y}{dt^2} + \omega^2 y = -A\omega^2 \cos(\omega t - \phi) + A\omega^2 \cos(\omega t - \phi) = 0 \).

9. We compute the first three derivatives of \( y(t) = \ln t \):

\[
\frac{dy}{dt} = \frac{1}{t}, \quad \frac{d^2y}{dt^2} = -\frac{1}{t^2}, \quad \frac{d^3y}{dt^3} = \frac{2}{t^3}.
\]

Therefore,

\[
2 \left( \frac{dy}{dt} \right)^3 = \frac{2}{t^3} = \frac{d^3y}{dt^3},
\]

as required.

11. We compute the first two derivatives of \( y(x) = e^x \sin x \):

\[
\frac{dy}{dx} = e^x (\sin x + \cos x) \quad \text{and} \quad \frac{d^2y}{dx^2} = 2e^x \cos x.
\]

Then

\[
2y \cot x - \frac{d^2y}{dx^2} = 2(e^x \sin x) \cot x - 2e^x \cos x = 0,
\]

as required.

13. After 4 p.m. In the first two hours after noon, the water temperature increased from 50° F to 55° F, an increase of five degrees. Because the temperature of the water has grown closer to the ambient air temperature, the temperature difference \( |T - T_m| \) is smaller, and thus, the rate of change of the temperature of the water grows smaller, according to Newton’s Law of Cooling. Thus, it will take longer for the water temperature to increase another five degrees. Therefore, the water temperature will reach 60° F more than two hours later than 2 p.m., or after 4 p.m.

15. Given a family of curves satisfies: \( x^2 + 4y^2 = c_1 \implies 2x + 8y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{4y} \).

Orthogonal trajectories satisfy:

\[
\frac{dy}{dx} = \frac{4y}{x} \implies \frac{1}{y} \frac{dy}{dx} = \frac{4}{x} \implies \frac{d}{dx}(\ln |y|) = \frac{4}{x} \implies \ln |y| = 4 \ln |x| + c_2 \implies y = kx^4, \text{ where } k = \pm e^{c_2}.
\]
17. Given family of curves satisfies: \( y = cx^2 \implies c = \frac{y}{x^2} \). Hence,

\[
\frac{dy}{dx} = 2cx = c \left( \frac{y}{x^2} \right) x = \frac{2y}{x}.
\]

Orthogonal trajectories satisfy:

\[
\frac{dy}{dx} = -x \implies 2y \frac{dy}{dx} = -x \implies \frac{d}{dx}(y^2) = -x \implies y^2 = -\frac{1}{2}x^2 + c_1 \implies 2y^2 + x^2 = c_2,
\]

where \( c_2 = 2c_1 \).
19. Given family of curves satisfies: \( y^2 = 2x + c \implies \frac{dy}{dx} = \frac{1}{y} \). Orthogonal trajectories satisfy:

\[
\frac{dy}{dx} = -y \implies y^{-1} \frac{dy}{dx} = -1 \implies \frac{d}{dx} (\ln |y|) = -1 \implies \ln |y| = -x + c_1 \implies y = c_2 e^{-x}.
\]

![Figure 0.0.3: Figure for Exercise 19](image)

21. \( y = mx + c \implies \frac{dy}{dx} = m \).

Orthogonal trajectories satisfy:

\[
\frac{dy}{dx} = -\frac{1}{m} \implies y = -\frac{1}{m} x + c_1.
\]

23. \( y^2 + mx^2 = c \implies 2y \frac{dy}{dx} + 2mx = 0 \implies \frac{dy}{dx} = -\frac{mx}{y} \).

Orthogonal trajectories satisfy:

\[
\frac{dy}{dx} = \frac{y}{mx} \implies y^{-1} \frac{dy}{dx} = \frac{1}{mx} \implies \frac{d}{dx} (\ln |y|) = \frac{1}{mx} \implies m \ln |y| = \ln |x| + c_1 \implies y^m = c_2 x.
\]

25. \( u = x^2 + 2y^2 \implies 0 = 2x + 4y \frac{dy}{dx} \implies \frac{dy}{dx} = -\frac{x}{2y} \).

Orthogonal trajectories satisfy:

\[
\frac{dy}{dx} = \frac{2y}{x} \implies y^{-1} \frac{dy}{dx} = \frac{2}{x} \implies \frac{d}{dx} (\ln |y|) = \frac{2}{x} \implies \ln |y| = 2 \ln |x| + c_1 \implies y = c_2 x^2.
\]

**Solutions to Section 1.2**
True-False Review:

1. **FALSE.** The order of a differential equation is the order of the *highest* derivative appearing in the differential equation.

3. **TRUE.** This is the content of Theorem 1.2.15.

5. **FALSE.** There are solutions to $y'' + y = 0$ that do not have the form $c_1 \cos x + 5c_1 \sin x$, such as $y(x) = \cos x + \sin x$. Therefore, $c_1 \cos x + 5c_1 \sin x$ does not meet the second requirement set form in Definition 1.2.11 for the general solution.

Problems:

1. 2, nonlinear.

3. 2, nonlinear.

5. 4, linear.

7. We can quickly compute the first two derivatives of $y(x)$:

$$y'(x) = (c_1 + 2c_2)e^x \cos 2x + (-2c_1 + c_2)e^x \sin 2x$$

and

$$y''(x) = (-3c_1 + 4c_2)e^x \cos 2x + (-4c_1 - 3c_2)e^x \sin 2x.$$

Then we have

$$y'' - 2y' + 5y =
\left[(-3c_1 + 4c_2)e^x \cos 2x + (-4c_1 - 3c_2)e^x \sin 2x\right] - 2 \left[(c_1 + 2c_2)e^x \cos 2x + (-2c_1 + c_2)e^x \sin 2x\right] + 5 \left(c_1 e^x \cos 2x + c_2 e^x \sin 2x\right),$$

which cancels to 0, as required. This solution is valid for all $x \in \mathbb{R}$.

9. $y(x) = \frac{1}{x + 4} \implies y' = -\frac{1}{(x + 4)^2} = -y^2$. Thus $y(x) = \frac{1}{x + 4}$ is a solution of the given differential equation for $x \in (-\infty, -4)$ or $x \in (-4, \infty)$. 
11. \( y(x) = c_1 e^{-x} \sin(2x) \implies y' = 2c_1 e^{-x} \cos(2x) - c_1 e^{-x} \sin(2x) \implies y'' = -3c_1 e^{-x} \sin(2x) - 4c_1 e^{-x} \cos(2x) \implies y'' + 2y' + 5y = -3c_1 e^{-x} \sin(2x) - 4c_1 e^{-x} \cos(2x) + 2[2c_1 e^{-x} \cos(2x) - c_1 e^{-x} \sin(2x)] + 5[c_1 e^{-x} \sin(2x)] = 0. \) Consequently the given function is a solution to the Legendre equation with all \( x \in \mathbb{R}. \)

13. \( y(x) = \frac{c_1}{x^3} + \frac{c_2}{x} \implies y' = -\frac{3c_1}{x^4} - \frac{c_2}{x^2} \implies y'' = \frac{12c_1}{x^5} + \frac{2c_2}{x^3} \implies x^2y'' + 5xy' + 3y = x^2 \left( \frac{12c_1}{x^5} + \frac{2c_2}{x^3} \right) + 5x \left( -\frac{3c_1}{x^4} - \frac{c_2}{x^2} \right) + 3 \left( \frac{c_1}{x^3} + \frac{c_2}{x} \right) = 0. \) Thus \( y(x) = \frac{c_1}{x^3} + \frac{c_2}{x} \) is a solution to the given differential equation for all \( x \in (-\infty, 0) \) or \( x \in (0, \infty). \)

15. \( y(x) = c_1 x^2 + c_2 x^3 - x^2 \sin x \implies y' = 2c_1 x + 3c_2 x^2 - x^2 \cos x - 2x \sin x \implies y'' = 2c_1 + 6c_2 x + x^2 \sin x - 2x \cos x - 2x \cos 2x \sin x. \) Substituting these results into the given differential equation yields
\[
x^2 y'' - 4xy' + 6y = x^2 (2c_1 + 6c_2 x + x^2 \sin x - 4x \cos x - 2 \sin x) - 4x (2c_1 x + 3c_2 x^2 - x^2 \cos x - 2x \sin x) + 6(c_1 x^2 + c_2 x^3 - x^2 \sin x)
\]
\[
= 2c_1 x^2 + 6c_2 x^3 + x^4 \sin x - 4x^3 \cos x - 2x^2 \sin x - 8c_1 x^2 - 12c_3 x^3 + 4x^3 \cos x + 8x^2 \sin x + 6c_1 x^2 + 6c_2 x^3 - 6x^2 \sin x
\]
\[
= x^4 \sin x.
\]
Hence, \( y(x) = c_1 x^2 + c_2 x^3 - x^2 \sin x \) is a solution to the differential equation for all \( x \in \mathbb{R}. \)

17. \( y(x) = e^{ax} (c_1 + c_2 x) \implies y' = e^{ax} (c_2) + ae^{ax} (c_1 + c_2 x) = e^{ax} (c_2 + ac_1 + ac_2 x) \implies y'' = ea^{ax} (ac_2) + ae^{ax} (c_2 + ac_1 + ac_2 x) = ae^{ax} (2c_2 + ac_1 + ac_2 x). \) Substituting these into the differential equation yields
\[
y'' - 2ay' + a^2 y = ae^{ax} (2c_2 + ac_1 + ac_2 x) - 2ae^{ax} (c_2 + ac_1 + ac_2 x) + a^2 e^{ax} (c_1 + c_2 x)
\]
\[
= ae^{ax} (2c_2 + ac_1 + ac_2 x - 2c_2 - 2ac_1 - 2ac_2 x + ac_1 + ac_2 x)
\]
\[
= 0.
\]
Thus, \( y(x) = e^{ax} (c_1 + c_2 x) \) is a solution to the given differential equation for all \( x \in \mathbb{R}. \)

19. \( y(x) = e^{rx} \implies y' = re^{rx} \implies y'' = r^2 e^{rx}. \) Substituting these results into the given differential equation yields \( e^{rx} (r^2 + 2r - 3) = 0, \) so that \( r \) must satisfy \( r^2 + 2r - 3 = 0, \) or \( (r + 3)(r - 1) = 0. \) Consequently \( r = -3 \) and \( r = 1 \) are the only values of \( r \) for which \( y(x) = e^{rx} \) is a solution to the given differential equation. The corresponding solutions are \( y(x) = e^{-3x} \) and \( y(x) = e^x. \)

21. \( y(x) = x^r \implies y' = r x^{r-1} \implies y'' = r(r-1) x^{r-2}. \) Substituting into the given differential equation yields \( x^r (r(r-1) + r - 1) = 0, \) so that \( r \) must satisfy \( r^2 - 1 = 0. \) Consequently \( r = -1 \) and \( r = 1 \) are the only values of \( r \) for which \( y(x) = x^r \) is a solution to the given differential equation. The corresponding solutions are \( y(x) = x^{-1} \) and \( y(x) = x. \)

23. \( y(x) = \frac{1}{2} (5x^2 - 3) = \frac{1}{2} (5x^2 - 3x) \implies y'' = \frac{1}{2} (15x^2 - 3) \implies y'' = 15x. \) Substitution into the Legendre equation with \( N = 3 \) yields \( (1 - x^2)y'' - 2xy' + 12y = (1 - x^2)(15x) + x(15x^2 - 3) + 6x(5x^2 - 3) = 0. \) Consequently the given function is a solution to the Legendre equation with \( N = 3. \)

25. \( x \sin y - e^x = c \implies x \cos y \frac{dy}{dx} + \sin y - e^x = 0 \implies \frac{dy}{dx} = e^{-x} - \sin y \frac{x \cos y}. \)

27. \( e^{xy} + x = c \implies e^{xy} \frac{dy}{dx} + y - 1 = 0 \implies xe^{xy} \frac{dy}{dx} + ye^{xy} = 1 \implies \frac{1 - ye^{xy}}{xe^{xy}}. \) Given \( y(1) = 0 \implies e^0(1) = 1 \implies c = 0. \) Therefore, \( e^{xy} - x = 0, \) so that \( y = \ln \frac{x}{x}. \)
29. \[ x^2y^2 - \sin x = c \implies 2x^2y \frac{dy}{dx} + 2xy^2 - \cos x = 0 \implies \frac{dy}{dx} = \frac{\cos x - 2xy^2}{2x^2y}. \] Since \( y(\pi) = \frac{1}{\pi} \), then \[ \pi^2 \left( 1 - \frac{1}{\pi} \right)^2 - \sin \pi = c \implies c = 1. \] Hence, \( x^2y^2 - \sin x = 1 \) so that \( y^2 = \frac{1 + \sin x}{x^2} \). Since \( y(\pi) = \frac{1}{\pi} \), take the branch of \( y \) where \( x < 0 \) so \( y(x) = \frac{\sqrt{1 + \sin x}}{x} \).

31. \[ \frac{dy}{dx} = x^{-1/2} \implies y(x) = 2x^{1/2} + c \text{ for all } x > 0. \]

33. \[ \frac{d^2y}{dx^2} = x^n, \text{ where } n \text{ is an integer.} \]

If \( n = -1 \) then \[ \frac{dy}{dx} = \ln |x| + c_1 \implies y(x) = x \ln |x| + c_1x + c_2 \text{ for all } x \in (-\infty, 0) \text{ or } x \in (0, \infty). \]

If \( n = -2 \) then \[ \frac{dy}{dx} = -x^{-1} + c_1 \implies y(x) = c_1x + c_2 - \ln |x| \text{ for all } x \in (-\infty, 0) \text{ or } x \in (0, \infty). \]

If \( n \neq -1 \) and \( n \neq -2 \) then \[ \frac{dy}{dx} = \frac{x^{n+1}}{n+1} + c_1 \implies \frac{x^{n+2}}{(n+1)(n+2)} + c_1x + c_2 \text{ for all } x \in \mathbb{R}. \]

35. \[ \frac{d^2y}{dx^2} = \cos x \implies \frac{dy}{dx} = \sin x + c_1 \implies y(x) = -\cos x + c_1x + c_2. \]

Thus, \( y'(0) = 1 \implies c_1 = 1 \), and \( y(0) = 2 \implies c_2 = 3 \). Thus, \( y(x) = 3 + x - \cos x. \)

37. \[ y'' = xe^x \implies y' = xe^x - e^x + c_1 \implies y = xe^x - 2e^x + c_1x + c_2. \]

Thus, \( y'(0) = 4 \implies c_1 = 5 \), and \( y(0) = 3 \implies c_2 = 5 \). Thus, \( y(x) = xe^x - 2e^x + 5x + 5. \)

39. \[ \frac{d^2y}{dx^2} = e^{-x} \implies \frac{dy}{dx} = -e^{-x} + c_1 \implies y(x) = e^{-x} + c_1x + c_2. \]

Thus, \( y(0) = 1 \implies c_2 = 0 \), and \( y(1) = 0 \implies c_1 = -\frac{1}{e}. \) Hence, \( y(x) = e^{-x} - \frac{1}{e}x. \)

41. \[ y(x) = c_1 \cos x + c_2 \sin x \]

(a) \( y(0) = 0 \implies 0 = c_1(1) + c_2(0) \implies c_1 = 0, y(\pi) = 1 \implies 1 = c_2(0), \text{ which is impossible. No solutions.} \)

(b) \( y(0) = 0 \implies 0 = c_1(1) + c_2(0) \implies c_1 = 0. y(\pi) = 0 \implies 0 = c_2(0), \text{ so } c_2 \text{ can be anything. Infinitely many solutions.} \)

42-47. Use some kind of technology to define each of the given functions. Then use the technology to simplify the expression given on the left-hand side of each differential equation and verify that the result corresponds to the expression on the right-hand side.

49. (a) \[ J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left( \frac{x}{2} \right)^{2k} = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + \ldots \]

(b) A Maple plot of \( J(0, x, 4) \) is given in the accompanying figure. From this graph, an approximation to the first positive zero of \( J_0(x) \) is 2.4. Using the Maple internal function BesselJZeros gives the approximation 2.404825558.

(c) A Maple plot of the functions \( J_0(x) \) and \( J(0, x, 4) \) on the interval \([0,2]\) is given in the accompanying figure. We see that to the printer resolution, these graphs are indistinguishable. On a larger interval, for example, \([0,3]\), the two graphs would begin to differ dramatically from one another.

(d) By trial and error, we find the smallest value of \( m \) to be \( m = 11 \). A plot of the functions \( J(0, x) \) and \( J(0, x, 11) \) is given in the accompanying figure.
Solutions to Section 1.3

True-False Review:

1. **TRUE.** This is precisely the remark after Theorem 1.3.2.

3. **FALSE.** This differential equation has equilibrium solutions $y(x) = 2$ and $y(x) = -2$.

5. **TRUE.** Equilibrium solutions are always horizontal lines. These are always parallel to each other.

7. **TRUE.** An equilibrium solution is a solution, and two solution curves to the differential equation $\frac{dy}{dx} = f(x, y)$ do not intersect.

Problems:

1. $y = cx^{-1} \implies c = xy$. Hence, $\frac{dy}{dx} = -cx^{-2} = -(xy)x^{-2} = -\frac{y}{x}$.
3. \( x^2 + y^2 = 2cx \implies \frac{x^2 + y^2}{2x} = c \). Hence, \( 2x + 2y \frac{dy}{dx} = 2c = \frac{x^2 + y^2}{x} \), so that, \( \frac{dy}{dx} = \frac{x^2 + y^2}{2x} - x \).

Consequently, \( \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \).

5. \( 2cy = x^2 - c^2 \implies c^2 + 2cy - x^2 = 0 \implies c = -\frac{2y \pm \sqrt{4y^2 + 4x^2}}{2} = -y \pm \sqrt{x^2 + y^2} \).

7. \( (x - c)^2 + (y - c)^2 = 2c^2 \implies x^2 - 2cx + y^2 - 2cy = 0 \implies c = \frac{x^2 + y^2}{2(x + y)} \). Differentiating the given equation yields \( 2(x - c) + 2(y - c) \frac{dy}{dx} = 0 \), so that \( 2 \left[ x - \frac{x^2 + y^2}{2(x + y)} \right] + 2 \left[ y - \frac{x^2 + y^2}{2(x + y)} \right] \frac{dy}{dx} = 0 \), that is \( \frac{dy}{dx} = \frac{-x^2 + 2xy - y^2}{y^2 + 2xy - x^2} \).

9. \( y = cx^3 \implies \frac{dy}{dx} = 3cx^2 = 3 \frac{y}{x^2} x^2 = \frac{3y}{x} \). The initial condition \( y(2) = 8 \implies 8 = c(2)^3 \implies c = 1 \). Thus the unique solution to the initial value problem is \( y = x^3 \).

11. \( (x - c)^2 + y^2 = c^2 \implies x^2 - 2cx + c^2 + y^2 = c^2 \), so that
\[
\begin{align*}
x^2 - 2cx + y^2 &= 0. \\
\end{align*}
\]

(0.0.1)

Differentiating with respect to \( x \) yields
\[
\begin{align*}
2x - 2c + 2y \frac{dy}{dx} &= 0. \\
\end{align*}
\]

(0.0.2)

But from (0.0.1), \( c = \frac{x^2 + y^2}{2x} \) which, when substituted into (0.0.2), yields \( 2x - \left( \frac{x^2 + y^2}{x} \right) + 2y \frac{dy}{dx} = 0 \), that is, \( \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \). Imposing the initial condition \( y(2) = 2 \implies \) from (0.0.1) \( c = 2 \), so that the unique solution to the initial value problem is \( y = +\sqrt{x(4 - x)} \).
13. \( \frac{dy}{dx} = \frac{x}{x^2 + 1}(y^2 - 9), \ y(0) = 3. \)
\[ f(x, y) = \frac{x}{x^2 + 1} (y^2 - 9), \text{ which is continuous for all } x, y \in \mathbb{R}. \]

\[ \frac{\partial f}{\partial y} = \frac{2xy}{x^2 + 1}, \text{ which is continuous for all } x, y \in \mathbb{R}. \]

So the IVP stated above has a unique solution on any interval containing (0, 3). By inspection we see that \( y(x) = 3 \) is the unique solution.

15. (a) \( f(x, y) = -2xy^2 \Rightarrow \frac{\partial f}{\partial y} = -4xy. \) Both of these functions are continuous for all \((x, y), \) and therefore the hypothesis of the uniqueness and existence theorem are satisfied for any \((x_0, y_0). \)

(b) \( y(x) = \frac{1}{x^2 + c} \Rightarrow y' = -\frac{2x}{(x^2 + c)^2} = -2xy^2. \)

(c) \( y(x) = \frac{1}{x^2 + c}. \)

(i) \( y(0) = 1 \Rightarrow 1 = \frac{1}{c} \Rightarrow c = 1. \) Hence, \( y(x) = \frac{1}{x^2 + 1}. \) The solution is valid on the interval \((-\infty, \infty). \)

(ii) \( y(1) = 1 \Rightarrow 1 = \frac{1}{1 + c} \Rightarrow c = 0. \) Hence, \( y(x) = \frac{1}{x^2}. \) This solution is valid on the interval \((0, \infty). \)

(iii) \( y(0) = -1 \Rightarrow -1 = \frac{1}{c} \Rightarrow c = -1. \) Hence, \( y(x) = \frac{1}{x^2 - 1}. \) This solution is valid on the interval \((-1, 1). \) (d) Since, by inspection, \( y(x) = 0 \) satisfies the given IVP, it must be the unique solution to the IVP.

17. \( y' = 4x. \) There are no equilibrium solutions. The slope of the solution curves is positive for \( x > 0 \) and is negative for \( x < 0. \) The isoclines are the lines \( x = \pm \frac{1}{2}. \)

<table>
<thead>
<tr>
<th>Slope of Solution Curve</th>
<th>Equation of Isocline</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>( x = -1 )</td>
</tr>
<tr>
<td>-2</td>
<td>( x = -1/2 )</td>
</tr>
<tr>
<td>0</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( x = 1/2 )</td>
</tr>
<tr>
<td>4</td>
<td>( x = 1 )</td>
</tr>
</tbody>
</table>

19. \( y' = x + y. \) There are no equilibrium solutions. The slope of the solution curves is positive for \( y > -x, \) and negative for \( y < -x. \) The isoclines are the lines \( y + x = k. \)
Figure 0.0.11: Figure for Exercise 15c(ii)

Figure 0.0.12: Figure for Exercise 15c(iii)

<table>
<thead>
<tr>
<th>Slope of Solution Curve</th>
<th>Equation of Isocline</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>$y = -x - 2$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$y = -x - 1$</td>
</tr>
<tr>
<td>$0$</td>
<td>$y = -x$</td>
</tr>
<tr>
<td>$1$</td>
<td>$y = -x + 1$</td>
</tr>
<tr>
<td>$2$</td>
<td>$y = -x + 2$</td>
</tr>
</tbody>
</table>
Since the slope of the solution curve along the isocline $y = -x - 1$ coincides with the slope of the isocline, it follows that $y = -x - 1$ is a solution to the differential equation. Differentiating the given differential equation yields: $y'' = 1 + y' = 1 + x + y$. Hence the solution curves are concave up for $y > -x - 1$, and concave down for $y < -x - 1$. Putting this information together leads to the slope field in the accompanying figure.

21. $y' = -\frac{4x}{y}$. Slope is zero when $x = 0 (y \neq 0)$. The solutions have a vertical tangent line at all points along the $x$-axis (except the origin). The isoclines are the lines $-\frac{4x}{y} = k$. Some values are given in the table below.
Differentiating the given differential equation yields: 
\[ y' = -\frac{4}{y} + \frac{4xy'}{y^2} = -\frac{4}{y} - \frac{16x^2}{y^3} = -\frac{4(y^2 + 4x^2)}{y}. \]

Consequently the solution curves are concave up for \( y < 0 \), and concave down for \( y > 0 \). Putting this information together leads to the slope field in the accompanying figure.

**Figure 0.0.15: Figure for Exercise 21**

23. \( y' = x^2 \cos y \). The slope is zero when \( x = 0 \). There are equilibrium solutions when \( y = (2k + 1) \frac{\pi}{2} \). The slope field is best sketched using technology. The accompanying figure gives the slope field for \(-\frac{\pi}{2} < y < \frac{3\pi}{2}\).

25. \( \frac{dT}{dt} = -\frac{1}{80}(T - 70) \). Equilibrium solution: \( T(t) = 70 \). The slope of the solution curves is positive for \( T > 70 \), and negative for \( T < 70 \). \( \frac{d^2T}{dt^2} = -\frac{1}{80} \frac{dT}{dt} = -\frac{1}{6400} (T - 70) \). Hence the solution curves are concave up for \( T > 70 \), and concave down for \( T < 70 \). The isoclines are the horizontal lines \(-\frac{1}{80}(T - 70) = k\).
27. \( y' = \frac{x \sin x}{1 + y^2} \).

29. \( y' = 2x^2 \sin y \).

31. \( y' = \frac{1 - y^2}{2 + 0.5x^2} \).

33. (a) Differentiating the given equation gives \( \frac{dy}{dx} = 2kx = 2\frac{y}{x} \). Hence the differential equation of the orthogonal trajectories is \( \frac{dy}{dx} = -\frac{x}{2y} \).

(b) The orthogonal trajectories appear to be ellipses. This can be verified by integrating the differential equation derived in (a).

**Solutions to Section 1.4**
Figure 0.0.18: Figure for Exercise 27

Figure 0.0.19: Figure for Exercise 29

Figure 0.0.20: Figure for Exercise 31

True-False Review:
1. **TRUE.** The differential equation \( \frac{dy}{dx} = f(x)g(y) \) can be written \( \frac{1}{g(y)} \frac{dy}{dx} = f(x) \), which is the proper form, according to Definition 1.4.1, for a separable differential equation.

3. **TRUE.** Newton’s Law of Cooling is usually expressed as \( \frac{dT}{dt} = -k(T - T_m) \), and this can be rewritten as

\[
\frac{1}{T - T_m} \frac{dT}{dt} = -k,
\]

and this form shows that the equation is separable.

5. **FALSE.** The expression \( x \sin(xy) \) cannot be separated in the form \( f(x)g(y) \), so the equation is not separable.

7. **TRUE.** We can write the given equation as \( (1 + y^2) \frac{dy}{dx} = \frac{1}{x^2} \), which is the proper form for a separable equation.

9. **TRUE.** We can write \( \frac{x^2y + x^2y^2}{x^2 + xy} = xy \), so we can write the given differential equation as \( \frac{1}{y} \frac{dy}{dx} = x \), which is the proper form for a separable equation.

**Problems:**

1. Separating the variables and integrating yields

\[
\int \frac{dy}{y} = 2 \int xdx \implies \ln|y| = x^2 + c_1 \implies y(x) = ce^{x^2}.
\]

3. Separating the variables and integrating yields

\[
\int e^ydy = \int e^{-x} dx = 0 \implies e^y + e^{-x} = c \implies y(x) = \ln(c - e^{-x}).
\]

5. Separating the variables and integrating yields

\[
\int \frac{dx}{x - 2} = \int \frac{dy}{y} \implies \ln|x - 2| - \ln|y| = c_1 \implies y(x) = c(x - 2).
\]
7. $y - x \frac{dy}{dx} = 3 - 2x^{2} \frac{dy}{dx} \implies x(2x - 1) \frac{dy}{dx} = (3 - y)$. Separating the variables and integrating yields

$$- \int \frac{dy}{y - 3} = \int \frac{dx}{x(2x - 1)} \implies - \ln |y - 3| = - \int \frac{dx}{x} + \int \frac{2}{2x - 1} \, dx$$

$$\implies - \ln |y - 3| = - \ln |x| + \ln |2x - 1| + c_{1}$$

$$\implies \frac{x}{(y - 3)(2x - 1)} = c_{2} \implies y(x) = \frac{cx - 3}{2x - 1}.$$  

9. $\frac{dy}{dx} = \frac{x(y^{2} - 1)}{2(x - 2)(x - 1)} \implies \int \frac{dy}{y + 1} = \frac{1}{2} \int \frac{x \, dx}{(x - 2)(x - 1)} \implies y(x) = \frac{c(x - 2)^{2}}{(x - 1) - c(x - 2)^{2}}$. By inspection we see that $y(x) = 1$, and $y(x) = -1$ are solutions of the given differential equation. The former is included in the above solution when $c = 0$.

11. $(x-a)(x-b) \frac{dy}{dx} - (y-c) = 0 \implies \int \frac{dy}{y - c} = \int \frac{dx}{(x-a)(x-b)} \implies \int \frac{dy}{y - c} = \frac{1}{a-b} \int \left( \frac{1}{x-a} - \frac{1}{x-b} \right) \, dx \implies y(x) = c + c_{2} \left( \frac{x-a}{x-b} \right)^{1/(a-b)}.$

13. $(1 - x^{2}) \frac{dy}{dx} + xy = a \implies \int \frac{dy}{y^{3}} = \int \frac{x \, dx}{x^{2}} \implies y(x) = a + c\sqrt{1-x^{2}}$, but $y(0) = 2a$ so $c = a$ and therefore, $y(x) = a(1 + \sqrt{1-x^{2}})$.

15. $\frac{dy}{dx} = y^{3} \sin x \implies \int \frac{dy}{y^{3}} = \int \sin x \, dx$ for $y \neq 0$. Thus $-\frac{1}{2y^{2}} = -\cos x + c$. However, we cannot impose the initial condition $y(0) = 0$ on the last equation since it is not defined at $y = 0$. But, by inspection, $y(x) = 0$ is a solution to the given differential equation and further, $y(0) = 0$; thus, the unique solution to the initial value problem is $y(x) = 0$.

17. (a) $m \frac{dv}{dt} = mg - kv^{2} \implies m \int \frac{dv}{k(v^{2} - \frac{mg}{k})} = dt$. If we let $a = \sqrt{\frac{mg}{k}}$ then the preceding equation can be written as $m \int \frac{1}{a^{2} - v^{2}} \, dv = f(t)$ which can be integrated directly to obtain

$$\frac{m}{2ak} \ln \left( \frac{a + v}{a - v} \right) = t + c,$$

that is, upon exponentiating both sides,

$$\frac{a + v}{a - v} = c_{1} e^{\frac{2ak}{m} t}.$$  

Imposing the initial condition $v(0) = 0$, yields $c = 0$ so that

$$\frac{a + v}{a - v} = e^{\frac{2ak}{m} t}.$$
Therefore,

\[ v(t) = a \left( \frac{e^{2 \alpha t^2} - 1}{e^{2 \alpha t^2} + 1} \right) \]

which can be written in the equivalent form

\[ v(t) = a \tanh \left( \frac{gt}{a} \right). \]

(b) From the chain rule we have \( \frac{dv}{dT} \) or equivalently, \( \frac{dv}{dt} \) and so \( v \) equation (1.4.17) yields \( \frac{dv}{dt} \) and so

\[ v \]

\[ dT \]

\[ \frac{dv}{T} \]

\[ \frac{dv}{t} \]

\[ dt \]

\[ dT \]

\[ (b) \]

\[ \frac{dv}{dt} \]

\[ \frac{dv}{t} \]

\[ dt \]

\[ (b) \]

\[ \frac{dv}{dt} \]

\[ \frac{dv}{t} \]

\[ dt \]

\[ (b) \]

\[ \frac{dv}{dt} \]

\[ \frac{dv}{t} \]

\[ dt \]

\[ (b) \]

\[ \frac{dv}{dt} \]

\[ \frac{dv}{t} \]

\[ dt \]

\[ (b) \]

\[ \frac{dv}{dt} \]

\[ \frac{dv}{t} \]

\[ dt \]

\[ (b) \]

\[ \frac{dv}{dt} \]

\[ \frac{dv}{t} \]

\[ dt \]

\[ (b) \]

\[ \frac{dv}{dt} \]

\[ \frac{dv}{t} \]

\[ dt \]

\[ (b) \]

\[ \frac{dv}{dt} \]

\[ \frac{dv}{t} \]

\[ dt \]

\[ (b) \]

\[ \frac{dv}{dt} \]

\[ \frac{dv}{t} \]

\[ dt \]

\[ (b) \]

\[ v(t) = a \tanh \left( \frac{gt}{a} \right). \]

19. The required curve is the solution curve to the IVP \( \frac{dy}{dx} = e^{2x} - y, y(3) = 1. \) Separating the variables in the differential equation yields \( e^{2x} \frac{dy}{dx} = e^x + c. \) Imposing the initial condition we obtain \( c = e - e^2, \) so that the solution curve has the equation \( e^x = e^x + e - e^2, \) or equivalently, \( y = \ln(e^x + e - e^2). \)

21. (a) Separating the variables in the given differential equation yields \( \frac{1}{1 + v^2} \frac{dv}{dt} = -dt. \) Integrating we obtain \( \tan^{-1}(v) = -t + c. \) The initial condition \( v(0) = v_0 \) implies that \( c = \tan^{-1}(v_0), \) so that \( \tan^{-1}(v) = -t + \tan^{-1}(v_0). \) The object will come to rest if there is time \( t, \) at which the velocity is zero. To determine \( t_r, \) we set \( v = 0 \) in the previous equation which yields \( \tan^{-1}(0) = t_r + \tan^{-1}(v_0). \) Consequently, \( t_r = \tan^{-1}(v_0). \)

The object does not remain at rest since we see from the given differential equation that \( \frac{dv}{dt} < 0 \) att \( t = t_r, \) and so \( v \) is decreasing with time. Consequently \( v \) passes through zero and becomes negative for \( t < t_r. \)

(b) From the chain rule we have \( \frac{dv}{dt} = \frac{dx}{dt}. \) Then \( \frac{dv}{dx} = v \frac{dv}{dx}. \) Substituting this result into the differential equation (1.4.17) yields \( \frac{dv}{dx} = -(1 + v^2). \) We now separate the variables \( \frac{v}{1 + v^2} \frac{dv}{dx} = -dx. \) Integrating we obtain \( \ln(1 + v^2) = -2x + c. \) Imposing the initial condition \( v(0) = v_0, x(0) = 0 \) implies that \( c = \ln(1 + v_0^2), \) so that \( \ln(1 + v^2) = -2x + \ln(1 + v_0^2). \) When the object comes to rest the distance traveled by the object is \( x = \frac{1}{2} \ln(1 + v_0^2). \)

23. Solving \( p = p_0 \frac{P}{p_0}^{1/\gamma}. \) Consequently the given differential equation can be written as \( \frac{dp}{dt} = -g p_0 \frac{P}{p_0}^{1/\gamma} dy, \) or equivalently, \( p^{-1/\gamma} \frac{dp}{dt} = -g \frac{P}{p_0}^{1/\gamma} dy. \) This can be integrated directly to obtain \( \frac{\gamma}{\gamma - 1} = -g \frac{P}{p_0}^{1/\gamma} dy + c. \) At the center of the Earth we have \( p = p_0. \) Imposing this initial condition on the preceding solution gives \( c = \frac{\gamma}{\gamma - 1} p_0. \) Substituting this value of \( c \) into the general solution to the differential equation we find, after some simplification, \( p^{(\gamma - 1)/\gamma} = p_0^{(\gamma - 1)/\gamma} \left[ 1 - \frac{(\gamma - 1) p_0 g y}{\gamma p_0} \right], \) so that \( p = p_0 \left[ 1 - \frac{(\gamma - 1) p_0 g y}{\gamma p_0} \right]^{(\gamma - 1)/\gamma}. \)

25. \( \frac{dT}{dt} = -k(T - 450) \implies T(t) = 450 + C e^{-kt}. T(0) = 50 \implies C = -400 \) so \( T(t) = 450 - 400 e^{-kt} \) and \( T(20) = 150 \implies k = \frac{1}{20} \ln \frac{4}{3}; \) hence, \( T(t) = 450 - 400 (\frac{3}{4})^{t/20}. \)
(i) \[ T(40) = 450 - 400(\frac{3}{4})^2 = 225^\circ F. \]

(ii) \[ T(t) = 350 = 450 - 400(\frac{3}{4})^{t/20} \implies (\frac{3}{4})^{t/20} = \frac{1}{4} \implies t = \frac{20 \ln 4}{\ln(4/3)} \approx 96.4 \text{ minutes}. \]

27. \[ T(t) = 75 + ce^{-kt}. T(10) = 415 \implies 75 + ce^{-10k} = 415 \implies 340 = ce^{-10k} \text{ and } T(20) = 347 \implies 75 + ce^{-20k} = 347 \implies 272 = ce^{-20k}. \]

Solving these two equations yields \( k = \frac{1}{10} \ln \frac{5}{4} \) and \( c = 425 \); hence, \[ T(t) = 75 + 425(\frac{4}{5})^{t/10} \]

(a) Furnace temperature: \( T(0) = 500^\circ F. \)

(b) If \( T(t) = 100 \) then \[ 100 = 75 + 425(\frac{4}{5})^{t/10} \implies t = \frac{10 \ln 17}{\ln \frac{5}{4}} \approx 126.96 \text{ minutes.} \]

Thus the temperature of the coal was 100°F at 6:07 p.m.

**Solutions to Section 1.5**

**True-False Review:**

1. TRUE. The differential equation for such a population growth is \( \frac{dP}{dt} = kP \), where \( P(t) \) is the population as a function of time, and this is the Malthusian growth model described at the beginning of this section.

3. TRUE. The differential equation governing the logistic model is \( \frac{dP}{dt}(C - P) = r. \) Likewise, the differential equation governing the Malthusian growth model is \( \frac{dP}{dt} = kP \), and this is separable as \( \frac{1}{P} \frac{dP}{dt} = k. \)

5. TRUE. Every five minutes, the population doubles (increase 2-fold). Over 30 minutes, this population will double a total of 6 times, for an overall \( 2^6 = 64 \)-fold increase.

7. FALSE. The growth rate is \( \frac{dP}{dt} = kP \), and so as \( P \) changes, \( \frac{dP}{dt} \) changes. Therefore, it is not always constant.

9. FALSE. If the initial population is in the interval \( (\frac{C}{2}, C) \), then although it is less than the carrying capacity, its concavity does not change. To get a true statement, it should be stated instead that the initial population is less than half of the carrying capacity.

**Problems:**

1. \( \frac{dP}{dt} = kP \implies P(t) = P_0e^{kt} \). Since \( P(0) = 10 \), then \( P = 10e^{kt} \). Since \( P(3) = 20 \), then \( 2 = e^{3k} \implies k = \frac{\ln 2}{3} \). Thus \( P(t) = 10e^{(t/3)\ln 3} \). Therefore, \( P(24) = 10e^{(24/3)\ln 3} = 10 \cdot 2^8 = 2560 \) bacteria.

3. From \( P(t) = P_0e^{kt} \) and \( P(0) = 2000 \) it follows that \( P(t) = 2000e^{kt} \). Since \( t_d = 4, k = \frac{1}{4} \ln 2 \) so \( P = 2000e^{t\ln 2/4} \). Therefore, \( P(t) = 10^6 \implies 10^6 = 2000e^{t\ln 2/4} \implies t \approx 35.86 \text{ h.} \)

5. \( P(t) = \frac{50C}{50 + (C - 50)e^{-rt}} \). In formulas (1.5.5) and (1.5.6) we have \( P_0 = 500, P_1 = 800, P_2 = 1000, t_1 = 5, \) and \( t_2 = 10. \) Hence, \( r = \frac{1}{5} \ln \left[ \frac{(1000)(300)}{(500)(200)} \right] = \frac{1}{5} \ln 3, C = \frac{(800)(1500) - 2(500)(1000)}{800^2 - (500)(1000)} \approx 1142.86, \) so
that \( P(t) = \frac{1142.86(500)}{500 + 642.86e^{-0.2t \ln 3}} \approx \frac{571430}{500 + 642.86e^{-0.2t \ln 3}} \). Inserting \( t = 15 \) into the preceding formula yields \( P(15) = 1091.7 \).

7. From equation (1.5.5) \( r > 0 \) requires \( \frac{P_2(P_1 - P_0)}{P_0(P_2 - P_1)} > 1 \). Rearranging the terms in this inequality and using the fact that \( P_2 > P_1 \) yields \( P_1 > \frac{2P_0P_2}{P_0 + P_2} \). Further, \( C > 0 \) requires that \( \frac{P_1(P_0 + P_2) - 2P_0P_2}{P_1^2 - P_0P_2} > 0 \). From \( P_1 > \frac{2P_0P_2}{P_0 + P_2} \), we see that the numerator in the preceding inequality is positive, and therefore the denominator must also be positive. Hence in addition to \( P_1 > \frac{2P_0P_2}{P_0 + P_2} \), we must also have \( P_2 > P_0P_2 \).

9. (a) Equilibrium solutions: \( P(t) = 0, P(t) = T \).
Slope: \( P > T \implies \frac{dP}{dt} > 0, 0 < P < T \implies \frac{dP}{dt} < 0 \).

Isoclines: \( r(P - T) = k \implies P^2 - TP - \frac{k}{r} = 0 \implies P = \frac{1}{2} \left( T \pm \sqrt{\frac{rT^2 + 4k}{r}} \right) \). We see that slope of the solution curves satisfies \( k \geq -\frac{rT^2}{4} \).

Concavity: \( \frac{d^2P}{dt^2} = r(2P - T)\frac{dP}{dt} = r^2(2P - T)(P - T)P \). Hence, the solution curves are concave up for \( P > \frac{T}{2} \), and are concave down for \( 0 < P < \frac{T}{2} \).

(b) See accompanying figure.

Figure 0.0.22: Figure for Exercise 9(b)

(c) For \( 0 < P_0 < T \), the population dies out with time. For \( P_0 > T \), there is a population growth. The term threshold level is appropriate since \( T \) gives the minimum value of \( P_0 \) above which there is a population growth.

11. \( \frac{dP}{dt} = r(C - P)(P - T)P, P(0) = P_0, r > 0, 0 < T < C \).
Equilibrium solutions: \( P(t) = 0, P(t) = T, P(t) = C \). The slope of the solution curves is negative for \( 0 < P < T \), and for \( P > C \). It is positive for \( T < P < C \).

Concavity: \( \frac{d^2P}{dt^2} = r^2[(C - P)(P - T) - (P - T)P + (C - P)P](C - P)(P - T)P \), which simplifies to
\[
\frac{d^2 P}{dt^2} = r^2 (-3P^2 + 2PT + 2CP - CT)(C - P)(P - T).
\] Hence changes in concavity occur when
\[
P = \frac{1}{3}(C + T \pm \sqrt{C^2 - CT + T^2}).
\] A representative slope field with some solution curves is shown in the accompanying figure. We see that for \(0 < P_0 < T\) the population dies out, whereas for \(T < P_0 < C\) the population grows and asymptotes to the equilibrium solution \(P(t) = C\). If \(P_0 > C\), then the solution decays towards the equilibrium solution \(P(t) = C\).

![Figure](image)

**Figure 0.0.23: Figure for Exercise 11**

**13.** Separating the variables in (1.5.8) yields
\[
\frac{1}{P(\ln C - \ln P)} \frac{dP}{dt} = r
\]
which can be integrated directly to obtain
\[
- \ln (\ln C - \ln P) = rt + c \text{ so that } \ln \left(\frac{C}{P}\right) = c_1 e^{-rt}.
\]
The initial condition \(P(0) = P_0\) requires that \(\ln \left(\frac{C}{P_0}\right) = c_1\). Hence, \(\ln \left(\frac{C}{P}\right) = e^{-rt} \ln \left(\frac{C}{P_0}\right)\) so that \(P(t) = C e^{\ln(P_0/k)e^{-rt}}\). Since \(\lim_{t \to \infty} e^{-rt} = 0\), it follows that \(\lim_{t \to \infty} P(t) = C\).

**15. (a)** More.

(b) Using the exponential decay model we have \(\frac{dP}{dt} = kP\), which is easily integrated to obtain \(P(t) = P_0 e^{kt}\).

The initial condition \(P(0) = 100,000\) requires that \(P_0 = 100,000\), so that \(P(t) = 100,000 e^{kt}\). We also know that \(P(10) = 80,000\). This requires that \(100,000 = 80,000 e^{10k}\) so that \(k = \frac{1}{10} \ln \left(\frac{4}{5}\right)\). Consequently,

\[
P(t) = 100,000 e^{\frac{t}{10} \ln \left(\frac{4}{5}\right)}.
\]

Using (0.0.3), the half-life is determined from
\[
50,000 = 100,000 e^{\frac{t_H}{10} \ln \left(\frac{4}{5}\right)} \implies t_H = 10 \frac{\ln 2}{\ln \left(\frac{4}{5}\right)} \approx 31.06 \text{ min.}
\]

(c) Using (0.0.3) there will be 15,000 fans left in the stadium at time \(t_0\), where
\[
15,000 = 100,000 e^{\frac{t_0}{10} \ln \left(\frac{4}{5}\right)} \implies t_0 = 10 \frac{\ln \left(\frac{3}{20}\right)}{\ln \left(\frac{4}{15}\right)} \approx 85.02 \text{ min.}
\]

**17.** Maple, or even a TI 92 plus, has no problem in solving these equations.
19. \( P(t) = \frac{50C}{50 + (C - 50)e^{-rt}} \). Imposing the conditions \( P(5) = 100 \), \( P(15) = 250 \) gives the pair of equations

\[
100 = \frac{50C}{50 + (C - 50)e^{-5r}} \quad \text{and} \quad 250 = \frac{50C}{50 + (C - 50)e^{-15r}}
\]

whose positive solutions are \( C \approx 370.32 \), \( r \approx 0.17 \). Using these values for \( C \) and \( r \) gives \( P(t) = \frac{18500}{50 + 18450e^{-0.17t}} \). From the figure we see that it will take approximately 52 years to reach 95% of the carrying capacity.

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\hline
\text{t} & 0 & 10 & 20 & 30 & 40 & 50 & 60 & 70 \\
\hline
\text{P(t)} & 0 & 50 & 100 & 150 & 200 & 250 & 300 & 350 \\
\hline
\end{array}
\]

Figure 0.0.24: Figure for Exercise 19

**Solutions to Section 1.6**

**True-False Review:**

1. **FALSE.** Any solution to the differential equation (1.6.7) serves as an integrating factor for the differential equation. There are infinitely many solutions to (1.6.7), taking the form \( I(x) = c_1 e^{\int p(x)dx} \), where \( c_1 \) is an arbitrary constant.

3. **TRUE.** Multiplying \( y' + p(x)y = q(x) \) by \( I(x) \) yields \( y'I + pIy = qI \). Assuming that \( I' = pI \), the requirement on the integrating factor, we have \( y'I + I'y = qI \), or by the product rule, \( (I \cdot y)' = qI \), as requested.

5. **TRUE.** Rewriting the differential equation as

\[
\frac{dy}{dx} + \frac{1}{x}y = x,
\]

we have \( p(x) = \frac{1}{x} \), and so an integrating factor must have the form \( I(x) = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x + c} = e^x \).

Since \( 5x \) does indeed have this form, it is an integrating factor.

**Problems:**

In this section the function \( I(x) = e^{\int p(x)dx} \) will represent the integrating factor for a differential equation of the form \( y' + p(x)y = q(x) \).

1. \( y' - y = e^{2x} \). \( I(x) = e^{-\int dx} = e^{-x} \implies \frac{d(e^{-x}y)}{dx} = e^x \implies e^{-x}y = e^x + c \implies y(x) = e^x(e^x + c) \).
3. \( y' + 2xy = 2x^3 \). \( I(x) = e^2 \int x \, dx = e^2 \Rightarrow \frac{d}{dx}(e^2y) = 2e^2x^4 \Rightarrow e^2y = 2 \int e^2x^3 \, dx \Rightarrow e^2y = e^2(x^2 - 1) + c \Rightarrow y(x) = x^2 - 1 + ce^{-x^2}.

5. \( y' + \frac{2x}{1 + x^2}y = \frac{4}{1 + x^2} \). \( I(x) = e^{\int \frac{2x}{1 + x^2} \, dx} = 1 + x^2 \Rightarrow \frac{d}{dx}[(1 + x^2)y] = \frac{4}{1 + x^2} \Rightarrow (1 + x^2)y = 4 \int \frac{dx}{1 + x^2} \Rightarrow 1 + x^2) = 4 \tan^{-1}x + c \Rightarrow y(x) = \frac{1}{1 + x^2}(4 \tan^{-1}x + c).

7. \( y' + \frac{1}{x + x^2} = 9x^2 \). \( I(x) = e^{\int \frac{1}{x + x^2} \, dx} = \ln x \Rightarrow \frac{d}{dx}(y \ln x) = 9 \int x^2 \ln x - x^3 + c \Rightarrow y(x) = \frac{3x^3 \ln x - x^3 + c}{\ln x} \) but \( y(e) = 2e^3 \) so \( c = 0 \); thus, \( y(x) = \frac{x^3(3 \ln x - 1)}{\ln x} \).

9. \( \frac{dx}{dt} + 2x = 4e^t \Rightarrow x' + 2 \frac{t}{x} = 4. \) \( I(x) = e^{\int \frac{4t}{x} \, dt} = t^2 \Rightarrow \frac{d}{dt}(t^2x) = 4te^t \Rightarrow t^2x = 4 \int te^t \, dt + c \Rightarrow t^2x = 4e^{t(t - 1) + c} \Rightarrow \frac{x}{t^2} = e^{4t - (t - 1) + c} \).

11. \((1 - y \sin x) \, dx - \cos x \, dy = 0 \Rightarrow y' + (\sin x \sec x) = \sec x. \( I(x) = e^{\int \sin x \sec x \, dx} = \sec x \Rightarrow \frac{d}{dx}(y \sec x) = \sec^2 x \Rightarrow y \sec x = \int \sec^2 x \, dx + c \Rightarrow y(x) = \tan x + c \Rightarrow y(x) = \sin x + c \cos x.

13. \( y' + ay = e^{\beta x}. \) \( I(x) = e^{\alpha \int \, dx} = e^{\alpha x} \Rightarrow \frac{d}{dx}(e^{\alpha x}y) = e^{(\alpha + \beta)x} \Rightarrow e^{\alpha x}y = \int e^{(\alpha + \beta)x} \, dx + c. \) If \( \alpha + \beta = 0 \), then \( e^{\alpha x}y = x + c \Rightarrow y(x) = e^{-\alpha}y(x + c). \) If \( \alpha + \beta \neq 0 \), then \( e^{\alpha x}y = e^{\frac{(\alpha + \beta)x}{\alpha + \beta}} + c \Rightarrow y(x) = \frac{e^{\beta x}}{\alpha + \beta} + ce^{-\alpha x}.

15. \( y' + \frac{2}{x}y = 4x. \) \( I(x) = e^{\int \frac{4x}{x} \, dx} = e^{2 \ln x} = x^2 \Rightarrow \frac{d}{dx}(x^2y) = 4x^3 = x^2y = 4 \int x^3 \, dx + c \Rightarrow x^2y = x^4 + c, \) but \( y(1) = 2 \) so \( c = 1 \); thus, \( y(x) = \frac{e^{4x}}{x^2} - 1.

17. \( x' + \frac{2}{4 - t} = 5. \) \( I(t) = e^{\int \frac{2}{4 - t} \, dt} = e^{2 \ln(4-t)} = (4-t)^{-2} \Rightarrow \frac{d}{dt}((4-t)^{-2}x) = (4-t)^{-2} = 5 \int (4-t)^{-2} \, dt + c \Rightarrow (4-t)^{-2}x = 5(4-t)^{-1} + c, \) but \( x(0) = 4 \) so \( c = -1 \); thus, \( x(t) = (4-t)^2[5(4-t)^{-1} - 1] \) or \( x(t) = (4-t)(1+t).

19. \( y' + y = f(x), y(0) = 3, \) \( f(x) = \begin{cases} 1, & \text{if } x < 1, \\ 0, & \text{if } x > 1. \end{cases} \)

\( I(x) = e^{\int \, dx} = e^x \Rightarrow \frac{d}{dx}(e^y) = e^x f(x) \Rightarrow [e^y]_0^x = \int_0^x e^x f(x) \, dx \Rightarrow e^y - y(0) = \int_0^x e^x f(x) \, dx \Rightarrow e^y - 3 = \int_0^x e^x \, dx \Rightarrow y(x) = e^{-x} \left[ 3 + \int_0^x e^x f(x) \, dx \right]. \)

If \( x \leq 1, \int_0^x e^x f(x) \, dx = \int_0^1 e^x \, dx = e^x - 1 \Rightarrow y(x) = e^{-x}(2 + e^x) \)

If \( x > 1, \int_0^x e^x f(x) \, dx = \int_0^x e^x \, dx - e - 1 \Rightarrow y(x) = e^{-x}(2 + e). \)

21. On \((-\infty, 1), y' - y = 1 \Rightarrow I(x) = e^{-x} \Rightarrow y(x) = c_1 e^{-x} - 1. \) Imposing the initial condition \( y(0) = 0 \) requires \( c_1 = 1 \), so that \( y(x) = e^{-x} - 1, \) for \( x < 1. \)

On \([1, \infty), y' - y = 2 - x \Rightarrow I(x) = e^{-x} \Rightarrow \frac{d}{dx}(e^{-x}y) = (2 - x)e^{-x} \Rightarrow y(x) = x - 1 + c_2 e^{-x}. \)
Continuity at $x = 1$ requires that $\lim_{x \to 1} y(x) = y(1)$. Consequently we must choose $c_2$ to satisfy $c_2e = e - 1$, so that $c_2 = 1 - e^{-1}$. Hence, for $x \geq 1$, $y(x) = x - 1 + (1 - e^{-1})e^x$.

23. The differential equation for Newton’s Law of Cooling is $\frac{dT}{dt} = -k(T - T_m)$. We can re-write this equation in the form of a first-order linear differential equation: $\frac{dT}{dt} + kT = kT_m$. An integrating factor for this differential equation is $I = e^{\int k \, dt} = e^{kt}$. Thus, $\frac{d}{dt}(Te^{kt}) = kT e^{kt}$. Integrating both sides, we get $Te^{kt} = T_m e^{kt} + c$, and hence, $T = T_m + ce^{-kt}$, which is the solution to Newton’s Law of Cooling.

25. $\frac{dT_m}{dt} = 10 \implies T_m = 10t + c_1$ but $T_m = 65$ when $t = 0$ so $c_1 = 65$ and $T_m = 10t + 65$. $\frac{dT}{dt} = -k(T - T_m) \implies \frac{dT}{dt} = -k(10t - 65)$, but $\frac{dT}{dt}(1) = 5$, so $k = \frac{1}{8}$. The last differential equation can be written $\frac{dT}{dt} + kT = k(10t + 65) \implies \frac{d}{dt}(e^{kt}T) = 5ke^{kt}(2t + 13) \implies e^{kt}T = 5ke^{kt}\left(\frac{2}{k}t - \frac{2}{k^2} + \frac{13}{k}\right) + c \implies T = 5(2t - \frac{2}{k} + 13) + ce^{-kt}$, but $k = \frac{1}{8}$ so $T(t) = 5(2t - 3) + ce^{-\frac{1}{8}t}$. Since $T(1) = 35, c = 40e^{\frac{1}{8}}$. Thus, $T(t) = 10t - 15 + 40e^{\frac{1}{8}(1-t)}$.

27. (a) The temperature varies from a minimum of $A - B$ at $t = 0$ to a maximum of $A + B$ when $t = 12$.

(b) First write the differential equation in the linear form $\frac{dT}{dt} + k_1 T = k_1(A - B \cos \omega t) + T_0$. Multiplying by the integrating factor $I = e^{k_1 t}$ reduces this differential equation to the integrable form

$$\frac{d}{dt}(e^{k_1 t}T) = k_1 e^{k_1 t}(A - B \cos \omega t) + T_0 e^{k_1 t}.$$ 

Consequently,

$$e^{k_1 t}T(t) = \left(Ae^{k_1 t} - Bk_1 \int e^{k_1 t} \cos \omega t \, dt + \frac{T_0}{k_1} e^{k_1 t} + c\right).$$ 

so that

$$T(t) = A + \frac{T_0}{k_1} - \frac{Bk_1}{k_1^2 + \omega^2} (k_1 \cos \omega t + \omega \sin \omega t) + ce^{-k_1 t}.$$ 

This can be written in the equivalent form

$$T(t) = A + \frac{T_0}{k_1} - \frac{Bk_1}{\sqrt{k_1^2 + \omega^2}} \cos (\omega t - \alpha) + ce^{-k_1 t}.$$
for an approximate phase constant $\alpha$.

29. The associated homogeneous equation is $\frac{dy}{dx} + x^{-1}y = 0$, with solution $y_H = cx^{-1}$. According to problem 28 we determine the function $u(x)$ such that $y(x) = x^{-1}u(x)$ is a solution to the given differential equation. We have $\frac{dy}{dx} = x^{-1}\frac{du}{dx} - x^{-2}u$. Substituting into $\frac{dy}{dx} + x^{-1}y = \cos x$ yields $x^{-1}\frac{du}{dx} - \frac{1}{x^2}u + x^{-1}(x^{-1}u) = \cos x$, so that $\frac{du}{dx} = x \cos x$. Integrating we obtain $u = x \sin x + \cos x + c$, so that $y(x) = x^{-1}(x \sin x + \cos x + c)$.

31. The associated homogeneous equation is $\frac{dy}{dx} + \cot x \cdot y = 0$, with solution $y_H = c \cdot \csc x$. According to problem 28 we determine the function $u(x)$ such that $y(x) = \csc x \cdot u(x)$ is a solution to the given differential equation. We have $\frac{dy}{dx} = \csc x \cdot \frac{du}{dx} - \csc x \cdot \cot x \cdot u$. Substituting into $\frac{dy}{dx} + \cot x \cdot y = 2 \cos x$ yields $\csc x \cdot \frac{du}{dx} - \csc x \cdot \cot x \cdot u + \csc x \cdot \cot x \cdot u = \cos x$, so that $\frac{du}{dx} = 2 \cos x \sin x$. Integrating we obtain $u = \sin^2 x + c$, so that $y(x) = \csc x(\sin^2 x + c)$.

Problems 33 - 39 are easily solved using a differential equation solver such as the `dsolve` package in Maple.

**Solutions to Section 1.7**

**True-False Review:**

1. **TRUE.** Concentration of chemical is defined as the ratio of mass to volume; that is, $c(t) = \frac{A(t)}{V(t)}$. Therefore, $A(t) = c(t)V(t)$.

2. **TRUE.** This is reflected in the fact that $c_1$ is always assumed to be a constant.

3. **FALSE.** Kirchoff’s second law states that the sum of the voltage drops around a closed circuit is zero, not that it is independent of time.

4. **TRUE.** Due to the negative exponential in the formula for the transient current, $i_T(t)$, it decays to zero as $t \to \infty$. Meanwhile, the steady-state current, $i_S(t)$, oscillates with the same frequency $\omega$ as the alternating current, albeit with a phase shift.

**Problems:**

1. Given $V(0) = 10, A(0) = 20, c_1 = 4, r_1 = 2$, and $r_2 = 1$. Then $\Delta V = r_1 \Delta t - r_2 \Delta t \implies \frac{dV}{dt} = 1 \implies V(t) = t + 10$ since $V(0) = 10$. $\Delta A \approx c_1r_1\Delta t - c_2r_2\Delta t \implies \frac{dA}{dt} = 8 - c_2 = 8 - \frac{A}{V} \implies \frac{dA}{dt} + \frac{1}{t + 10}A = 8 \implies (t + 10)A = 4(t + 10)^2 + c_1$. Since $A(0) = 20 \implies c_1 = -200$ so $A(t) = \frac{4}{t + 10}[(t + 10)^2 - 50]$. Therefore, $A(40) = 196$ g.

3. Given $V(0) = 20, A(0) = 0, c_1 = 10, r_1 = 4$, and $r_2 = 2$. Then $\Delta V = r_1 \Delta t - r_2 \Delta t \implies \frac{dV}{dt} = 2 \implies V = 2(t + 10)$ since $V(0) = 20$. Thus $V(t) = 40$ for $t = 10$, so we must find $A(10)$. $\Delta A \approx c_1r_1\Delta t - c_2r_2\Delta t \implies \frac{dA}{dt} = 40 - 2c_2 = 40 - \frac{2A}{V} = 40 - \frac{A}{t + 10} \implies \frac{dA}{dt} + \frac{1}{t + 10}A = 40 \implies \frac{dA}{dt}[(t + 10)A] = 40(t + 10)dt \implies$
(t + 10)A = 20(t + 10)^2 + c. Since A(0) = 0 \implies c = -2000 so A(t) = \frac{20}{t + 10}|(t + 10)^2 - 100| and A(10) = 300 g.

5. Given V(0) = 10, A(0) = 0, c_1 = 0.5, r_1 = 3, r_2 = 1, and A(5)/V(5) = 0.2.

(a) ΔV = r_1Δt - r_2Δt \implies dV \implies V(t) = t + 10 since V(0) = 10. Then ΔA \approx c_1r_1Δt - c_2r_2Δt \implies \frac{dA}{dt} = -2c_2 = -2A = \frac{dA}{V} = \frac{2dt}{t + 10} \implies \ln |A| = -2\ln |t + 10| + c \implies A = k(t + 10)^{-2}. Then A(5) = 3 since V(5) = 15 and A(5)/V(5) = 0.2. Thus, k = 675 and A(t) = \frac{675}{(t + 10)^2}. In particular, A(0) = 6.75 g.

(b) Find V(t) when \frac{A(t)}{V(t)} = 0.1. From part (a) A(t) = \frac{675}{(t + 10)^2} and V(t) = t + 10 \implies A(t)/V(t) = \frac{675}{(t + 10)^3}.

Since \frac{A(t)}{V(t)} = 0.1 \implies (t + 10)^3 = 6750 \implies t + 10 = 15\sqrt{2} so V(t) = t + 10 = 15\sqrt{2} L.

7. Given V(0) = w, c_1 = k, r_1 = r, r_2 = r, and A(0) = A_0. Then ΔV = r_1Δr - r_2Δt \implies dV = 0 \implies V(t) = V(0) = w for all t. Then ΔA = c_1r_1Δt - c_2r_2Δt \implies \frac{dA}{dt} = kr - rA = kr - rA/V = kr - \frac{r}{w}A \implies \frac{dA}{dt} + \frac{r}{w}A = kr \implies \frac{d}{dt}(e^{-rt/w}A) = kwe^{-rt/w} \implies A(t) = kw + ce^{-rt/w}. Since A(0) = A_0 so c = A_0 - kw \implies A(t) = e^{-rt/w}[kw(e^{rt/w} - 1) + A_0].

(b) \lim_{t \to \infty} \frac{A(t)}{V(t)} = \lim_{t \to \infty} e^{-rt/w}[kw(e^{rt/w} - 1) + A_0] = \lim_{t \to \infty}[k + \left(\frac{A_0}{w} - k\right)e^{-rt/w}] = k. This is reasonable since the volume remains constant, and the solution in the tank is gradually mixed with and replaced by the solution of concentration k flowing in.

9. Let E(t) = 20, R = 4 and L = \frac{1}{5}. Then \frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}E(t) \implies \frac{di}{dt} + 40i = 200 \implies d\frac{e^{40t}}{dt} = 200e^{40t} \implies i(t) = 5 + ce^{-40t}. But i(0) = 0 \implies c = -5. Consequently i(t) = (1 - e^{-40t}).

11. Let R = 2, L = \frac{2}{3} and E(t) = 10\sin 4t. Then \frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}E(t) \implies \frac{di}{dt} + 3i = 15\sin 4t \implies \frac{d}{dt}(e^{3t}i) = 15e^{3t}\sin 4t \implies e^{3t}i = \frac{3e^{3t}}{5}(3\sin 4t - 4 \cos 4t) + c \implies i(t) = 3\left(\frac{3}{5}\sin 4t - \frac{4}{5}\cos 4t\right) + ce^{-3t}, but i(0) = 0 \implies c = \frac{12}{5} so i(t) = \frac{3}{5}(3\sin 4t - 4 \cos 4t + e^{-3t}).

13. In an RC circuit for t > 0 the differential equation is given by \frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}. If E(t) = 0 then \frac{dq}{dt} + \frac{1}{RC}q = 0 \implies \frac{d}{dt}(e^{t/RC}q) = 0 \implies q = ce^{t/RC} and if q(0) = 0 then q(t) = 5e^{-t/RC}. Then lim_{t \to \infty} q(t) = 0. Yes, this is reasonable. As the time increases and E(t) = 0, the charge will dissipate to zero.

15. In an RL circuit, \frac{di}{dt} + \frac{R}{L}i = \frac{E(t)}{L} and since E(t) = E_0\sin \omega t, then \frac{di}{dt} + \frac{R}{L}i = \frac{E_0}{L}\sin \omega t \implies \frac{d}{dt}(e^{Rt/L}i) = \frac{E_0}{R}e^{Rt/L}\sin \omega t \implies i(t) = \frac{E_0}{R^2 + L^2\omega^2}\left[R\sin \omega t - \omega L \cos \omega t\right] + Ae^{-Rt/L}. We can write this
as $i(t) = \frac{E_0}{\sqrt{R^2 + L^2 \omega^2}} \left[ \frac{R}{\sqrt{R^2 + L^2 \omega^2}} \sin \omega t - \frac{\omega L}{\sqrt{R^2 + L^2 \omega^2}} \cos \omega t \right] + Ae^{-Rt/L}$. Defining the phase $\phi$ by

$$\cos \phi = \frac{R}{\sqrt{R^2 + L^2 \omega^2}}, \sin \phi = \frac{\omega L}{\sqrt{R^2 + L^2 \omega^2}},$$

we have $i(t) = \frac{E_0}{\sqrt{R^2 + L^2 \omega^2}} (\cos \phi \sin \omega t - \sin \phi \cos \omega t) + Ae^{-Rt/L}$.

That is, $i(t) = \frac{E_0}{\sqrt{R^2 + L^2 \omega^2}} \sin (\omega t - \phi) + Ae^{-Rt/L}$.

**Transient part of the solution:** $i_T(t) = Ae^{-Rt/L}$.

**Steady state part of the solution:** $i_s(t) = \frac{E_0}{\sqrt{R^2 + L^2 \omega^2}} \sin (\omega t - \phi)$.

17. $\frac{dq}{dt} + \frac{1}{RC}q = \frac{E(t)}{L} \implies \frac{dq}{dt} + \frac{1}{RC}q = \frac{E_0}{R} e^{-at} \implies \frac{d}{dt}(e^{t/RC}q) = \frac{E_0}{R} e^{(1/RC-a)t}$.

$q(t) = e^{-t/RC} \left[ \frac{E_0C}{1-aRC} e^{(1/RC-a)t} + k \right] \implies q(t) = \frac{E_0C}{1-aRC} e^{-at} + ke^{-t/RC}$. Imposing the initial condition $q(0) = 0$ (capacitor initially uncharged) requires $k = -\frac{E_0C}{1-aRC}$, so that $q(t) = \frac{E_0C}{1-aRC} (e^{-at} - e^{-t/RC})$.

Thus $i(t) = \frac{dq}{dt} = \frac{E_0C}{1-aRC} \left( \frac{1}{RC} e^{-t/RC} - ae^{-at} \right)$.

19. $\frac{d^2q}{dt^2} + \frac{1}{LC}q = \frac{E_0}{L} \implies \frac{d^2q}{dt^2} = \frac{di}{dq} \frac{di}{dt} dt = \frac{di}{dq} \frac{d^2q}{dt^2} = \frac{i}{2} \frac{d^2q}{dt^2} + \frac{q^2}{2LC} = \frac{E_0q}{L} + A$. Since $i(0) = 0$ and $q(0) = q_0$ then $A = \frac{q_0^2}{2LC} - \frac{E_0q_0}{L}$. Here we get that $i = \left[ 2A + \frac{2E_0q}{L} - \frac{q^2}{LC} \right]^{1/2} \implies \sqrt{LC} \sin^{-1} \left( \frac{q - E_0C}{D \sqrt{LC}} \right) = t + B$. Then since $q(0) = 0$ so $B = \sqrt{LC} \sin^{-1} \left( \frac{q - E_0C}{D \sqrt{LC}} \right)$ and therefore

$q - E_0C \frac{D \sqrt{LC}}{q - E_0C} = \sin \left( \frac{t + B}{\sqrt{LC}} \right) \implies q(t) = D \sqrt{LC} \sin \left( \frac{t + B}{\sqrt{LC}} \right) + E_0c \implies i = \frac{dq}{dt} = D \cos \left( \frac{t + B}{\sqrt{LC}} \right)$. Since $D^2 = \frac{2A + (E_0C)^2}{LC}$ and $A = \frac{q_0^2}{2LC} - \frac{E_0q_0}{L}$ we can substitute to eliminate $A$ and obtain $D = \pm \frac{|q_0 - E_0C| \sqrt{LC}}{\sqrt{LC}}$.

Thus $q(t) = \pm |q_0 - E_0C| \sin \left( \frac{t + B}{\sqrt{LC}} \right) + E_0c$.

**Solutions to Section 1.8**

**True-False Review:**

1. **TRUE.** We have

$$f(tx, ty) = \frac{2(tx)(yt) - (xt)^2}{2(tx)(yt) + (yt)^2} = \frac{2xyt^2 - x^2t^2}{2xyt^2 + y^2t^2} = \frac{2xy - x^2}{2xy + y^2} = f(x, y),$$

so $f$ is homogeneous of degree zero.
3. FALSE. Setting \( f(x, y) = \frac{1 + xy^2}{1 + x^2} \), we have
\[
f(tx, ty) = \frac{(tx)^2 - (ty)^2}{1 + (tx)^2(3ty)} = \frac{1 + xy^2 + 3x^2y^2}{3x^4 + y^4} \neq f(x, y),
\]
so \( f \) is not homogeneous of degree zero. Therefore, the differential equation is not homogeneous.

5. TRUE. This is verified in the calculation leading to Theorem 1.8.5.

7. TRUE. We can rewrite the equation as
\[
y' - \sqrt{xy} = \sqrt{xy}^{1/2},
\]
which is the proper form for a Bernoulli equation, with \( p(x) = -\sqrt{x} \), \( q(x) = \sqrt{x} \), and \( n = 1/2 \).

9. FALSE. Setting \( x = \sqrt{\sin x} \), we have
\[
f(x, y) = \frac{\sin \frac{x}{y} - y \cos \frac{x}{y}}{y} = f(x, y).
\]
Thus \( f \) is homogeneous of degree zero. If \( x \neq 0 \), 
\[
f(x, y) = \frac{\sin \frac{x}{y} - y \cos \frac{x}{y}}{y} = \frac{\sin \frac{1}{x} - v \cos \frac{1}{x}}{v} = F(v).
\]

5. FALSE. Setting \( x = \sqrt{\sin x} \), we have
\[
f(x, y) = \frac{ty}{x - 1}.
\]
Thus \( f \) is not homogeneous.

7. TRUE. We can rewrite the equation as
\[
y' = \sqrt{x^2 + y^2} = \sqrt{\frac{x^2 + y^2}{x}} = \sqrt{\frac{x}{1 + (\frac{x}{y})^2}} = \sqrt{\frac{1}{1 + \frac{y^2}{x^2}}} = \sqrt{1 + v^2} = F(v).
\]

9. \( 3x - 2y \) \( \frac{dy}{dx} = 3y \implies (3 - 2 \frac{y}{x}) \frac{dy}{dx} = 3 \frac{y}{x} \implies (3 - 2v) \left(v + \frac{dv}{dx}\right) = 3v \implies \frac{dv}{dx} = -3 \frac{v}{2y} \implies \frac{3 - 2v}{2v} dv = \int \frac{dx}{x} \implies -3 \frac{v}{2y} - \ln |v| = \ln |x| + c_1 \implies -3 \frac{v}{2y} - \ln \left|\frac{y}{x}\right| = \ln |x| + c_1 \implies \ln y = -3 \frac{v}{2y} + c_2 \implies y^2 = ce^{-3x/y}.
\]

11. \sin \left( \frac{y}{x} \right) \left( \frac{dy}{dx} - y \right) = x \cos \left( \frac{y}{x} \right) \implies \sin \left( \frac{x}{y} \right) \left( \frac{dy}{dx} - y \right) = \cos \left( \frac{y}{x} \right) \implies \sin v \left( v + \frac{dv}{dx} - y \right) = \cos v \implies \sin v \left( \frac{dv}{dx} - y \right) = \cos v \implies \sin \left( \frac{y}{x} \right) \left( \frac{dy}{dx} - y \right) = \cos v \implies \frac{dy}{dx} - y = \ln |\cos v| = \ln |x| + c_1 \implies \left| x \cos \left( \frac{y}{x} \right) \right| = c_2 \implies y(x) = x \cos^{-1} \left( \frac{c}{x} \right).
13. We first rewrite the given differential equation in the equivalent form \( y' = \frac{\sqrt{9x^2 + y^2}}{x} \). Factoring out an \( x^2 \) from the square root yields \( y' = \frac{|x|\sqrt{9 + \left(\frac{y}{x}\right)^2}}{x} \). Since we are told to solve the differential equation on the interval \( x > 0 \) we have \( |x| = x \), so that \( y' = 9 + \left(\frac{y}{x}\right)^2 + \frac{2}{y} \), which we recognize as being homogeneous. We therefore let \( y = xV \), so that \( y' = xV' + V \). Substitution into the preceding differential equation yields \( xV' + V = \sqrt{9 + V^2} + V \), that is \( xV' = \sqrt{9 + V^2} \). Separating the variables in this equation we obtain \( \frac{1}{\sqrt{9 + V^2}} dV = \frac{1}{x} dx \). Integrating we obtain \( \frac{1}{2} \ln (V + \sqrt{9 + V^2}) = \ln c_1x \). Exponentiating both sides yields \( V + \sqrt{9 + V^2} = c_1x \). Substituting \( \frac{y}{x} = V \) and multiplying through by \( x \) yields the general solution \( y + \sqrt{9x^2 + y^2} = c_1x^2 \).

15. \( x \frac{dy}{dx} + y \ln x = y \ln y \Rightarrow \frac{dy}{dx} = \frac{y \ln x}{x} \Rightarrow v + x \frac{dv}{dx} = v \ln v \Rightarrow \int \frac{dv}{v(\ln v - 1)} \int \frac{dx}{x} \Rightarrow \ln |v| - 1 = \ln |x| + c \Rightarrow \ln \frac{x}{v} - 1 = c \Rightarrow y(x) = xe^{1+cx}.

17. \( 2xydy - (x^2e^{-y^2/x^2} + 2y^2)dx = 0 \Rightarrow 2y \frac{dy}{dx} - x^2 \left( e^{-y^2/x^2} + 2 \left(\frac{y}{x}\right)^2 \right) = 0 \Rightarrow 2y \left( v + x \frac{dv}{dx} \right) - (e^{-v^2} + 2v^2) = 0 \Rightarrow 2v \frac{dv}{dx} = e^{-v^2} \Rightarrow \int e^{-v^2} dv = \int \frac{dx}{x} \Rightarrow e^{-v^2} = \ln |x| + c \Rightarrow e^{-v^2/x^2} = \ln (cx) \Rightarrow y^2 = x^2 \ln (cx) \).

19. \( \frac{dy}{dx} = \sqrt{x^2 + y^2} - x \) \( \frac{y}{x} \) \( \frac{dy}{dx} = \sqrt{1 + \left(\frac{y}{x}\right)^2 - 1} \Rightarrow v + x \frac{dv}{dx} = \sqrt{1 + v^2 - v} \Rightarrow \frac{dv}{\sqrt{1 + v^2 - 1 - v^2}} \int \frac{dx}{x} \Rightarrow \ln |1 - u| = \ln |x| + c \Rightarrow |x(1 - u)| = c_2 \Rightarrow 1 - u = c \Rightarrow u^2 = \frac{c^2}{x^2} - 2c = \frac{c^2}{x^2} - 2c \Rightarrow y^2 = c^2 - 2cx \).

21. \( x \frac{dy}{dx} = x \tan \left(\frac{y}{x}\right) + y \Rightarrow v + x \frac{dv}{dx} = \tan v + v \Rightarrow x \frac{dv}{dx} = \tan v \Rightarrow \int \cot v dv = \int \frac{dx}{x} \Rightarrow \ln |\sin v| = \ln |x| + c \Rightarrow \sin v = cx \Rightarrow v = \sin^{-1} (cx) \Rightarrow y(x) = x \sin^{-1} (cx) \).

23. The given differential equation can be written as \((x-4y)dy = (4x+y)dx\). Converting to polar coordinates we have \( x = r \cos \theta \Rightarrow dx = \cos \theta dr - r \sin \theta d\theta \), and \( y = r \sin \theta dr + r \cos \theta d\theta \). Substituting these results into the preceding differential equation and simplifying yields the separable equation \( 4r^{-1} dr = d\theta \) which can be integrated directly to yield \( 4 \ln r = \theta + c \), so that \( r = c_1 e^{\theta/4} \).

25. \( \frac{dy}{dx} = \frac{2x-y}{x+4y} \Rightarrow \frac{dy}{dx} = \frac{2 - \frac{x}{x+4y}}{1 + 4y} \Rightarrow v + x \frac{dv}{dx} = \frac{2 - v}{1 + 4v} \Rightarrow x \frac{dv}{dx} = \frac{2 - 2v - 4v^2}{1 + 4v} \Rightarrow \frac{1}{2} \int \frac{1 + 4v}{2v^2 + v - 1} dv = \frac{1}{2} \left[ \frac{1}{2} \ln |2v^2 + v - 1| = - \ln |x| + c \right] = \frac{1}{2} \ln x^2 (2v^2 + v - 1) = c \Rightarrow \frac{1}{2} \ln |2y^2 + yx - x^2| = c \), but \( y(1) = 1 \) so \( c = \frac{1}{2} \ln 2 \). Thus \( \frac{1}{2} \ln |2y^2 + yx - x^2| = \frac{1}{2} \ln 2 \) and since \( y(1) = 1 \) it must be the case that \( 2y^2 + yx - x^2 = 2 \).

27. \( \frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{4 - \left(\frac{y}{x}\right)^2}}{x} \Rightarrow v + x \frac{dv}{dx} = v + \sqrt{4 - v^2} \Rightarrow \int \frac{dv}{\sqrt{4 - v^2}} = \int \frac{dx}{x} \Rightarrow \sin^{-1} \frac{v}{2} = \ln |x| + c \Rightarrow \)
\[
\sin^{-1} \frac{y}{2x} = \ln x + c \quad \text{since} \quad x > 0.
\]

29. Given family of curves satisfies: \(x^2 + y^2 = 2cy \Rightarrow \frac{c}{2y} = \frac{x^2 + y^2}{2y^2}.\) Hence \(2x + 2y \frac{dy}{dx} = 2c \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{c-y} = \frac{2xy}{x^2-y^2}.\) Orthogonal trajectories satisfies: \(\frac{dy}{dx} = \frac{y^2-x^2}{2xy}.\) Let \(y = vx\) so that \(\frac{dy}{dx} = v + x \frac{dv}{dx}.\) Substituting these results into the last equation yields \(x \frac{dv}{dx} = -\frac{v^2+1}{2v} \Rightarrow |v^2+1| = -\ln |x| + c_1 \Rightarrow\)
\[
\frac{y^2}{x^2} + 1 = \frac{c_2}{x} \Rightarrow x^2 + y^2 = 2kx.
\]

\[\text{Figure 0.0.26: Figure for Exercise 29}\]

31. (a) Let \(r\) represent the radius of one of the circles with center at \((a, ma)\) and passing through \((0,0)\).
\(r = \sqrt{(a-0)^2 + (ma-0)^2} = |a|\sqrt{1+m^2}.\) Thus, the circle’s equation can be written as \((x-a)^2 + (y-ma)^2 = (|a|\sqrt{1+m^2})^2\) or \((x-a)^2 + (y-ma)^2 = a^2(1+m^2)\).

(b) \((x-a)^2 + (y-ma)^2 = a^2(1+m^2) \Rightarrow a = \frac{x^2+y^2}{2(x+my)}.\) Differentiating the first equation with respect \(x\) and solving we obtain \(\frac{dy}{dx} = \frac{a-x}{y-ma}.\) Substituting for \(a\) and simplifying yields \(\frac{dy}{dx} = \frac{y^2-x^2-2mxy}{my^2-mx^2+2xy}.
\)

Orthogonal trajectories satisfies: \(\frac{dy}{dx} = \frac{mx^2-my^2-2xy}{y^2-x^2-2mxy} \Rightarrow \frac{dy}{dx} = \frac{m-m\frac{y}{x}^2-2\frac{y}{x}}{\left(\frac{y}{x}\right)^2-1-2m\frac{y}{x}}.\) Let \(y = vx\) so that \(\frac{dy}{dx} = v + x \frac{dv}{dx}.\) Substituting these results into the last equation yields \(v + x \frac{dv}{dx} = \frac{m-mv^2-2v}{v^2-1-2mv} \Rightarrow xdv = \frac{(m-v)(1+v^2)}{v^2-2mv-1} \Rightarrow v^2-2mv-1 = \int \frac{dv}{x} \Rightarrow \int \frac{dv}{v-m} + \int \frac{2v}{1+v^2} = \frac{x}{x} \Rightarrow \ln |v-m| - \ln (1+v^2) = \ln |x| + c_1 \Rightarrow v-m = c_2 x(1+v^2) \Rightarrow y-mx = c_2 x^2 + c_2 y^2 \Rightarrow x^2 + y^2 + cmx - cy = 0.
\]

Completing the square we obtain \((x + cm/2)^2 + (y - c/2)^2 = c^2/4(m^2 + 1).\) Now letting \(b = c/2,\) the last
equation becomes \((x + bm)^2 + (y - b)^2 = b^2(m^2 + 1)\) which is a family of circles lying on the line \(y = -my\) and passing through the origin.
(c) See the accompanying figure.

![Figure for Exercise 31(c)](image)

33. \(y = cx^6 \implies \frac{dy}{dx} = 6y/x = m_2\).
\[ m_1 = \frac{m_2 - \tan \left( \frac{x}{2} \right)}{1 + m_2 \tan \left( \frac{x}{2} \right)} = \frac{6y/x - 1}{1 + 6y/x} = \frac{6y/x}{6y/x} \]
Let \(y = vx\) so that
\[
\frac{dy}{dx} = v + x \frac{dv}{dx}.
\]
Substitute these results into the last equation yields
\[
\frac{dv}{dx} = \frac{6v - 1}{6v + 1} \implies x \frac{dv}{dx} = \frac{9}{6v + 1} - \frac{8}{3v - 1} - \frac{2v - 1}{2v - 1} \]
\[
\frac{9}{6v + 1} - \frac{8}{3v - 1} = 3 \ln |3v - 1| - 4 \ln |2v - 1| = \ln |x| + c_1 \implies \]
Oblique trajectories \((3y - x)^3 = k(2y - x)^3\).

35. \(y = cx^{-1} \implies \frac{dy}{dx} = -cx^{-2} = -y/x\).
\[ m_1 = \frac{m_2 - \tan \alpha}{1 + m_2 \tan \alpha} = \frac{-y/x - \tan \alpha}{1 - y/x \tan \alpha} \]
Let \(y = vx\) so that
\[
\frac{dy}{dx} = v + x \frac{dv}{dx}.
\]
Substituting these results into the last equation yields
\[
v + x \frac{dv}{dx} = \frac{\tan \alpha + v}{v \tan \alpha - 1} \implies \frac{2v \tan \alpha - 2}{v \tan \alpha - 2} \frac{dv}{dx} = \frac{-2x}{x} \implies \ln |v^2 \tan \alpha 2v - \tan \alpha - 2| = -2 \ln |x| + c_1 \implies (y^2 - x^2) \tan \alpha - \frac{2xy}{v^2 \tan \alpha} = k\).
(b) See the accompanying figure.

37. \(\frac{dy}{dx} - \frac{1}{x}y = 4x^2 y^{-1} \cos x\). This is a Bernoulli equation. Multiplying both sides \(y\) results in
\[
\frac{dy}{dx} - \frac{1}{x}y^2 = 4x^2 \cos x.
\]
Let \(u = y^2\) so \(\frac{du}{dx} = 2y \frac{dy}{dx}\) or \(\frac{dy}{dx} = \frac{1}{2} \frac{du}{dx}\). Substituting these results into \(\frac{dy}{dx} - \frac{1}{x}y^2 = 4x^2 \cos x\) yields
\[
\frac{du}{dx} - \frac{2}{x}u = 8x^2 \cos x \] which has an integrating factor \(I(x) = x^{-2} \implies \frac{d}{dx}(x^{-2}u) = 8 \cos x \implies x^{-2}u = 8 \cos x + c \implies u = x^2(8 \sin x + c) \implies y^2 = x^2(8 \sin x + c).\]
39. \( \frac{dy}{dx} - \frac{3}{2x} y = 6y^{1/3}x^2 \ln x \) or \( \frac{1}{y^{1/3}} \frac{dy}{dx} - \frac{3}{2x} y^{2/3} = 6x^2 \ln x \). Let \( u = y^{2/3} \Rightarrow \frac{dy}{dx} = \frac{2}{3} y^{-1/3} \frac{du}{dx} \). Substituting these results into \( \frac{1}{y^{1/3}} \frac{dy}{dx} - \frac{3}{2x} y^{2/3} = 6x^2 \ln x \) yields \( \frac{du}{dx} - \frac{1}{x} u = 4x^2 \ln x \). An integrating factor for this equation is \( I(x) = \frac{1}{x} \) so \( \frac{d}{dx}(x^{-1} u) = 4x \ln x \Rightarrow x^{-1} u = 4 \int x \ln x \, dx + c \Rightarrow x^{-1} u = 2x^2 \ln x - x^2 + c \Rightarrow u(x) = x(2x^2 - x^2 + c) \Rightarrow y^{2/3} = x(2x^2 - x^2 + c).

41. \( \frac{dy}{dx} + \frac{2}{x} y = 6y^2 x^4 \) or \( y^{-2} \frac{dy}{dx} + \frac{2}{x} y^{-1} = 6x^4 \). Let \( u = y^{-1} \Rightarrow \frac{du}{dx} = -y^{-2} \frac{dy}{dx} \). Substituting these results into \( y^{-2} \frac{dy}{dx} + \frac{2}{x} y^{-1} = 6x^4 \) yields \( \frac{du}{dx} - \frac{2}{x} u = -6x^4 \). An integrating factor for this equation is \( I(x) = x^{-2} \) so \( \frac{d}{dx}(x^{-2} u) = -6x^2 \Rightarrow x^{-2} u = -2x^3 + c \Rightarrow u = -2x^5 + cx^2 \Rightarrow y^{-1} = -2x^5 + cx^2 \Rightarrow y(x) = \frac{1}{x^2(c - 2x^3)} \).

43. \( (x - a)(x - b) \left( \frac{dy}{dx} - y^{1/2} \right) = 2(b - a)y \) or \( y^{-1/2} \frac{dy}{dx} - \frac{2(b - a)}{(x - a)(x - b)} y^{1/2} = 1 \). Let \( u = y^{1/2} \Rightarrow \frac{du}{dx} = \frac{y^{-1/2} \frac{dy}{dx} - \frac{2(b - a)}{(x - a)(x - b)} y^{1/2}}{1} \). Substituting these results into \( y^{-1/2} \frac{dy}{dx} - \frac{2(b - a)}{(x - a)(x - b)} y^{1/2} = 1 \) yields \( \frac{du}{dx} - \frac{(b - a)}{(x - 1)(x - b)} u = \frac{1}{2} \). An integrating factor for this equation is \( I(x) = \frac{x - a}{x - b} \) so \( \frac{d}{dx} \left( \frac{x - a}{x - b} u \right) = \frac{x - a}{x - b} \Rightarrow \frac{x - a}{x - b} u = \frac{1}{2} \left[ x + (b - a) \ln |x - b| + c \right] \Rightarrow y^{1/2} = \frac{x - b}{2(x - a)} [x + (b - a) \ln |x - b| + c] \Rightarrow y(x) = \frac{1}{4} \left( \frac{x - b}{x - a} \right)^2 \left[ x + (b - a) \ln |x - b| + c \right]^2.

45. \( \frac{dy}{dx} + 4xy = 4x^3y^{1/2} \) or \( y^{-1/2} \frac{dy}{dx} + 4xy^{1/2} = 4x^3 \). Let \( u = y^{1/2} \Rightarrow 2 \frac{du}{dx} = y^{-1/2} \frac{dy}{dx} \). Substituting these results into \( y^{-1/2} \frac{dy}{dx} + 4xy^{1/2} = 4x^3 \) yields \( \frac{du}{dx} + 2xu = 2x^3 \). An integrating factor for this equation is \( I(x) = \)
\( e^x \) so \( \frac{d}{dx}(e^x u) = 2e^x x^3 \implies e^x u = e^x(x^2 - 1) + c \implies y^{1/2} = x^2 - 1 + ce^{-x^2} \implies y(x) = [(x^2 - 1) + ce^{-x^2}]^2.

47. \( \frac{dy}{dx} - \frac{1}{(\pi - 1)x} y = \frac{3}{(\pi - 1)} xy'' + y' \). Substituting these results into \( y^\pi dx - \frac{1}{(\pi - 1)x} y^\pi = \frac{3x}{1 - \pi} \). Let \( u = y^{1 - \pi} \implies \frac{du}{dx} = y^{-\pi} \frac{dy}{dx} \)

49. \( (1 - \sqrt{3}) \frac{dy}{dx} + y \sqrt{3} \sec x = \frac{y}{(1 - \sqrt{3})} y^{-\sqrt{3} dy} + y^{1 - \sqrt{3} \sec x} \implies \sec x \). Let \( u = y^{1 - \sqrt{3}} \implies \frac{du}{dx} = (1 - \sqrt{3}) y^{-\sqrt{3} \frac{dy}{dx}}. \) Substituting these results into \( (1 - \sqrt{3}) y^{-\sqrt{3} \frac{dy}{dx}} + y^{1 - \sqrt{3} \sec x} \) yields \( \frac{du}{dx} + u \sec x = \sec x \).

51. \( \frac{dy}{dx} + y \cot x = y^3 \sin^3 x \) or \( y^{-3} \frac{dy}{dx} + y^{-2} \cot x = \sin^3 x \). Let \( u = y^{-2} \implies -\frac{1}{2} \frac{du}{dx} = y^{-3} \frac{dy}{dx} \). Substituting these results into \( y^{-3} \frac{dy}{dx} + y^{-2} \cot x = \sin^3 x \) yields \( \frac{du}{dx} - 2u \cot x = -2 \sin^3 x \). An integrating factor for this equation is \( I(x) = \sec x + \tan x \) so \( \frac{d}{dx}[(sec x + tan x)u] = \sec x(sec x + tan x) \Rightarrow (sec x + tan x)u = \tan x + \sec x + c \implies \frac{1}{sec x + tan x} \frac{du}{dx} = \frac{1}{1 + \frac{c}{sec x + tan x}} \frac{y(x)}{1/(1-\sqrt{3})} \). Thus \( y^2 = \frac{1}{\sin^2 x(2 \cos x + 1)} \).

53. \( \frac{dy}{dx} = (9x - y^2) \). Let \( v = 9x - y \) so that \( \frac{dy}{dx} = 9 - \frac{dv}{dx} \implies \frac{dv}{dx} = 9 - v^2 \implies \frac{dv}{9 - v^2} = \int \frac{dx}{3} \tan^{-1} \frac{v}{3} = x + c \) but \( y(0) = 0 \) so \( c = 0 \). Thus, \( \tan^{-1} (3x - y/3) = 3x \) or \( y(x) = 3(3x - \tan 3x) \).

55. \( \frac{dy}{dx} = \sin^2 (3x - 3y + 1) \). Let \( v = 3x - 3y + 1 \) so that \( \frac{dy}{dx} - \frac{1}{3} \frac{dv}{dx} \implies 1 - \frac{1}{3} \frac{dv}{dx} = \sin^2 v \Rightarrow \frac{dv}{dx} = 3 \cos^2 v \Rightarrow \int \sec^2 v dv = 3 \int \sec v = \tan v = 3x + c \implies \tan (3x - 3y + 1) = 3x + c \Rightarrow \frac{y(x)}{3} = \tan^{-1} (3x + c + 1) \).

57. Substituting into \( \frac{1}{V[F(V) + 1]} \frac{dV}{dx} = \frac{1}{x} \) for \( F(V) = \ln V - 1 \) yields \( \frac{1}{\ln V} \frac{dV}{dx} = \ln \ln V = \ln c \) \( \Rightarrow V = e^{ex} \Rightarrow y(x) = \frac{1}{x} e^{ex} \).

59. \( a) y = y(x) + v^{-1}(x) \Rightarrow y' = y'(x) - v^{-2}(x) v'(x) \). Now substitute into the given differential equation and simplify algebraically to obtain \( y'(x) + p(x) y(x) + q(x) Y^2(x) - v^{-2}(x) v'(x) + v^{-1}(x) p(x) + q(x) Y^2(x) v^{-1}(x) + v^{-2}(x) = r(x) \). We are told that \( Y(x) \) is a particular solution to the given differential equation, and therefore \( Y'(x) + p(x) Y(x) + q(x) Y^2(x) = r(x) \). Consequently the transformed differential equation reduces to \( -v^{-2}(x) v'(x) + v^{-1}(x) p(x) + q(x) Y(x) v^{-1}(x) + v^{-2}(x) \) = 0, or equivalently \( v' - [p(x) + 2Y(x) q(x)] v = q(x) \).
(b) The given differential equation can be written as \( y' - x^{-1}y - y^2 = x^{-2} \), which is a Riccati differential equation with \( p(x) = -x^{-1} \), \( q(x) = -1 \), and \( r(x) = x^{-2} \). Since \( y(x) = -x^{-1} \) is a solution to the given differential equation, we make a substitution \( y(x) = -x^{-1} + v(x) \). According to the result from part (a), the given differential equation then reduces to \( v' - (x^{-1} + 2x^{-1})v = -1 \), or equivalently \( v' - 3x^{-1}v = -1 \). This linear differential equation has an integrating factor \( I(x) = x^{-1} \), so that \( v(x) = x(c - \ln x) \). Hence the solution to the original equation is \( y(x) = -\frac{1}{x} + \frac{1}{x(c - \ln x)} = \frac{1}{x} \left( \frac{1}{c - \ln x} - 1 \right) \).

61. (a) \( y = x^{-1} + w(x) \Rightarrow y' = -x^{-2} + w' \). Substituting into the given differential equation yields \((-x^{-2} + w') + 7x^{-1}(x^{-1} + w) - 3(x^{-2} + 2x^{-1}w + w^2) = 3x^{-2} \), which simplifies to \( w' + x^{-1}w - 3w^2 = 0 \). According to Definition 1.9.2., \( 3 \). FALSE.

(b) The preceding equation can be written in the equivalent form \( w^{-2}w' + x^{-1}w^{-1} = 3 \). We let \( u = w^{-1} \), so that \( u' = -w^{-2}w' \). Substitution into the differential equation gives, after simplification, \( u' - x^{-1}u = -3 \). An integrating factor for this linear differential equation is \( I(x) = x^{-1} \), so that the differential equation can be written in the integrable form \( \frac{d}{dx}(x^{-1}u) = -3x^{-1} \). Integrating we obtain \( u(x) = x(-3\ln x + c) \), so that \( w(x) = \frac{1}{x(c - 3\ln x)} \). Consequently the solution to the original Riccati equation is \( y(x) = \frac{1}{x} \left( 1 + \frac{1}{c - 3\ln x} \right) \).

63. \( y^{-1}\frac{dy}{dx} - \frac{2}{x} \ln y = \frac{1 - 2\ln x}{x^2} \). Let \( u = \ln y \) so using the technique of the preceding problem, \( \frac{du}{dx} - \frac{2}{x} u = \frac{1 - 2\ln x}{x^2} \) and \( u = e^{\int \frac{1 - 2\ln x}{x^2} dx} + c_1 = x^2 \left[ e^{\int \frac{1 - 2\ln x}{x^2} dx} + c_1 \right] = \ln x + cx^2 \), and since \( u = \ln y, \ln y = \ln x + cx^2 \). Now \( y(1) = e \) so \( c = 1 \Rightarrow y(x) = xe^{x^2} \).

65. \( \sec^2 \frac{dy}{dx} + \frac{1}{2\sqrt{1+x}} \tan y = \frac{1}{2\sqrt{1+x}} \). Let \( u = \tan y \) so that \( \frac{du}{dx} = \sec^2 y \frac{dy}{dx} \) and the given equation becomes \( \frac{du}{dx} + \frac{1}{2\sqrt{1+x}} u = \frac{1}{2\sqrt{1+x}} \) which is first order linear. An integrating factor for this equation is \( I(x) = e^{\sqrt{1+x}} \Rightarrow \frac{d}{dx}(e^{\sqrt{1+x}}u) = e^{\sqrt{1+x}} \frac{d}{dx}(e^{\sqrt{1+x}}u) = e^{\sqrt{1+x}} \Rightarrow e^{\sqrt{1+x}}u = \frac{e^{\sqrt{1+x}}}{2\sqrt{1+x}} + c \Rightarrow u = 1 + ce^{-\sqrt{1+x}} \). But \( u = \tan y \) so \( \tan y = 1 + ce^{-\sqrt{1+x}} \) or \( y(x) = \tan^{-1}(1 + ce^{-\sqrt{1+x}}) \).

Solutions to Section 1.9

True-False Review:

1. FALSE. The requirement, as stated in Theorem 1.9.4, is that \( M_y = N_x, \) not \( M_x = N_y, \) as stated.

3. FALSE. According to Definition 1.9.2, \( M(x)dx + N(y)dy = 0 \) is only exact if there exists a function \( \phi(x, y) \) such that \( \phi_x = M \) and \( \phi_y = N \) for all \( (x, y) \) in a region \( R \) of the \( xy \)-plane.

5. FALSE. If \( \phi(x, y) \) is a potential function for \( M(x, y)dx + N(x, y)dy = 0, \) then so is \( \phi(x, y) + c \) for any constant \( c. \)
7. TRUE. We have
\[ M_y = \frac{(x^2 + y)^2(-2x) + 4xy(x^2 + y)}{(x^2 + y)^4} = N_x, \]
and so this equation is exact.

9. FALSE. We have
\[ M_y = e^x \sin y \cos y + xe^x \sin y \cos y \quad \text{and} \quad N_x = \cos y \sin ye^x \sin y, \]
and since \( M_y \neq N_x \), we conclude that this equation is not exact.

Problems:

1. \((y + 3x^2)dx + xdy = 0. \) \( M = y + 3x^2 \) and \( N = x \implies M_y = 1 \) and \( N_x = 1 \implies M_y = N_x \implies \) the differential equation is exact.

3. \( ye^{xy}dx + (2y - x e^{-xy})dy = 0. \) \( M = ye^{xy} \) and \( N = 2y - x e^{-xy} \implies M_y = yxe^{xy} + e^{xy} \) and \( N_x = yxe^{-xy} - e^{-xy} \implies M_y \neq N_x \implies \) the differential equation is not exact.

5. \((y^2 + \cos x)dx + (2xy + \sin y)dy = 0. \) \( M = y^2 + \cos x \) and \( N = 2xy + \sin y \implies M_y = 2y \) and \( N_x = 2y \implies M_y = N_x \implies \) the differential equation is exact so there exists a potential function \( \phi \) such that (a) \( \frac{\partial \phi}{\partial x} = y^2 + \cos x \) and (b) \( \frac{\partial \phi}{\partial y} = 2xy + \sin y. \) From (a) \( \phi(x, y) = xy^2 + \sin x + h(y) \implies \frac{\partial \phi}{\partial y} = 2xy + \frac{dh(y)}{dy} \) so from (b) \( 2xy + \frac{dh(y)}{dy} = 2xy + \sin y \implies \frac{dh}{dy} = \sin y \implies h(y) = -\cos y \) where the constant of integration has been set to zero since we just need one potential function. \( \phi(x, y) = xy^2 + \sin x - \cos y = xy^2 + \sin x - \cos y = c. \)

7. Given \((4e^{2x} + 2xy - y^2)dx + (x - y^2)dy = 0 \) then \( M_y = N_x = 2y \) so the differential equation is exact and there exists a potential function \( \phi \) such that (a) \( \frac{\partial \phi}{\partial x} = 4e^{2x} + 2xy - y^2 \) and (b) \( \frac{\partial \phi}{\partial y} = (x - y)^2. \) From (b) \( \phi(x, y) = x^2y - xy^2 + \frac{y^3}{3} + h(x) \implies \frac{\partial \phi}{\partial x} = 2xy - y^2 + \frac{dh(x)}{dx} \) so from (a) \( 2xy - y^2 + \frac{dh(x)}{dx} = 4e^{2x} + 2xy - y^2 \implies \frac{dh(x)}{dx} = 4e^{2x} \implies h(x) = 2e^{2x} \) where the constant of integration has been set to zero since we need just one potential function. \( \phi(x, y) = x^2y - xy^2 + \frac{y^3}{3} + 2e^{2x} \implies x^2y - xy^2 + \frac{y^3}{3} + 2e^{2x} = c_1 \implies 6e^{2x} + 3x^2y - 3xy^2 + y^3 = c. \)

9. Given \( \left(1 - \frac{y}{x^2 + y^2}\right)dx + \frac{x}{x^2 + y^2}dy = 0 \) then \( M_y = N_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} \) so the differential equation is exact and there exists a potential function \( \phi \) such that (a) \( \frac{\partial \phi}{\partial x} = \frac{1}{x} - \frac{y}{x^2 + y^2} \) and (b) \( \frac{\partial \phi}{\partial y} = \frac{x}{x^2 + y^2}. \) From (b) \( \phi(x, y) = \tan^{-1}(y/x) + h(x) \implies \frac{\partial \phi}{\partial x} = -\frac{y}{x^2 + y^2} + \frac{dh(x)}{dx} \) so from (a) \(-\frac{y}{x^2 + y^2} + \frac{dh(x)}{dx} = \frac{1}{x} - \frac{y}{x^2 + y^2} \implies \frac{dh}{dx} = x^{-1} \implies h(x) = \ln|x| \) where the constant of integration is set to zero since we only need one potential function. \( \phi(x, y) = \tan^{-1}(y/x) + \ln|x| \implies \tan^{-1}(y/x) + \ln|x| = c. \)

11. Given \([y \cos(xy) - \sin x]dx + x \cos(xy)dy = 0 \) then \( M_y = N_x = -xy \sin(xy) + \cos(xy) \) so the differential equation is exact so there exists a potential function \( \phi \) such that (a) \( \frac{\partial \phi}{\partial x} = y \cos(xy) - \sin x \) and (b)
\[
\frac{\partial \phi}{\partial x} = x \cos(xy). \quad \text{From (b) } \phi(x, y) = \sin(xy) + h(x) \Rightarrow \frac{\partial \phi}{\partial x} = y \cos(xy) + \frac{dh(x)}{dx} \quad \text{so from (a) } y \cos(xy) + \frac{dh(x)}{dx} = y \cos(xy) - \sin x \Rightarrow \frac{dh(x)}{dx} = -\sin x \Rightarrow h(x) = \cos x \text{ where the constant of integration is set to zero since we only need one potential function. } \phi(x, y) = \sin(xy) + \cos x \Rightarrow \sin(xy) + \cos x = c.
\]

13. Given \((3x^2 \ln x + x^2 - y)dx - x dy = 0\) then \(M_y = N_x = -1\) so the differential equation is exact so there exists a potential function \(\phi\) such that \((a) \frac{\partial \phi}{\partial x} = 3x^2 \ln x + x^2 - y\) and \((b) \frac{\partial \phi}{\partial y} = -x\). From (b) \(\phi(x, y) = -xy + h(x) \Rightarrow \frac{\partial \phi}{\partial x} = -y + \frac{dh(x)}{dx}\) so from (a) \(-y + \frac{dh(x)}{dx} = 3x^2 \ln x + x^2 - y \Rightarrow \frac{dh(x)}{dx} = 3x^2 \ln x + x^2 \Rightarrow h(x) = x^3 \ln x\) where the constant of integration has been set to zero since we only need one potential function. \(\phi(x, y) = -xy + x^3 \ln x \Rightarrow -xy + x^3 \ln x = c\). Now since \(y(1) = 5, c = -5\); thus, \(x^3 \ln x - xy = -5\) or \(y(x) = \frac{x^3 \ln x + 5}{x}\).

15. Given \((ye^{xy} + \cos x)dx + x e^{xy} dy = 0\) then \(M_y = N_x = xy e^{xy} + e^{xy}\) so the differential equation is exact so there exists a potential function \(\phi\) such that \((a) \frac{\partial \phi}{\partial x} = ye^{xy} + \cos x\) and \((b) \frac{\partial \phi}{\partial y} = xe^{xy}\). From (b) \(\phi(x, y) = e^{xy} + h(x) \Rightarrow \frac{\partial \phi}{\partial x} = ye^{xy} + \frac{dh(x)}{dx}\) so from (a) \(ye^{xy} + \cos x \Rightarrow \frac{dh(x)}{dx} = \cos x \Rightarrow h(x) = \sin x\) where the constant of integration is set to zero since we only need one potential function. \(\phi(x, y) = e^{xy} + \sin x \Rightarrow e^{xy} + \sin x = c\). Now since \(y(\pi/2) = 0, c = 2\); thus, \(e^{xy} + \sin x = 2\) or \(y(x) = \ln(2 - \sin x)/x\).

17. \(M = \cos(xy)[\tan(xy) + xy]\) and \(N = x^2 \cos(xy) \Rightarrow M_y = 2x \cos(xy) - x^2 y \sin(xy) = N_x \Rightarrow M_y = N_x \Rightarrow Mdx - Ndy = 0\) is exact so \(I(x, y) = \cos(xy)\) is an integrating factor for \([\tan(xy) + xy]dx + x^2 dy = 0\).

19. \(M = e^{-x/y}(x^2 y^{-1} - 2x)\) and \(N = -e^{-x/y}x^3 y^{-2} \Rightarrow M_y = e^{-x/y}(x^3 y^{-3} - 3x^2 y^{-2}) = N_x \Rightarrow Mdx + Ndy = 0\) is exact so \(I(x, y) = -e^{-x/y}\) is an integrating factor for \([x^2 - 2xy]dy - x^3 dx = 0\).

21. Given \(ydx - (2x + y^4)dy = 0\) then \(M = y\) and \(N = -(2x + y^4)\). Thus \(M_y = 1\) and \(N_x = -2\) so \(\frac{M_y - N_x}{M} = 3y^{-1} = g(y)\) is a function of \(y\) alone so \(I(y) = e^{-\int g(y)dy} = 1/y^3\) is an integrating factor for the given differential equation. Multiplying the given equation by \(I(y)\) results in the exact equation \(y^{-2}dx - (2xy^{-3} + y)dy = 0\). We find that \(\phi(x, y) = xy^{-2} - y^2/2\) and hence, the general solution of our differential equation is \(xy^{-2} - y^2/2 = c_1 \Rightarrow 2x - y^4 = cy^2\).

23. Given \((y - x^2)dx + 2xdy = 0\) then \(M = y - x^2\) and \(N = 2x\). Thus \(M_y = 1\) and \(N_x = 2\) so \(\frac{M_y - N_x}{N} = -\frac{1}{2x} = f(x)\) is a function of \(x\) alone so \(I(x) = e^{\int f(x)dx} = \frac{1}{\sqrt{x}}\) is an integrating factor for the given equation. Multiplying the given equation by \(I(x)\) results in the exact equation \((x^{-1/2}y - x^{3/2})dx + 2x^{1/2}dy = 0\). We find that \(\phi(x, y) = 2x^{1/2}y - \frac{2x^{3/2}}{5}\) and hence the general solution of our differential equation is \(2x^{1/2}y - \frac{2x^{3/2}}{5} = c\) or \(y(x) = \frac{c + 2x^{5/2}}{10\sqrt{x}}\).

25. Given \(\frac{dy}{dx} + \frac{2xy}{1 + x^2} = \frac{1}{(1 + x^2)^2} \Rightarrow (2xy + 2x^3 y - 1)dx + (1 + x^2)^2 dy = 0\) then \(M = 2xy + 2x^3 y - 1\) and
\[ N = (1 + x^2)^2. \] Thus \( M_y = 2x + 2x^3 \) and \( N_x = 4x(1 + x^2) \) so \( \frac{M_y - N_x}{N} = -\frac{2x}{1 + x^2} = f(x) \) is a function of \( x \) alone so \( I(x) = e^{\int f(x) \, dx} = \frac{1}{1 + x^2} \) is an integrating factor for the given equation. Multiplying the given equation by \( I(x) \) yields the exact equation \( \left(2xy - \frac{1}{1 + x^2}\right) \, dx + (1 + x^2) \, dy = 0 \). We find that \( \phi(x, y) = (1 + x^2)y - \tan^{-1}x \) and hence the general solution of our differential equation is \((1 + x^2)y - \tan^{-1}x = c \) or \( y(x) = \frac{\tan^{-1}x + c}{1 + x^2} \).

27. Given \( y_1 = y^2 - 2y^3 \) \( \Rightarrow \) \( x^r y^s \) \( \Rightarrow \) \( x^r \) \( y^s \). The equation is exact if and only if \( M_y = N_x \Rightarrow x^r y^s \Rightarrow x^r y^s = (r + 1)x^r y^s - 2rx^{-1}y^{-1} \Rightarrow s - r - 2 = \frac{s}{y^2} - \frac{r + 1}{y^2} = \frac{2r}{xy} \Rightarrow s - r - 2 = \frac{s}{y^2} - \frac{2r}{xy} \). From the last equation we require that \( s - r - 2 = 0 \) and \( s - 2r = 0 \). Solving this system yields \( r = 2 \) and \( s = 4 \).

29. Given \( 2y(y + 2x^2) \) \( \Rightarrow \) \( x^r y^s \) \( \Rightarrow \) \( x^r y^s \Rightarrow x^r y^s \Rightarrow x^r y^s = 4 (s + 1) = 4(r + 1) = 4(s + 1) = 4(r + 1) + 4(r + 3) \). From this last equation we require that \( 2(s + 2) = 4(r + 1) \) and \( 4(s + 1) = 3(r + 3) \) solving this system yields \( r = 1 \) and \( s = 2 \).

31. (a) Note \( \frac{dy}{dx} + py = q \) can be written in the differential form as \((py-q)dx + dy = 0 \) (31.1). This has \( M = py - q \) and \( N = 1 \) so that \( \frac{M_y - N_x}{N} = p(x) \). Consequently, an integrating factor for \((py-q)dx + dy = 0 \) is \( I(x) = e^{\int p(t) \, dt} \).

(b) Multiplying (31.1) by \( I(x) = e^{\int p(t) \, dt} \) yields the exact equation \( e^{\int p(t) \, dt} (py - q) \, dx + e^{\int p(t) \, dt} \, dy = 0 \). Hence, there exists a potential function \( \phi \) such that (i) \( \frac{\partial \phi}{\partial x} = e^{\int p(t) \, dt} (py - q) \) and (ii) \( \frac{\partial \phi}{\partial y} = e^{\int p(t) \, dt} \). From (i) \( p(x)y e^{\int p(t) \, dt} + \frac{dh}{dx} = e^{\int p(t) \, dt} (py - q) \Rightarrow \frac{dh}{dx} = -q(x)e^{\int p(t) \, dt} \Rightarrow h(x) = -\int q(x)e^{\int p(t) \, dt} \, dx \) where the constant of integration has been set to zero since we just need one potential function. Consequently, \( \phi(x, y) = ye^{\int p(t) \, dt} - \int q(x)e^{\int p(t) \, dt} \, dx \Rightarrow y(x) = I^{-1} (\int x \, dt + c) \), where \( I(x) = e^{\int p(t) \, dt} \).

Solutions to Section 1.10

True-False Review:

1. TRUE. This is well-illustrated by the calculations shown in Example 1.10.1.

3. FALSE. It is possible, depending on the circumstances, for the errors associated with Euler’s method to decrease from one step to the next.

Problems:
1. Applying Euler’s method with \( y' = 4y - 1 \), \( x_0 = 0 \), \( y_0 = 1 \), and \( h = 0.05 \) we have \( y_{n+1} = y_n + 0.05(4y_n - 1) \). This generates the sequence of approximants given in the table below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>1.15</td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>1.33</td>
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<tr>
<td>3</td>
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<td>1.546</td>
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<tr>
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<td>2.489</td>
</tr>
<tr>
<td>7</td>
<td>0.35</td>
<td>2.937</td>
</tr>
<tr>
<td>8</td>
<td>0.40</td>
<td>3.475</td>
</tr>
<tr>
<td>9</td>
<td>0.45</td>
<td>4.120</td>
</tr>
<tr>
<td>10</td>
<td>0.50</td>
<td>4.894</td>
</tr>
</tbody>
</table>

Consequently the Euler approximation to \( y(0.5) \) is \( y_{10} = 4.894 \). (Actual value: \( y(0.05) = 5.792 \) rounded to 3 decimal places).

3. Applying Euler’s method with \( y' = x - y^2 \), \( x_0 = 0 \), \( y_0 = 2 \), and \( h = 0.05 \) we have \( y_{n+1} = y_n + 0.05(x_n - y_n^2) \). This generates the sequence of approximants given in the table below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
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<tr>
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</tr>
<tr>
<td>10</td>
<td>0.50</td>
<td>1.048</td>
</tr>
</tbody>
</table>

Consequently the Euler approximation to \( y(0.5) \) is \( y_{10} = 1.048 \). (Actual value: \( y(0.05) = 1.088 \) rounded to 3 decimal places).

5. Applying Euler’s method with \( y' = 2xy^2 \), \( x_0 = 0 \), \( y_0 = 1 \), and \( h = 0.1 \) we have \( y_{n+1} = y_n + 0.1x_n y_n^2 \). This generates the sequence of approximants given in the table below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
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<tr>
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</tr>
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</tr>
<tr>
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<td>0.9</td>
<td>0.754</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>0.858</td>
</tr>
</tbody>
</table>
Consequently the Euler approximation to \( y(1) \) is \( y_{10} = 0.856 \). (Actual value: \( y(1) = 1 \)).

7. Applying the modified Euler method with \( y' = -\frac{2x y}{1 + x^2}, x_0 = 0, y_0 = 1 \), and \( h = 0.1 \) we have \( y_{n+1} = y_n - 0.2 \frac{x_n y_n}{1 + x_n^2} \).

\[
y_{n+1} = y_n + 0.05 \left[ -\frac{x_n y_n}{1 + x_n^2} - 2 \frac{x_{n+1} y_{n+1}}{1 + x_{n+1}^2} \right].
\]

This generates the sequence of approximants given in the table below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
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<td>8</td>
<td>0.8</td>
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<td>0.9</td>
<td>0.5536</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>0.5012</td>
</tr>
</tbody>
</table>

Consequently the modified Euler approximation to \( y(1) \) is \( y_{10} = 0.5012 \). (Actual value: \( y(1) = 0.5 \)).

9. Applying the modified Euler method with \( y' = -x^2 y, x_0 = 0, y_0 = 1 \), and \( h = 0.2 \) we have \( y_{n+1} = y_n - 0.2x_n^2 y_n \).

\[
y_{n+1} = y_n - 0.1[x_n^2 y_n + x_{n+1}^2 y_{n+1}].
\]

This generates the sequence of approximants given in the table below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
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<td>0.4</td>
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<tr>
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<td>1.0</td>
<td>0.7114</td>
</tr>
</tbody>
</table>

Consequently the modified Euler approximation to \( y(1) \) is \( y_5 = 0.7114 \). (Actual value: \( y(1) = 0.7165 \) rounded to 4 decimal places).

11. We have \( y' = 4y - 1, x_0 = 0, y_0 = 1 \), and \( h = 0.05 \). So, \( k_1 = 0.05(4y_n - 1), k_2 = 0.05[4(y_n + \frac{1}{2}k_1) - 1], k_3 = 0.05[4(y_n + \frac{1}{2}k_2) - 1], k_4 = 0.05[4(y_n + \frac{1}{2}k_3) - 1] \).

\[
y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4).
\]

This generates the sequence of approximants given in the table below (computations rounded to five decimal places).
Consequently the Runge-Kutta approximation to \( y(0.5) \) is \( y_{10} = 5.79167 \). (Actual value: \( y(0.5) = 5.79179 \) rounded to 5 decimal places).

13. We have \( y' = x - y^2 \), \( x_0 = 0 \), \( y_0 = 2 \), and \( h = 0.05 \). So, \( k_1 = 0.05(x_n - y_n^2) \), \( k_2 = 0.05[x_n + 0.025 - (y_n + \frac{k_1}{2})^2] \), \( k_3 = 0.05[x_n + 0.025 - (y_n + \frac{k_2}{2})^2] \), \( k_4 = 0.05[x_{n+1} - (y_n + k_3)^2] \),
\[ y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4). \]
This generates the sequence of approximants given in the table below (computations rounded to six decimal places).

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<td>5</td>
<td>0.25</td>
<td>2.28868</td>
</tr>
<tr>
<td>6</td>
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<td>2.74005</td>
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<tr>
<td>7</td>
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<tr>
<td>8</td>
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</tr>
<tr>
<td>9</td>
<td>0.45</td>
<td>4.78714</td>
</tr>
<tr>
<td>10</td>
<td>0.50</td>
<td>5.79167</td>
</tr>
</tbody>
</table>

Consequently the Runge-Kutta approximation to \( y(0.5) \) is \( y_{10} = 1.087845 \). (Actual value: \( y(0.5) = 1.087845 \) rounded to 6 decimal places).

15. We have \( y' = 2xy^2 \), \( x_0 = 0 \), \( y_0 = 1 \), and \( h = 0.1 \). So, \( k_1 = 0.2x_n - y_n^2 \), \( k_2 = 0.2(x_n + 0.05)(y_n + \frac{k_1}{2})^2 \), \( k_3 = 0.2(x_n + 0.05)(y_n + \frac{k_2}{2})^2 \), \( k_4 = 0.2x_{n+1}(y_n + k_3)^2 \),
\[ y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4). \]
This generates the sequence of approximants given in the table below (computations rounded to six decimal places).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.81936</td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>1.671135</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
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<tr>
<td>4</td>
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<td>1.125263</td>
</tr>
<tr>
<td>10</td>
<td>0.50</td>
<td>1.087845</td>
</tr>
</tbody>
</table>

Consequently the Runge-Kutta approximation to \( y(0.5) \) is \( y_{10} = 1.087845 \). (Actual value: \( y(0.5) = 1.087845 \) rounded to 6 decimal places).
<table>
<thead>
<tr>
<th>n</th>
<th>x_n</th>
<th>y_n</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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<td>9</td>
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<td>0.840336</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>0.999996</td>
</tr>
</tbody>
</table>

Consequently the Runge-Kutta approximation to \( y(1) \) is \( y_{10} = 0.999996 \). (Actual value: \( y(1) = 1 \).

### Solutions to Section 1.11

**Problems:**

1. \[
\frac{d^2y}{dx^2} = \frac{2}{x} \frac{dy}{dx} + 4x^2. \]
   Let \( u = \frac{dy}{dx} \) so that \( \frac{du}{dx} = \frac{d^2y}{dx^2} \). Substituting these results into the first equation yields \( \frac{du}{dx} = -\frac{2}{x} u + 4x^2 \Rightarrow \frac{du}{dx} - \frac{2}{x} u = 4x^2 \). An appropriate integrating factor for this equation is \( I(x) = e^{-\int \frac{2}{x} dx} = x^{-2} \). Substituting these results into the equation gives \( d(x^{-2}u) = 4 \Rightarrow x^{-2} u = 4 \int dx = 4x + c_1 \Rightarrow u = 4x^3 + c_1 x^2 \Rightarrow \frac{dy}{dx} = 4x^3 + c_1 x^2 \Rightarrow y(x) = c_1 x^3 + x^4 + c_2 \).

2. \[
\frac{d^2y}{dx^2} + \frac{2}{y} (\frac{dy}{dx})^2 = \frac{dy}{dx}. \]
   Let \( u = dy/dx \) so that \( \frac{du}{dx} = u \frac{dy}{dx} = \frac{d^2y}{dx^2} \). Substituting these results into the first equation yields \( \frac{du}{dx} + \frac{2}{y} u^2 = u \Rightarrow u = 0 \) or \( \frac{du}{dx} - \frac{2}{y} u = 1 \). An appropriate integrating factor for the last equation is \( I(y) = e^{\int \frac{2}{y} dy} = y^2 \Rightarrow \frac{d}{dy}(y^3 u) = y^2 \Rightarrow y^3 u = \int y^2 dy \Rightarrow y^2 u = \frac{y^3}{3} + c_1 \Rightarrow \frac{dy}{dx} = \frac{y^3}{3} + \frac{c_1}{y^2} \Rightarrow \ln |y^3 + c_2| = x + c_3 \Rightarrow y(x) = \sqrt[3]{c_4 e^x + c_5} \).

3. \[
\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2. \]
   Let \( u = \frac{dy}{dx} \) so that \( \frac{du}{dx} = \frac{d^2y}{dx^2} \). Substituting these results into the first equation yields \( \frac{du}{dx} + \tan x u = u^2 \) which is a Bernoulli equation. Letting \( z = u^{-1} \) gives \( \frac{1}{u^2} = \frac{du}{dx} = -\frac{dz}{dx} \). Substituting these results into the last equation yields \( \frac{dz}{dx} - \tan x z = -1 \). Then an integrating factor for this equation is \( I(x) = e^{-\int \tan x dx} = \cos x \Rightarrow \frac{d}{dx} (\cos x) = -\cos x \Rightarrow \cos x = -\int \cos x dx \Rightarrow z = -\sin x + c_1 \Rightarrow u = \frac{\cos x}{c_1 - \sin x} \Rightarrow \frac{dy}{dx} = \frac{\cos x}{c_1 - \sin x} \Rightarrow y(x) = c_2 - \ln |c_1 - \sin x| \).

4. \[
\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} = 6x^4. \]
   Let \( u = \frac{dy}{dx} \) so that \( \frac{du}{dx} = \frac{d^2y}{dx^2} \). Substituting these results into the first equation
yields \( \frac{du}{dx} = -\frac{2}{x} u = 6x^4 \). An appropriate integrating factor for this equation is \( I(x) = e^{-\int \frac{dx}{x}} = x^{-2} \Rightarrow \frac{d}{dx}(x^{-2}u) = 6x^2 \Rightarrow x^{-2}u = 6 \int x^2 dx \Rightarrow u = 2x^5 + cx^2 \Rightarrow \frac{dy}{dx} = 2x^5 + cx^2 \Rightarrow y(x) = \frac{1}{3}x^6 + c_1x^3 + c_2.

9. \( \frac{d^2y}{dx^2} - \alpha \left( \frac{dy}{dx} \right)^2 - \beta \frac{dy}{dx} = 0 \). Let \( u = \frac{dy}{dx} \) so that \( \frac{du}{dx} = \frac{d^2y}{dx^2} \). Substituting these results into the first equation yields \( \frac{d}{dx} - \beta u = \alpha u^2 \) which is a Bernoulli equation. If \( u = 0 \) then \( y \) is a constant and satisfies the equation. Now suppose that \( u \neq 0 \). Let \( z = u^{-1} \) so that \( \frac{dz}{dx} = -u^{-2} \frac{du}{dx} \). Substituting these results into the last equation yields \( \frac{dz}{dx} + \beta z = -\alpha \). The an integrating factor for this equation is \( I(x) = e^{\int \frac{dz}{dx} = e^{\beta x}} \Rightarrow e^{\beta x}z = -\alpha \int e^{\beta x}dx \Rightarrow -\frac{\alpha}{\beta} + c = \frac{\beta e^{\beta x}}{c_\beta - c} \Rightarrow \frac{dy}{dx} = \frac{\beta e^{\beta x}}{c_\beta - c} \Rightarrow y = \int \frac{\beta e^{\beta x}}{c_\beta - c} dx \Rightarrow y(x) = -\frac{1}{\alpha} \ln |c_1 + c_2 e^{\beta x}|.

11. \( \frac{d^2y}{dx^2} = -\frac{2x}{1+x^2} \frac{dy}{dx} \). Let \( u = \frac{dy}{dx} \) so that \( \frac{du}{dx} = \frac{d^2y}{dx^2} \). If \( u = 0 \) then \( y \) is a constant and satisfies the equation. Now suppose that \( u \neq 0 \). Substituting these results into the first equation yields \( \frac{du}{dx} = -\frac{2x}{1+x^2} u \Rightarrow \ln |u| = -\ln (1+x^2) + c \Rightarrow u = \frac{c_1}{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{c_1}{1+x^2} \Rightarrow y(x) = c_1 \tan^{-1} x + c_2 \).

13. \( \frac{d^2y}{dx^2} - \tan x \frac{dy}{dx} = 1 \). Let \( u = \frac{dy}{dx} \) so that \( \frac{du}{dx} = \frac{d^2y}{dx^2} \). Substituting these results into the first equation yields \( \frac{du}{dx} - u \tan x = 1 \). An appropriate integrating factor for this equation is \( I(x) = e^{-\int \tan x dx} = \cos x \Rightarrow \frac{d}{dx}(u \cos x) = \cos x \Rightarrow u \cos x = \sin x + c \Rightarrow u(x) = \tan x + c \sec x \Rightarrow \frac{dy}{dx} = \tan x + c \sec x \Rightarrow y(x) = \ln \sec x + c_1 \ln (\sec x + \tan x) + c_2 \).

15. \( \frac{d^2y}{dx^2} = \omega^2 y \) where \( \omega > 0 \). Let \( u = \frac{dy}{dx} \) so that \( \frac{du}{dx} = \frac{d^2y}{dx^2} \). Substituting these results into the first equation yields \( \frac{du}{dy} = \omega^2 \Rightarrow u^2 = \omega^2 y^2 + c_2 \). Using the given that \( y(0) = a \) and \( y'(0) = 0 \) we find that \( c_2 = \omega^2 a^2 \). Thus \( \frac{dy}{dx} = \pm \omega \sqrt{y^2 - a^2} \Rightarrow \frac{1}{\omega} \cosh^{-1} (y/a) = \pm x + c \Rightarrow y(x) = a \cosh [\omega (c \pm x)] \Rightarrow y' = \pm \omega \sinh [\omega (c \pm x)] \) and since \( y'(0) = 0, c = 0 \); hence, \( y(x) = a \cosh (\omega x) \).

17. \( \frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} = q(x) \). Let \( u = \frac{dy}{dx} \) so that \( \frac{du}{dx} = \frac{d^2y}{dx^2} \). Substituting these results into the first equation gives us the equivalent system: \( \frac{du}{dx} + p(x)u = q(x) \) which has a solution \( u = e^{-\int p(x)dx} \left[ \int e^{-\int p(x)dx} q(x)dx + c_1 \right] \) so \( \frac{dy}{dx} = e^{-\int p(x)dx} \left[ \int e^{-\int p(x)dx} q(x)dx + c_1 \right] \). Thus \( y = \int \left\{ e^{-\int p(x)dx} \left[ \int e^{-\int p(x)dx} q(x)dx + c_1 dx \right] \right\} + c_2 \) is a solution to the original equation.

19. Given \( \frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0, \theta(0) = \theta_0, \) and \( \frac{d\theta}{dt}(0) = 0 \).
(a) \( \frac{d^2\theta}{dt^2} + \frac{g}{L} \theta = 0 \). Let \( u = \frac{d\theta}{dt} \) so that \( \frac{du}{dt} = \frac{d^2\theta}{dt^2} = \frac{du}{d\theta} \frac{d\theta}{dt} = u \frac{d^2\theta}{d\theta dt} \). Substituting these results into the last equation yields \( u^2 + \frac{g}{L} \theta = 0 \Rightarrow u^2 = -\frac{g}{L} \theta^2 + c_1 \), but \( \frac{d\theta}{dt}(0) = 0 \) and \( \theta(0) = \theta_0 \) so
\[
c_1 = \frac{g}{L} \theta_0^2 \Rightarrow u^2 = \frac{g}{L} (\theta_0^2 - \theta^2) \Rightarrow u = \pm \sqrt{\frac{g}{L}} \sqrt{\theta_0^2 - \theta^2} \Rightarrow \sin^{-1} \left( \frac{\theta}{\theta_0} \right) = \pm \sqrt{\frac{g}{L}} t + c_2, \text{ but } \theta(0) = \theta_0 \text{ so}
\]
c_2 = \frac{\pi}{2} \Rightarrow \sin^{-1} \left( \frac{\theta}{\theta_0} \right) = \frac{\pi}{2} \pm \sqrt{\frac{g}{L}} t \Rightarrow \theta = \theta_0 \sin \left( \frac{\pi}{2} \pm \sqrt{\frac{g}{L}} t \right) \Rightarrow \theta = \theta_0 \cos \left( \sqrt{\frac{g}{L}} t \right). \text{ Yes, the predicted motion is reasonable.}

(b) \( \frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0 \). Let \( u = \frac{d\theta}{dt} \) so that \( \frac{du}{dt} = \frac{d^2\theta}{dt^2} = \frac{du}{d\theta} \frac{d\theta}{dt} = u \frac{d^2\theta}{d\theta dt} \). Substituting these results into the last equation yields \( u^2 + \frac{g}{L} \sin \theta = 0 \Rightarrow u^2 = \frac{2g}{L} \cos \theta + c \). Since \( \theta(0) = \theta_0 \) and \( \frac{d\theta}{dt}(0) = 0 \), then \( c = \frac{2g}{L} \cos \theta_0 \)
and so \( u^2 = \frac{2g}{L} \cos \theta - \frac{2g}{L} \cos \theta_0 \Rightarrow \frac{d\theta}{dt} = \pm \sqrt{\frac{2g}{L}} \cos \theta - \frac{2g}{L} \cos \theta_0 \Rightarrow \frac{d\theta}{dt} = \pm \sqrt{\frac{2g}{L}} \left[ \cos \theta - \cos \theta_0 \right]^{1/2}. \)

(c) From part (b), \( \sqrt{\frac{L}{2g}} \left[ \cos \theta - \cos \theta_0 \right]^{1/2} = \pm dt \). When the pendulum goes from \( \theta = \theta_0 \) to \( \theta = 0 \)
(which corresponds to one quarter of a period) \( \frac{d\theta}{dt} \) is negative; hence, choose the negative sign. Thus,
\[
T = -\sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\left[ \cos \theta - \cos \theta_0 \right]^{1/2}} \Rightarrow T = \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\cos \theta - \cos \theta_0}^{1/2};
\]

(d) \( T = \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\cos \theta - \cos \theta_0}^{1/2} \Rightarrow \)
\[
T = \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\left[ 2 \sin^2 \left( \frac{\theta_0}{2} \right) - 2 \sin^2 \left( \frac{\theta}{2} \right) \right]^{1/2}} = \frac{1}{2} \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sin^2 \left( \frac{\theta_0}{2} \right) - \sin^2 \left( \frac{\theta}{2} \right)}^{1/2};
\]

Let \( k = \sin \left( \frac{\theta_0}{2} \right) \) so that
\[
T = \frac{1}{2} \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{[k^2 - \sin^2 \left( \frac{\theta}{2} \right)]^{1/2}}. \tag{0.0.4}
\]

Now let \( \sin \theta/2 = k \sin u \). When \( \theta = 0, u = 0 \) and when \( \theta = \theta_0, u = \pi/2 \); moreover, \( d\theta = \frac{2k \cos (u) du}{\cos (\theta/2)} \Rightarrow \)
d\( \frac{d\theta}{2k} \sqrt{1 - \sin^2 (u)} du = \frac{d\theta}{2k} \sqrt{1 - k^2 \sin^2 (u)} du \Rightarrow \)
d\( \theta = \frac{2k \sqrt{1 - \sin^2 (u)} du}{\sqrt{1 - k^2 \sin^2 (u)}} \). Making this
change of variables in equation (0.0.4) yields
\[
T = \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{du}{\sqrt{1 - k^2 \sin^2 (u)}} \text{ where } k = \sin \theta_0/2.
\]

Solutions to Section 1.12
3. We first determine the slope of the given family at the point \((x, y)\). Differentiating

\[ y = cx^3 \quad (0.0.5) \]

with respect to \(x\) yields

\[ \frac{dy}{dx} = 3cx^2. \quad (0.0.6) \]

From (0.0.5) we have \(c = \frac{y}{x^3}\) which, when substituted into Equation (0.0.6) yields

\[ \frac{dy}{dx} = \frac{3y}{x}. \]

Consequently, the differential equation for the orthogonal trajectories is

\[ \frac{dy}{dx} = -\frac{x}{3y}. \]

Separating the variables and integrating gives

\[ \frac{3}{2}y^2 = -\frac{1}{2}x^2 + C, \]

which can be written in the equivalent form

\[ x^2 + 3y^2 = k. \]

5. We first determine the slope of the given family at the point \((x, y)\). Differentiating

\[ y = \ln(cx) \quad (0.0.7) \]

with respect to \(x\) yields

\[ \frac{dy}{dx} = \frac{1}{x}. \]

Consequently, the differential equation for the orthogonal trajectories is

\[ \frac{dy}{dx} = -x. \]

which can be integrated directly to obtain

\[ y = -\frac{1}{2}x^2 + k. \]

7. (a) We first determine the slope of the given family at the point \((x, y)\). Differentiating

\[ x^2 + 3y^2 = 2cy \quad (0.0.8) \]

with respect to \(x\) yields

\[ 2x + 6y \frac{dy}{dx} = 2c \frac{dy}{dx} \]

so that

\[ \frac{dy}{dx} = \frac{x}{c - 3y}. \quad (0.0.9) \]
From (0.0.8) we have \( c = \frac{x^2 + 3y^2}{2y} \), which, when substituted into Equation (0.0.9) yields

\[
\frac{dy}{dx} = \frac{x}{x^2 + 3y^2 - 3y} = \frac{2xy}{x^2 - 3y^2},
\]

(0.0.10)

as required. (b) It follows from Equation (0.0.10) that the differential equation for the orthogonal trajectories is

\[
\frac{dy}{dx} = -\frac{3y^2 - x^2}{2xy}.
\]

This differential equation is first-order homogeneous. Substituting \( y = xV \) into the preceding differential equation gives

\[
x \frac{dV}{dx} + V = \frac{3V^2 - 1}{2V}
\]

which simplifies to

\[
\frac{dV}{dx} = \frac{V^2 - 1}{2V}.
\]

Separating the variables and integrating we obtain

\[
\ln(V^2 - 1) = \ln x + C,
\]

or, upon exponentiation,

\[
V^2 - 1 = kx.
\]

Inserting \( V = y/x \) into the preceding equation yields

\[
\frac{y^2}{x^2} - 1 = kx,
\]

that is,

\[
y^2 - x^2 = kx^3.
\]

9. See accompanying figure.

11. (a) Separating the variables in Equation (1.12.6) yields

\[
\frac{mv}{mg - kv^2} \frac{dv}{dy} = 1
\]

which can be integrated to obtain

\[
-\frac{m}{2k} \ln(mg - kv^2) = y + c.
\]

Multiplying both sides of this equation by \(-1\) and exponentiating gives

\[
mg - kv^2 = c_1 e^{-\frac{2k}{m}y}.
\]

The initial condition \( v(0) = 0 \) requires that \( c_1 = mg \), which, when inserted into the preceding equation yields

\[
mg - kv^2 = mge^{-\frac{2k}{m}y},
\]

or equivalently,

\[
v^2 = \frac{mg}{k} \left( 1 - e^{-\frac{2k}{m}y} \right),
\]
as required.
(b) See accompanying figure.

13. The given differential equation is first-order linear. We first divide by $x$ to put the differential equation
in standard form:
\[
\frac{dy}{dx} - \frac{2}{x}y = 2x \ln x.
\] (0.0.11)

An integrating factor for this equation is
\[
I = e^{\int \left(-\frac{2}{x}\right) dx} = x^{-2}.
\]
Multiplying Equation (0.0.11) by \(x^{-2}\) reduces it to
\[
\frac{d}{dx}(x^{-2}y) = 2x^{-1} \ln x,
\]
which can be integrated to obtain
\[
x^{-2}y = (\ln x)^2 + c
\]
so that
\[
y(x) = x^2[(\ln x)^2 + c].
\]

15. We first rewrite the given differential equation as
\[
\frac{dy}{dx} = \frac{y^2 + 3xy + x^2}{x^2},
\]
which is first order homogeneous. Substituting \(y = xV\) into the preceding equation yields
\[
x \frac{d}{dx} + V = V^2 + 3V + 1
\]
so that
\[
x \frac{dV}{dx} = V^2 + 2V + 1 = (V + 1)^2,
\]
or, in separable form,
\[
\frac{1}{(V + 1)^2} \frac{dV}{dx} = \frac{1}{x}.
\]
This equation can be integrated to obtain
\[
-(V + 1)^{-1} = \ln x + c
\]
so that
\[
V + 1 = \frac{1}{c_1 - \ln x}.
\]
Inserting \(V = y/x\) into the preceding equation yields
\[
\frac{y}{x} + 1 = \frac{1}{c_1 - \ln x},
\]
so that
\[
y(x) = \frac{1}{c_1 - \ln x} - x.
\]

17. The given differential equation is linear with integrating factor
\[
I = e^{\int \frac{x^2}{1+e^{2x}} dx} = e^{\ln(1+e^{2x})} = 1 + e^{2x}.
\]
Multiplying the given differential equation by $1 + e^{2x}$ yields

$$
\frac{d}{dx} [(1 + e^{2x})y] = \frac{e^{2x} + 1}{e^{2x} - 1} = -1 + \frac{2e^{2x}}{e^{2x} - 1}
$$

which can be integrated directly to obtain

$$(1 + e^{2x})y = -x + \ln|e^{2x} - 1| + c,$$

so that

$$y(x) = \frac{-x + \ln|e^{2x} - 1| + c}{1 + e^{2x}}.$$ 

19. We first rewrite the given differential equation in the equivalent form

$$(\sin y + y \cos x + 1)dx - (1 - x \cos y - \sin x)dy = 0.$$ 

Then

$$M_y = \cos y + \cos x = N_x$$

so that the differential equation is exact. Consequently, there is a potential function satisfying

$$\frac{\partial \phi}{\partial x} = \sin y + y \cos x + 1, \quad \frac{\partial \phi}{\partial y} = -(1 - x \cos y - \sin x).$$

Integrating these two equations in the usual manner yields

$$\phi(x, y) = x - y + x \sin y + y \sin x,$$

so that the differential equation can be written as

$$d(x - y + x \sin y + y \sin x) = 0,$$

and therefore has general solution

$$x - y + x \sin y + y \sin x = c.$$ 

21. The given differential equation can be written in the equivalent form

$$\frac{dy}{dx} = \frac{e^{x-y}}{e^{2x+y}} = e^{-x}e^{-2y},$$

which we recognize as being separable. Separating the variables gives

$$e^{2y} \frac{dy}{dx} = e^{-x}$$

which can be integrated to obtain

$$\frac{1}{2}e^{2y} = -e^{-x} + c$$

so that

$$y(x) = \frac{1}{2} \ln(c_1 - 2e^{-x}).$$
23. Writing the given differential equation as
\[ \frac{dy}{dx} + \frac{2e^{x}}{1 + e^{x}} y = 2y^{1/2} e^{-x}, \]
we see that it is a Bernoulli equation with \( n = 1/2 \). We therefore divide the equation by \( y^{1/2} \) to obtain
\[ y^{-1/2} \frac{dy}{dx} + \frac{2e^{x}}{1 + e^{x}} y^{1/2} = 2e^{-x}. \]
We now make the change of variables \( u = y^{1/2} \), in which case, \( \frac{du}{dx} = \frac{1}{2} y^{-1/2} \frac{dy}{dx} \). Inserting these results into the preceding differential equation yields
\[ 2 \frac{du}{dx} + \frac{2e^{x}}{1 + e^{x}} u = 2e^{-x}, \]
or, in standard form,
\[ \frac{du}{dx} + \frac{e^{x}}{1 + e^{x}} u = e^{-x}. \]
An integrating factor for this linear differential equation is
\[ I = e^{\int \frac{e^{x}}{1 + e^{x}} dx} = e^{\ln(1 + e^{x})} = 1 + e^{x}. \]
Multiplying the previous differential equation by \( 1 + e^{x} \) reduces it to
\[ \frac{d}{dx} [(1 + e^{x}) u] = e^{-x}(1 + e^{x}) = e^{-x} + 1 \]
which can be integrated directly to obtain
\[ (1 + e^{x}) u = -e^{-x} + x + c \]
so that
\[ u = \frac{x - e^{-x} + c}{1 + e^{x}}. \]
Making the replacement \( u = y^{1/2} \) in this equation gives
\[ y^{1/2} = \frac{x - e^{-x} + c}{1 + e^{x}}. \]

25. For the given differential equation we have
\[ M(x, y) = 1 + 2xe^{y}, \quad N(x, y) = -(e^{y} + x), \]
so that
\[ \frac{M_y - N_x}{M} = \frac{1 + 2xe^{y}}{1 + 2xe^{y}} = 1. \]
Consequently, an integrating factor for the given differential equation is
\[ I = e^{-\int dy} = e^{-y}. \]
Multiplying the given differential equation by \( e^{-y} \) yields the exact differential equation
\[ (2x + e^{-y}) dx - (1 + xe^{-y}) dy = 0. \]
Therefore, there exists a potential function $\phi$ satisfying
\[
\frac{\partial \phi}{\partial x} = 2x + e^{-y}, \quad \frac{\partial \phi}{\partial y} = -(1 + xe^{-y}).
\]

Integrating these two equations in the usual manner yields
\[
\phi(x, y) = x^2 - y + xe^{-y}.
\]

Therefore Equation (0.0.12) can be written in the equivalent form
\[
d(x^2 - y + xe^{-y}) = 0
\]
with general solution
\[
x^2 - y + xe^{-y} = c.
\]

27. For the given differential equation we have
\[
M(x, y) = 3y^2 + x^2, \quad N(x, y) = -2xy,
\]
so that
\[
\frac{M_y - N_x}{N} = -\frac{4y}{x}.
\]
Consequently, an integrating factor for the given differential equation is
\[
I = e^{-\int \frac{4y}{x} dx} = x^{-4}.
\]

Multiplying the given differential equation by $x^{-4}$ yields the exact differential equation
\[
(3y^2x^{-4} + x^{-2})dx - 2yx^{-3}dy = 0. \quad (0.0.13)
\]

Therefore, there exists a potential function $\phi$ satisfying
\[
\frac{\partial \phi}{\partial x} = 3y^2x^{-4} + x^{-2}, \quad \frac{\partial \phi}{\partial y} = -2yx^{-3}.
\]

Integrating these two equations in the usual manner yields
\[
\phi(x, y) = -y^2x^{-3} - x^{-1}.
\]

Therefore Equation (0.0.13) can be written in the equivalent form
\[
d(-y^2x^{-3} - x^{-1}) = 0
\]
with general solution
\[-y^2x^{-3} - x^{-1} = c,
\]
or equivalently,
\[
x^2 + y^2 = c_1x^3.
\]

Notice that the given differential equation can be written in the equivalent form
\[
\frac{dy}{dx} = \frac{3y^2 + x^2}{2xy},
\]
which is first-order homogeneous. Another equivalent way of writing the given differential equation is

\[ \frac{dy}{dx} - \frac{3}{2x}y = \frac{1}{2}xy^{-1}, \]

which is a Bernoulli equation.

29. Separating the variables in the given differential equation yields

\[ \frac{1}{y} \frac{dy}{dx} = \frac{2 + x}{1 + x} = 1 + \frac{1}{1 + x}, \]

which can be integrated to obtain

\[ \ln |y| = x + \ln |1 + x| + c. \]

Exponentiating both sides of this equation gives

\[ y(x) = c_1(1 + x)e^x. \]

31. The given differential equation can be written in the equivalent form

\[ [y \sec^2(xy) + 2x]dx + x \sec^2(xy)dy = 0 \]

Then

\[ M_y = \sec^2(xy) + 2xy \sec^2(x) \tan(xy) = N_x \]

so that the differential equation is exact. Consequently, there is a potential function satisfying

\[ \frac{\partial \phi}{\partial x} = y \sec^2(xy) + 2x, \quad \frac{\partial \phi}{\partial y} = x \sec^2(xy). \]

Integrating these two equations in the usual manner yields

\[ \phi(x, y) = x^2 + \tan(xy), \]

so that the differential equation can be written as

\[ d(x^2 + \tan(xy)) = 0, \]

and therefore has general solution

\[ x^2 + \tan(xy) = c, \]

or equivalently,

\[ y(x) = \tan^{-1}\left(\frac{c - x^2}{x}\right). \]

33. CHANGE PROBLEM IN TEXT TO

\[ \frac{dy}{dx} = \frac{x^2}{x^2 + y^2} + \frac{y}{x} \]

then the answer is correct. The given differential equation is first-order homogeneous. Inserting \( y = xV \) into the given equation yields

\[ \frac{dV}{dx} + V = \frac{1}{1 + V^2} + V, \]
that is,

$$(1 + V^2) \frac{dV}{dx} = \frac{1}{x}.$$ 

Integrating we obtain

$$V + \frac{1}{3}V^3 = \ln |x| + c.$$ 

Inserting $V = y/x$ into the preceding equation yields

$$\frac{y}{x} + \frac{y^3}{3x^3} = \ln |x| + c.$$ 

35. The given differential equation is a Bernoulli equation with $n = -1$. We therefore divide the equation by $y^{-1}$ to obtain

$$y \frac{dy}{dx} + \frac{1}{x} y^2 = \frac{25 \ln x}{2x^3}.$$ 

We now make the change of variables $u = y^2$, in which case, $\frac{du}{dx} = 2y \frac{dy}{dx}$. Inserting these results into the preceding differential equation yields

$$\frac{1}{2} \frac{du}{dx} + \frac{1}{x} u = \frac{25 \ln x}{2x^3},$$

or, in standard form,

$$\frac{du}{dx} + \frac{2}{x} u = 25x^{-3} \ln x.$$ 

An integrating factor for this linear differential equation is

$$I = e^{\int \frac{2}{x} \, dx} = x^2.$$ 

Multiplying the previous differential equation by $x^2$ reduces it to

$$\frac{d}{dx} (x^2 u) = 25x^{-1} \ln x,$$

which can be integrated directly to obtain

$$x^2 u = \frac{25}{2} (\ln x)^2 + c$$

so that

$$u = \frac{25(\ln x)^2 + c}{2x^2}.$$ 

Making the replacement $u = y^2$ in this equation gives

$$y^2 = \frac{25(\ln x)^2 + c}{2x^2}.$$ 

37. The given differential equation can be written in the equivalent form

$$\frac{dy}{dx} - \frac{\cos x}{\sin x} y = -\cos x$$
which is first order linear with integrating factor

\[ I = e^{-\int \frac{\cos x}{\sin x} \, dx} = e^{-\ln(\sin x)} = \frac{1}{\sin x}. \]

Multiplying the preceding differential equation by \( \frac{1}{\sin x} \) reduces it to

\[ \frac{d}{dx} \left( \frac{1}{\sin x} \cdot y \right) = -\frac{\cos x}{\sin x} \]

which can be integrated directly to obtain

\[ \frac{1}{\sin x} \cdot y = -\ln(\sin x) + c \]

so that

\[ y(x) = \sin x[c - \ln(\sin x)]. \]

39. The given differential equation can be written in the equivalent form

\[ e^{-6y} \frac{dy}{dx} = -e^{-4x} \]

which is separable. Integrating both sides of the preceding equation yields

\[ -\frac{1}{6} e^{-6y} = \frac{1}{4} e^{-4x} + c \]

so that

\[ y(x) = -\frac{1}{6} \ln \left( c_1 - \frac{3}{2} e^{-4x} \right). \]

Imposing the initial condition \( y(0) = 0 \) requires that

\[ 0 = \ln \left( c_1 - \frac{3}{2} \right). \]

Hence, \( c_1 = \frac{5}{2} \), and so

\[ y(x) = -\frac{1}{6} \ln \left( \frac{5 - 3e^{-4x}}{2} \right). \]

41. The given differential equation is linear with integrating factor

\[ I = e^{-\int \sin x \, dx} = e^{\cos x}. \]

Multiplying the given differential equation by \( e^{\cos x} \) reduces it to the integrable form

\[ \frac{d}{dx} (e^{\cos x} \cdot y) = 1, \]

which can be integrated directly to obtain

\[ e^{\cos x} \cdot y = x + c. \]
Hence,
\[ y(x) = e^{-\cos x}(x + c). \]

Imposing the given initial condition \( y(0) = \frac{1}{e} \) requires that \( c = 1 \). Consequently,
\[ y(x) = e^{-\cos x}(x + 1). \]

43. In Newton’s Law of Cooling we have
\[ T_m = 180^\circ F, \quad T(0) = 80^\circ F, \quad T(3) = 100^\circ F. \]

We need to determine the time, \( t_0 \) when \( T(t_0) = 140^\circ F \). The temperature of the sandals at time \( t \) is governed by the differential equation
\[ \frac{dT}{dt} = -k(T - 180). \]

This separable differential equation is easily integrated to obtain
\[ T(t) = 180 + ce^{-kt}. \]

Since \( T(0) = 80 \) we have
\[ 80 = 180 + c \implies c = -100. \]

Hence,
\[ T(t) = 180 - 100e^{-kt}. \]

Imposing the condition \( T(3) = 100 \) requires
\[ 100 = 180 - 100e^{-3k}. \]

Solving for \( k \) we find \( k = \frac{1}{3} \ln \left( \frac{5}{4} \right) \). Inserting this value for \( k \) into the preceding expression for \( T(t) \) yields
\[ T(t) = 180 - 100e^{-\frac{t}{3} \ln \left( \frac{5}{4} \right)}. \]

We need to find \( t_0 \) such that
\[ 140 = 180 - 100e^{-t_0 \ln \left( \frac{5}{4} \right)}. \]

Solving for \( t_0 \) we find
\[ t_0 = 3 \frac{\ln \left( \frac{5}{4} \right)}{\ln \left( \frac{5}{4} \right)} \approx 12.32 \text{ min.} \]

45. Let \( T(t) \) denote the temperature of the object at time \( t \), and let \( T_m \) denote the temperature of the surrounding medium. Then we must solve the initial-value problem
\[ \frac{dT}{dt} = k(T - T_m)^2, \quad T(0) = T_0, \]

where \( k \) is a constant. The differential equation can be written in separated form as
\[ \frac{1}{(T - T_m)^2} \frac{dT}{dt} = k. \]
Integrating both sides of this differential equation yields

\[-\frac{1}{T - T_m} = kt + c\]

so that

\[T(t) = T_m - \frac{1}{kt + c}.\]

Imposing the initial condition \(T(0) = T_0\) we find that

\[c = \frac{1}{T_m - T_0},\]

which, when substituted back into the preceding expression for \(T(t)\) yields

\[T(t) = T_m - \frac{1}{kt + \frac{1}{T_m - T_0}} = T_m - \frac{T_m - T_0}{k(T_m - T_0)t + 1}.

As \(t \to \infty\), \(T(t)\) approaches \(T_m\).

47. If we let \(C(t)\) denote the number of sandhill cranes in the Platte River valley \(t\) days after April 1, then \(C(t)\) is governed by the differential equation

\[
\frac{dC}{dt} = -kC
\]

(0.0.14)

together with the auxiliary conditions

\[C(0) = 500,000; \quad C(15) = 100,000.\]

(0.0.15)

Separating the variables in the differential equation (0.0.14) yields

\[
\frac{1}{C} \frac{dC}{dt} = -k,
\]

which can be integrated directly to obtain

\[\ln C = -kt + c.\]

Exponentiation yields

\[C(t) = c_0 e^{-kt}.
\]

The initial condition \(C(0) = 500,000\) requires \(c_0 = 500,000\), so that

\[C(t) = 500,000 e^{-kt}.
\]

(0.0.16)

Imposing the auxiliary condition \(C(15) = 100,000\) yields

\[100,000 = 500,000 e^{-15k}.
\]

Taking the natural logarithm of both sides of the preceding equation and simplifying we find that \(k = \frac{1}{15} \ln 5\). Substituting this value for \(k\) into (0.0.16) gives

\[C(t) = 500,000 e^{-\frac{t}{15} \ln 5}.
\]

(0.0.17)
(a) \( C(3) = 500,000e^{-2 \ln 5} = 500,000 \cdot \frac{1}{25} = 20,000 \) sandcranes.

(b) \( C(35) = 500,000e^{-35 \ln 5} \approx 11696 \) sandcranes.

(c) We need to determine \( t_0 \) such that

\[
1000 = 500,000e^{-\frac{t_0}{\ln 5}}
\]

that is,

\[
e^{-\frac{t_0}{\ln 5}} = \frac{1}{500}
\]

Taking the natural logarithm of both sides of this equation and simplifying yields

\[
t_0 = 15 \cdot \frac{\ln 500}{\ln 5} \approx 57.9 \text{ days after April 1}.
\]

49. The differential equation for determining \( q(t) \) is

\[
\frac{dq}{dt} + \frac{5}{4}q = \frac{3}{2} \cos 2t,
\]

which has integrating factor \( I = e^{\int \frac{5}{4} dt} = e^{\frac{5}{4} t} \). Multiplying the preceding differential equation by \( e^{\frac{5}{4} t} \) reduces it to the integrable form

\[
\frac{d}{dt} \left( e^{\frac{5}{4} t} \cdot q \right) = \frac{3}{2} e^{\frac{5}{4} t} \cos 2t.
\]

Integrating and simplifying we find

\[
q(t) = \frac{6}{89} (5 \cos 2t + 8 \sin 2t) + ce^{-\frac{5}{4} t}.
\] (0.0.18)

The initial condition \( q(0) = 3 \) requires

\[
3 = \frac{30}{89} + c,
\]

so that \( c = \frac{237}{89} \). Making this replacement in (0.0.18) yields

\[
q(t) = \frac{6}{89} (5 \cos 2t + 8 \sin 2t) + \frac{237}{89} e^{-\frac{5}{4} t}.
\]

The current in the circuit is

\[
i(t) = \frac{dq}{dt} = \frac{12}{89} (8 \cos 2t - 5 \sin 2t) - \frac{1185}{356} e^{-\frac{5}{4} t}.
\]

Answer in text has incorrect exponent.

51. We are given:

\[
r_1 = 6 \text{ L/min}, \quad c_1 = 3 \text{ g/L}, \quad r_2 = 4 \text{ L/min}, \quad V(0) = 30 \text{ L}, \quad A(0) = 0 \text{ g},
\]

and we need to determine the amount of salt in the tank when \( V(t) = 60 \text{ L} \). Consider a small time interval \( \Delta t \). Using the preceding information we have:

\[
\Delta V = 6\Delta t - 4\Delta t = 2\Delta t,
\]
and

\[ \Delta A \approx 18\Delta t - 4 \frac{A}{V} \Delta t. \]

Dividing both of these equations by \( \Delta t \) and letting \( \Delta t \to 0 \) yields

\[ \frac{dV}{dt} = 2. \]  \hspace{1cm} (0.0.19)

\[ \frac{dA}{dt} + 4 \frac{A}{V} = 18. \]  \hspace{1cm} (0.0.20)

Integrating (0.0.19) and imposing the initial condition \( V(0) = 30 \) yields

\[ V(t) = 2(t + 15). \]  \hspace{1cm} (0.0.21)

We now insert this expression for \( V(t) \) into (0.0.20) to obtain

\[ \frac{dA}{dt} + \frac{2}{t + 15} A = 18. \]

An integrating factor for this differential equation is

\[ I = e^{\int \frac{2}{t + 15} dt} = (t + 15)^2. \]

Multiplying the preceding differential equation by \((t + 15)^2\) reduces it to the integrable form

\[ \frac{d}{dt} [(t + 15)^2 A] = 18(t + 15)^2. \]

Integrating and simplifying we find

\[ A(t) = \frac{6(t + 15)^3 + c}{(t + 15)^2}. \]

Imposing the initial condition \( A(0) = 0 \) requires

\[ 0 = \frac{6(15)^3 + c}{(15)^2}, \]

so that \( c = -20250 \). Consequently,

\[ A(t) = \frac{6(t + 15)^3 - 20250}{(t + 15)^2}. \]

We need to determine the time when the solution overflows. Since the tank can hold 60 L of solution, from (0.0.21) overflow will occur when

\[ 60 = 2(t + 15) \implies t = 15. \]

The amount of chemical in the tank at this time is

\[ A(15) = \frac{6(30)^3 - 20250}{(30)^2} \approx 157.5 \text{ g}. \]

53. Applying Euler’s method with \( y' = \frac{3x}{y} + 2, x_0 = 1, y_0 = 2, \) and \( h = 0.05 \) we have

\[ y_{n+1} = y_n + 0.05 \left( \frac{3x_n}{y_n} + 2 \right). \]

This generates the sequence of approximants given in the table below.
Consequently, the Euler approximation to \( y(1.5) \) is \( y_{10} = 3.67185 \).

55. Applying the modified Euler method with \( y' = \frac{3x}{y} + 2, x_0 = 1, y_0 = 2 \), and \( h = 0.05 \) generates the sequence of approximants given in the table below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.05</td>
<td>2.1750</td>
</tr>
<tr>
<td>2</td>
<td>1.10</td>
<td>2.34741</td>
</tr>
<tr>
<td>3</td>
<td>1.15</td>
<td>2.51770</td>
</tr>
<tr>
<td>4</td>
<td>1.20</td>
<td>2.68622</td>
</tr>
<tr>
<td>5</td>
<td>1.25</td>
<td>2.85323</td>
</tr>
<tr>
<td>6</td>
<td>1.30</td>
<td>3.01894</td>
</tr>
<tr>
<td>7</td>
<td>1.35</td>
<td>3.18353</td>
</tr>
<tr>
<td>8</td>
<td>1.40</td>
<td>3.34714</td>
</tr>
<tr>
<td>9</td>
<td>1.45</td>
<td>3.50988</td>
</tr>
<tr>
<td>10</td>
<td>1.50</td>
<td>3.67185</td>
</tr>
</tbody>
</table>

Consequently, the modified Euler approximation to \( y(1.5) \) is \( y_{10} = 3.66576 \). Comparing this to the corresponding Euler approximation from Problem 53 we have

\[
|y_{ME} - y_E| = |3.66576 - 3.67185| = 0.00609.
\]

57. Applying the Runge-Kutta method with \( y' = \frac{3x}{y} + 2, x_0 = 1, y_0 = 2 \), and \( h = 0.05 \) generates the sequence of approximants given in the table below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.05</td>
<td>2.17369</td>
</tr>
<tr>
<td>2</td>
<td>1.10</td>
<td>2.34506</td>
</tr>
<tr>
<td>3</td>
<td>1.15</td>
<td>2.51452</td>
</tr>
<tr>
<td>4</td>
<td>1.20</td>
<td>2.68235</td>
</tr>
<tr>
<td>5</td>
<td>1.25</td>
<td>2.84880</td>
</tr>
<tr>
<td>6</td>
<td>1.30</td>
<td>3.01404</td>
</tr>
<tr>
<td>7</td>
<td>1.35</td>
<td>3.17823</td>
</tr>
<tr>
<td>8</td>
<td>1.40</td>
<td>3.34151</td>
</tr>
<tr>
<td>9</td>
<td>1.45</td>
<td>3.50396</td>
</tr>
<tr>
<td>10</td>
<td>1.50</td>
<td>3.66568</td>
</tr>
</tbody>
</table>
Consequently the Runge-Kutta approximation to \( y(1.5) \) is \( y_{10} = 3.66568 \). Comparing this to the corresponding Euler approximation from Problem 53 we have

\[
|y_{RK} - y_E| = |3.66568 - 3.67185| = 0.00617.
\]

Last digit in answer the text needs changing.

**Solutions to Section 2.1**

**True-False Review:**

1. **TRUE.** A diagonal matrix has no entries below the main diagonal, so it is upper triangular. Likewise, it has no entries above the main diagonal, so it is also lower triangular.

3. **TRUE.** Since \( A \) is symmetric, \( A = A^T \). Thus, \((A^T)^T = A = A^T\), so \( A^T \) is symmetric.

5. **TRUE.** If \( A \) is skew-symmetric, then \( A^T = -A \). But \( A \) and \( A^T \) contain the same entries along the main diagonal, so for \( A^T = -A \), both \( A \) and \( -A \) must have the same main diagonal. This is only possible if all entries along the main diagonal are 0.

7. **TRUE.** Both matrix functions are defined for values of \( t \) such that \( t > 0 \).

9. **TRUE.** Each numerical entry of the matrix function is a constant function, which has domain \( \mathbb{R} \).

**Problems:**

1. \( a_{31} = 0, a_{24} = -1, a_{14} = 2, a_{32} = 2, a_{21} = 7, a_{34} = 4 \).

3. \[
\begin{bmatrix}
2 & 1 & -1 \\
0 & 4 & -2
\end{bmatrix}; 2 \times 3 \text{ matrix.}
\]

5. \[
\begin{bmatrix}
1 & -3 & -2 \\
3 & 6 & 0 \\
2 & 7 & 4 \\
-4 & -1 & 5
\end{bmatrix}; 4 \times 3 \text{ matrix.}
\]

7. \( \text{tr}(A) = 1 + 3 = 4 \).

9. \( \text{tr}(A) = 2 + 2 + (-5) = -1 \).

11. Column vectors: \[
\begin{bmatrix}
1 \\
-1 \\
2
\end{bmatrix},
\begin{bmatrix}
3 \\
-2 \\
6
\end{bmatrix},
\begin{bmatrix}
-4 \\
5 \\
7
\end{bmatrix}.
\]

Row vectors: \([1 \ 3 \ -4], [-1 \ -2 \ 5], [2 \ 6 \ 7] \).

13. \( A = \begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 1
\end{bmatrix} \). Column vectors: \[
\begin{bmatrix}
1 \\
3 \\
5
\end{bmatrix},
\begin{bmatrix}
2 \\
4 \\
1
\end{bmatrix}.
\]

15. \( A = [a_1, a_2, \ldots, a_p] \) has \( p \) columns and each column \( q \)-vector has \( q \) rows, so the resulting matrix has dimensions \( q \times p \).

17. One example: \[
\begin{bmatrix}
2 & 3 & 1 & 2 \\
0 & 5 & 6 & 2 \\
0 & 0 & 3 & 5 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]
19. One example: \[
\begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 5
\end{bmatrix}.
\]

21. One example: \[
\begin{bmatrix}
\frac{1}{\sqrt{t}} & \sqrt{t+2} & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

23. One example: \[
\begin{bmatrix}
t^2 + 1 & 1 & 1 & 1
\end{bmatrix}.
\]

25. One example: Let \(A\) and \(B\) be 1 \times 1 matrix functions given by
\[A(t) = [t]\quad\text{and}\quad B(t) = [t^2].\]

27. Since \(A\) is skew-symmetric,
\[a_{11} = a_{22} = a_{33} = 0,\]
\[a_{12} = -a_{21} = -1,\]
\[a_{13} = -a_{31} = -3,\]
\[a_{32} = -a_{23} = 1.\]
Consequently,
\[A = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}.
\]

**Solutions to Section 2.2**

**True-False Review:**

1. **FALSE.** The correct statement is \((AB)C = A(BC)\), the associative law. A counterexample to the particular statement given in this review item can be found in Problem 7.

3. **TRUE.** We have \((A + B)^T = A^T + B^T = A + B\), so \(A + B\) is symmetric.

5. **FALSE.** The correct equation is \((A + B)^2 = A^2 + AB + BA + B^2\). The statement is false since \(AB + BA\) does not necessarily equal \(2AB\). For instance, if \(A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) and \(B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\), then \((A+ B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\) and \(A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq (A + B)^2\).

7. **FALSE.** For example, let \(A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\) and let \(B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\). Then \(A\) is not upper triangular, despite the fact that \(AB\) is the zero matrix, hence automatically upper triangular.

9. **TRUE.** The derivative of each entry of the matrix is zero, since in each entry, we take the derivative of a constant, thus obtaining zero for each entry of the derivative of the matrix.

11. **FALSE.** For instance, the matrix function \(A = \begin{bmatrix} 2e^t & 0 \\ 0 & 3e^t \end{bmatrix}\) satisfies \(A = \frac{dA}{dt}\), but \(A\) does not have the form \(\begin{bmatrix} ce^t & 0 \\ 0 & ce^t \end{bmatrix}\).

**Problems:**

1. \(2A = \begin{bmatrix} 2 & 4 & -2 \\ 6 & 10 & 4 \end{bmatrix}\), \(-3B = \begin{bmatrix} -6 & 3 & -9 \\ -3 & -12 & -15 \end{bmatrix}\),
\[A - 2B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 6 \\ 2 & 8 & 10 \end{bmatrix}\]
\[= \begin{bmatrix} -3 & 4 & -7 \\ 1 & -3 & -8 \end{bmatrix}\]
\[3A + 4B = \begin{bmatrix} 3 & 6 & -3 \\ 9 & 15 & 6 \end{bmatrix} + \begin{bmatrix} 8 & -4 & 12 \\ 4 & 16 & 20 \end{bmatrix} = \begin{bmatrix} 11 & 2 & 9 \\ 13 & 31 & 26 \end{bmatrix}.\]

2. Solving for \(D\), we have

\[2A + B - 3C + 2D = A + 4C\]
\[2D = -A - B + 7C\]
\[D = \frac{1}{2}(-A - B + 7C).\]

When appropriate substitutions are made for \(A, B,\) and \(C\), we obtain:

\[D = \begin{bmatrix} -5 & -2.5 & 2.5 \\ 1.5 & 6.5 & 9 \\ -2.5 & 2.5 & -0.5 \end{bmatrix}.\]

3.

\[AB = \begin{bmatrix} 5 & 10 & -3 \\ 27 & 22 & 3 \end{bmatrix}, \quad BC = \begin{bmatrix} 9 \\ 8 \\ -6 \end{bmatrix}, \quad DC = [10],\]
\[DB = [6 14 -4], \quad CD = \begin{bmatrix} 2 & -2 & 3 \\ -2 & 2 & -3 \\ 4 & -4 & 6 \end{bmatrix}.\]

\(CA\) and \(AD\) cannot be computed.

5.

\[AB = \begin{bmatrix} 3 + 2i & 2 - 4i \\ 5 + i & -1 + 3i \end{bmatrix}, \quad \begin{bmatrix} -1 + i & 3 + 2i \\ 4 - 3i & 1 + i \end{bmatrix}\]
\[= \begin{bmatrix} (3 + 2i)(-1 + i) + (2 - 4i)(4 - 3i) & (3 + 2i)(3 + 2i) + (2 - 4i)(1 + i) \\ (5 + i)(-1 + i) + (-1 + 3i)(4 - 3i) & (5 + i)(3 + 2i) + (-1 + 3i)(1 + i) \end{bmatrix}\]
\[= \begin{bmatrix} -9 - 21i & 11 + 10i \\ -1 + 19i & 9 + 15i \end{bmatrix}.\]

7.

\[ABC = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -2 & 3 & 4 & 6 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 4 & -3 \\ -1 & 6 \end{bmatrix} C\]
\[= \begin{bmatrix} 7 & 9 \\ 7 & 35 \end{bmatrix}, \quad \begin{bmatrix} -3 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -12 & -22 \\ 14 & -126 \end{bmatrix}.\]

\[CAB = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 2 & 3 \\ -2 & 3 & 4 & 6 \end{bmatrix} B\]
\[= \begin{bmatrix} -7 & 9 & 2 & 3 \\ 9 & -13 & -14 & -21 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 4 & -3 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} -7 & 43 \\ -21 & -131 \end{bmatrix}.\]
9. \[ A_c = \begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ -5 \end{bmatrix} + (-2) \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -38 \end{bmatrix}. \]

11. \[ A_c = \begin{bmatrix} -1 & 2 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = 5 \begin{bmatrix} -1 \\ 4 \end{bmatrix} + (-1) \begin{bmatrix} 7 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 13 \end{bmatrix}. \]

13. (a): \[ A^2 = AA = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}. \]
\[ A^3 = A^2 A = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix}. \]
\[ A^4 = A^3 A = \begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -31 & -24 \\ 48 & 17 \end{bmatrix}. \]

(b): \[ A^2 = AA = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & -1 \end{bmatrix}. \]
\[ A^3 = A^2 A = \begin{bmatrix} -2 & 0 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 0 \\ 6 & 4 & -3 \\ -12 & 3 & 4 \end{bmatrix}. \]
\[ A^4 = A^3 A = \begin{bmatrix} 4 & -3 & 0 \\ 6 & 4 & -3 \\ -12 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -3 \\ -20 & 9 & 4 \\ 10 & -16 & 3 \end{bmatrix}. \]

15. \[ A^2 - 2A - 8I_2 = \begin{bmatrix} 14 & -2 \\ -10 & 6 \end{bmatrix} - 2 \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix} - 8 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -2 \\ -10 & 6 \end{bmatrix} + \begin{bmatrix} -6 & 2 \\ 10 & -2 \end{bmatrix} + \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix} = 0_2. \]

17. In order that \( A^2 = A \), we require \[ \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix} \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix} = \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}, \] that is, \[ \begin{bmatrix} x^2 - x - 2 & x + y - 1 \\ -2x - 2y + 2 & y^2 - y - 2 \end{bmatrix} = 0_2. \] Since corresponding elements of equal matrices are equal, it follows that \[ x^2 - x - 2 = 0 \implies x = -1 \text{ or } x = 2, \] and \[ y^2 - y - 2 = 0 \implies y = -1 \text{ or } y = 2. \]

Two cases arise from \( x + y - 1 = 0 \):
(a): If \( x = -1 \), then \( y = 2 \).
(b): If \( x = 2 \), then \( y = -1 \). Thus, \[ A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \text{ or } A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}. \]
19. 

$$[A, B] = AB - BA$$

$$= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \\ 10 & 4 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 8 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 1 \\ 2 & 6 \end{bmatrix} \neq 0_2.$$

21. 

$$[A_1, A_2] = A_1A_2 - A_2A_1$$

$$= \frac{1}{4} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = A_3.$$

$$[A_2, A_3] = A_2A_3 - A_3A_2$$

$$= \frac{1}{4} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \frac{1}{4} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = A_1.$$

$$[A_3, A_1] = A_3A_1 - A_1A_3$$

$$= \frac{1}{4} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} i & 0 \\ i & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A_2.$$

23.

**Proof that** $A(BC) = (AB)C$: Let $A = [a_{ij}]$ be of size $m \times n$, $B = [b_{jk}]$ be of size $n \times p$, and $C = [c_{kl}]$ be of size $p \times q$. Consider the $(i, j)$-element of $(AB)C$:

$$[(AB)C]_{ij} = \sum_{k=1}^{p} \left( \sum_{h=1}^{n} a_{ih} b_{hk} \right) c_{kj} = \sum_{h=1}^{n} a_{ih} \left( \sum_{k=1}^{p} b_{hk} c_{kj} \right) = [A(BC)]_{ij}.$$
Proof that \( A(B + C) = AB + AC \): We have

\[
[A(B + C)]_{ij} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})
\]

\[
= \sum_{k=1}^{n} (a_{ij}b_{kj} + a_{ik}c_{kj})
\]

\[
= \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}
\]

\[
= [AB + AC]_{ij},
\]

25. We have

\[
(IA)_{ij} = \sum_{k=1}^{m} \delta_{ik}a_{kj} = \delta_{ii}a_{ij} = a_{ij},
\]

for \( 1 \leq i \leq m \) and \( 1 \leq j \leq p \). Thus, \( I_m A_{m \times p} = A_{m \times p} \).

27.

\[
A^T = \begin{bmatrix}
1 & 2 & 3 \\
-1 & 0 & 4 \\
1 & 2 & -1 \\
4 & -3 & 0
\end{bmatrix},
\]

\[
B^T = \begin{bmatrix}
0 & -1 & 1 & 2 \\
1 & 2 & 1 & 1
\end{bmatrix},
\]

\[
AA^T = \begin{bmatrix}
1 & -1 & 1 & 4 \\
2 & 0 & 2 & -3 \\
3 & 4 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
-1 & 0 & 4 \\
1 & 2 & -1 \\
4 & -3 & 0
\end{bmatrix}
= \begin{bmatrix}
19 & -8 & -2 \\
-8 & 17 & 4 \\
-2 & 4 & 26
\end{bmatrix},
\]

\[
AB = \begin{bmatrix}
1 & -1 & 1 & 4 \\
2 & 0 & 2 & -3 \\
3 & 4 & -1 & 0 \\
3 & 4 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
-1 & 2 \\
1 & 1 \\
2 & 1
\end{bmatrix}
= \begin{bmatrix}
10 & 4 \\
-4 & 1 \\
-5 & 10
\end{bmatrix},
\]

\[
B^T A^T = \begin{bmatrix}
0 & -1 & 1 & 2 \\
1 & 2 & 1 & 1 \\
4 & -3 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
-1 & 0 & 4 \\
1 & 2 & -1 \\
4 & -3 & 0
\end{bmatrix}
= \begin{bmatrix}
10 & -4 & -5 \\
4 & 1 & 10
\end{bmatrix}.
\]

29.

(a): \[
\begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}.
\]

(b): \[
\begin{bmatrix}
7 & 0 & 0 \\
0 & 7 & 0 \\
0 & 0 & 7
\end{bmatrix}.
\]

31. We have

\[
S^T = \left[ \frac{1}{2}(A + A^T) \right]^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + A) = S
\]
and
\[
T^T = \left[ \frac{1}{2} (A - A^T) \right]^T = \frac{1}{2} (A^T - A) = -\frac{1}{2} (A - A^T) = -T.
\]
Thus, \( S \) is symmetric and \( T \) is skew-symmetric.

33. If \( A \) is an \( n \times n \) symmetric matrix, then \( A^T = A \), so it follows that
\[
T = \frac{1}{2} (A - A^T) = \frac{1}{2} (A - A) = 0_n.
\]
If \( A \) is an \( n \times n \) skew-symmetric matrix, then \( A^T = -A \) and it follows that
\[
S = \frac{1}{2} (A + A^T) = \frac{1}{2} (A + (-A)) = 0_n.
\]

35. If \( A = [a_{ij}] \) and \( D = \text{diag}(d_1, d_2, \ldots, d_n) \), then we must show that the \((i, j)\)-entry of \( DA \) is \( d_i a_{ij} \). In index notation, we have
\[
(DA)_{ij} = \sum_{k=1}^{n} d_i \delta_{ik} a_{kj} = d_i a_{ij} = d_i a_{ij}.
\]
Hence, \( DA \) is the matrix obtained by multiplying the \( i \)th row vector of \( A \) by \( d_i \), where \( 1 \leq i \leq n \).

37. \( A'(t) = \left[ \begin{array}{c} -2e^{-2t} \\ \cos t \end{array} \right] \).  
39. \( A'(t) = \left[ \begin{array}{ccc} e^t & 2e^{2t} & 2t \\ 2e^t & 8e^{2t} & 10t \end{array} \right] \).

41. We show that the \((i, j)\)-entry of both sides of the equation agree. First, recall that the \((i, j)\)-entry of \( AB \) is \( \sum_{k=1}^{n} a_{ik} b_{kj} \), and therefore, the \((i, j)\)-entry of \( \frac{d}{dt}(AB) \) is (by the product rule)
\[
\sum_{k=1}^{n} a'_{ik} b_{kj} + a_{ik} b'_kj = \sum_{k=1}^{n} a'_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} b'_{kj}.
\]
The former term is precise the \((i, j)\)-entry of the matrix \( \frac{dA}{dt} B \), while the latter term is precise the \((i, j)\)-entry of the matrix \( A \frac{dB}{dt} \). Thus, the \((i, j)\)-entry of \( \frac{d}{dt}(AB) \) is precisely the sum of the \((i, j)\)-entry of \( \frac{dA}{dt} B \) and the \((i, j)\)-entry of \( A \frac{dB}{dt} \). Thus, the equation we are proving follows immediately.

43. We have
\[
\int_{0}^{1} \left[ \begin{array}{cc} e^t & e^{-t} \\ 2e^t & 5e^{-t} \end{array} \right] dt = \left[ \begin{array}{cc} e^t & -e^{-t} \\ 2e^t & -5e^{-t} \end{array} \right] \bigg|_{0}^{1} = \left[ \begin{array}{cc} e - 1/e \\ 2e - 5/e \end{array} \right] - \left[ \begin{array}{cc} 1 \\ 2 - 5 \end{array} \right] = \left[ \begin{array}{cc} e - 1 \quad 1 - 1/e \\ 2e - 2 \\ 5 - 5/e \end{array} \right].
\]

45. We have
\[
\int_{0}^{1} \left[ \begin{array}{ccc} e^t & e^{2t} & t^2 \\ 2e^t & 4e^{2t} & 5t^2 \end{array} \right] dt = \left[ \begin{array}{ccc} e^t & 1/2e^{2t} & \frac{5}{3}t^3 \\ 2e^t & 2e^{2t} & 5/3 \end{array} \right] \bigg|_{0}^{1} = \left[ \begin{array}{ccc} e - 1 \\ 2e - 2 \\ e^2/2 \end{array} \right] = \left[ \begin{array}{ccc} 1/3 \\ 2 \\ 2 \end{array} \right] = \left[ \begin{array}{ccc} e - 1 \quad e^2/2 \\ 2e - 2 \\ 5/3 \end{array} \right].
\]

47. \( \int \left[ \begin{array}{ccc} \sin t & \cos t & 0 \\ -\cos t & \sin t & t \\ 0 & 3t & 1 \end{array} \right] dt = \left[ \begin{array}{ccc} -\cos t & \sin t & 0 \\ -\sin t & -\cos t & t^2/2 \\ 0 & 3t^2/2 & t \end{array} \right]. \)
49. \[
\int \begin{bmatrix}
e^{2t} & \sin 2t \\
t^2 - 5 & te^t \\
\sec^2 t & 3t - \sin t
\end{bmatrix}
\] \[dt = \begin{bmatrix}
\frac{1}{2}e^{2t} & -\frac{1}{2}\cos 2t \\
t^2 - 5t & te^t - e^t \\
t \tan t & \frac{3}{2}t^2 + \cos t
\end{bmatrix}.
\]

**Solutions to Section 2.3**

**True-False Review:**

1. **FALSE.** The last column of the augmented matrix corresponds to the constants on the right-hand side of the linear system, so if the augmented matrix has \(n\) columns, there are only \(n-1\) unknowns under consideration in the system.

3. **FALSE.** The right-hand side vector must have \(m\) components, not \(n\) components.

5. **TRUE.** The augmented matrix for a linear system has one additional column (containing the constants on the right-hand side of the equation) beyond the matrix of coefficients.

**Problems:**

1.
\[
2 \cdot 1 - 3(-1) + 4 \cdot 2 = 13, \\
1 + (-1) - 2 = -2, \\
5 \cdot 1 + 4(-1) + 2 = 3.
\]

3.
\[
(1 - t) + (2 + 3t) + (3 - 2t) = 6, \\
(1 - t) - (2 + 3t) - 2(3 - 2t) = -7, \\
3(1 - t) + (2 + 3t) - (3 - 2t) = 4.
\]

5. The lines \(2x + 3y = 1\) and \(2x + 3y = 2\) are parallel in the \(xy\)-plane, both with slope \(-2/3\); thus, the system has no solution.

7. \(A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & -3 & 7 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, A^\# = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & -3 & 7 \end{bmatrix}.\)

9. It is acceptable to use any variable names. We will use \(x_1, x_2, x_3, x_4\):
\[
x_1 - x_2 + 2x_3 + 3x_4 = 1,  \\
x_1 + x_2 - 2x_3 + 6x_4 = -1,  \\
3x_1 + x_2 + 4x_3 + 2x_4 = 2.
\]

11. Given \(Ax = 0\) and \(Ay = 0\), and an arbitrary constant \(c\),

(a): we have
\[
Az = A(x + y) = Ax + Ay = 0 + 0 = 0
\]

and
\[
Aw = A(cx) = c(Ax) = c0 = 0.
\]

(b): No, because
\[
A(x + y) = Ax + Ay = b + b = 2b \neq b,
\]
and

\[ A(cx) = c(Ax) = cb \neq b \]

in general.

13. \[
\begin{bmatrix}
x'_1 \\
x'_2 \\
x'_3
\end{bmatrix} = \begin{bmatrix}
t^2 & -t \\
\sin t & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}.
\]

15. \[
\begin{bmatrix}
x'_1 \\
x'_2 \\
x'_3
\end{bmatrix} = \begin{bmatrix}
0 & -\sin t & 1 \\
-e^t & 0 & t^2 \\
-t & t^2 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
t \\
t^3
\end{bmatrix}.
\]

17. We have

\[ x'(t) = \begin{bmatrix} 4(-2e^{-2t}) + 2\cos t \\ 3(-2e^{-2t}) + \sin t \end{bmatrix} = \begin{bmatrix} -8e^{-2t} + 2\cos t \\ -6e^{-2t} + \sin t \end{bmatrix} \]

and

\[
Ax + b = \begin{bmatrix}
1 & -4 \\
-3 & 2
\end{bmatrix} \begin{bmatrix}
4e^{-2t} + 2\sin t \\
3e^{-2t} - \cos t
\end{bmatrix} + \begin{bmatrix}
-2(\cos t + \sin t) \\
7\sin t + 2\cos t
\end{bmatrix} + \begin{bmatrix}
4e^{-2t} + 2\sin t - 4(3e^{-2t} - \cos t) - 2(\cos t + \sin t) \\
-3(4e^{-2t} + 2\sin t) + 2(3e^{-2t} - \cos t) + 7\sin t + 2\cos t
\end{bmatrix} = \begin{bmatrix} -8e^{-2t} + 2\cos t \\ -6e^{-2t} + \sin t \end{bmatrix}.
\]

**Solutions to Section 2.4**

**True-False Review:**

1. **TRUE.** The precise row-echelon form obtained for a matrix depends on the particular elementary row operations (and their order). However, Theorem 2.4.15 states that there is a unique reduced row-echelon form for a matrix.

3. **TRUE.** The pivots in a row-echelon form of an \(n \times n\) matrix must move down and to the right as we look from one row to the next beneath it. Thus, the pivots must occur on or to the right of the main diagonal of the matrix, and thus all entries below the main diagonal of the matrix are zero.

5. **FALSE.** If \(A\) is a nonzero matrix and \(B = -A\), then \(A + B = 0\), so \(\text{rank}(A + B) = 0\), but \(\text{rank}(A)\), \(\text{rank}(B) \geq 1\) so \(\text{rank}(A) + \text{rank}(B) \geq 2\).

7. **TRUE.** A matrix of rank zero cannot have any pivots, hence no nonzero rows. It must be the zero matrix.

9. **TRUE.** The matrices \(A\) and \(2A\) have the same reduced row-echelon form, since we can move between the two matrices by multiplying the rows of one of them by 2 or 1/2, a matter of carrying out elementary row operations.

**Problems:**

1. Row-echelon form.
2. Reduced row-echelon form.
3. Reduced row-echelon form.
4. Reduced row-echelon form.
5. Reduced row-echelon form.
6. Reduced row-echelon form.

9. \[
\begin{bmatrix}
2 & 1 \\
1 & -3
\end{bmatrix} \sim \begin{bmatrix}
1 & -3 \\
2 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & -3 \\
0 & 7
\end{bmatrix} \sim \begin{bmatrix}
1 & -3 \\
0 & 1
\end{bmatrix}, \text{Rank} (A) = 2.
\]
11. \[
\begin{bmatrix}
2 & 1 & 4 \\
2 & -3 & 4 \\
3 & -2 & 6
\end{bmatrix}
\sim
\begin{bmatrix}
3 & -2 & 6 \\
2 & -3 & 4 \\
2 & 1 & 4
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 2 \\
2 & -3 & 4 \\
0 & -1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 2 \\
0 & -5 & 0 \\
0 & -5 & 0
\end{bmatrix}, \text{Rank}(A) = 2.
\]

13. \[
\begin{bmatrix}
2 & -1 \\
3 & 2 \\
2 & 5
\end{bmatrix}
\sim
\begin{bmatrix}
3 & 2 \\
2 & -1 \\
2 & 5
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 3 \\
2 & -1 \\
2 & 5
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 3 \\
0 & -7 \\
0 & -7
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 3 \\
0 & 1 \\
0 & 0
\end{bmatrix}, \text{Rank}(A) = 2.
\]

15. \[
\begin{bmatrix}
2 & -1 & 3 & 4 \\
1 & -2 & 1 & 3 \\
1 & -5 & 0 & 5
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -2 & 1 & 3 \\
2 & -1 & 3 & 4 \\
1 & -5 & 0 & 5
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -2 & 1 & 3 \\
0 & 3 & 1 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -2 & 1 & 3 \\
0 & 1 & \frac{1}{3} & -\frac{2}{3} \\
0 & 0 & 0 & 0
\end{bmatrix}, \text{Rank}(A) = 2.
\]

17. \[
\begin{bmatrix}
4 & 7 & 4 & 7 \\
3 & 5 & 3 & 5 \\
2 & -2 & 2 & -2 \\
5 & -2 & 5 & -2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 1 & 2 \\
3 & 5 & 3 & 5 \\
2 & -2 & 2 & -2 \\
5 & -2 & 5 & -2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 1 & 2 \\
0 & -1 & 0 & -1 \\
0 & -6 & 0 & -6 \\
0 & -12 & 0 & -12
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 1 & 2 \\
0 & 1 & 0 & 1 \\
0 & -6 & 0 & -6 \\
0 & -12 & 0 & -12
\end{bmatrix}, \text{Rank}(A) = 2.
\]

19. \[
\begin{bmatrix}
3 & 2 \\
1 & -1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -1 \\
3 & 2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -1 \\
0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -1 \\
0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = J_2, \text{Rank}(A) = 2.
\]

1. P_{12}  2. A_{12}(-2)  3. M_2(\frac{1}{2})
21. \[
\begin{bmatrix}
3 & -3 & 6 \\
2 & -2 & 4 \\
6 & -6 & 12
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \text{Rank } (A) = 1.
\]

1. \(M_1(\frac{1}{3}), A_{12}(-2), A_{13}(-6)\)

23. \[
\begin{bmatrix}
1 & -1 & -1 & 2 \\
3 & -2 & 0 & 7 \\
2 & -1 & 2 & 4 \\
4 & -2 & 3 & 8
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -1 & -1 & 2 \\
0 & 1 & 3 & 1 \\
0 & 1 & 4 & 0 \\
0 & 2 & 7 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & 3 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 1 & -2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & 5 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & -1
\end{bmatrix}
= I_4, \text{Rank } (A) = 4.
\]

1. \(A_{12}(-3), A_{13}(-2), A_{14}(-4)\) 2. \(A_{21}(1), A_{23}(-1), A_{24}(-2)\) 3. \(A_{31}(-2), A_{32}(-3), A_{34}(-1)\) 4. \(M_4(-1)\) 5. \(A_{41}(-5), A_{42}(-4), A_{43}(1)\)

25. \[
\begin{bmatrix}
0 & 1 & 2 & 1 \\
0 & 3 & 1 & 2 \\
0 & 2 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
0 & 1 & 2 & 1 \\
0 & 0 & -6 & -2 \\
0 & 0 & -4 & -1
\end{bmatrix}
\sim
\begin{bmatrix}
0 & 1 & 2 & 1 \\
0 & 0 & 1 & 1/3 \\
0 & 0 & -4 & -1
\end{bmatrix}
\sim
\begin{bmatrix}
0 & 1 & 0 & 1/3 \\
0 & 0 & 1 & 1/3 \\
0 & 0 & 0 & 1/3
\end{bmatrix}
\sim
\begin{bmatrix}
0 & 1 & 0 & 1/3 \\
0 & 0 & 1 & 1/3 \\
0 & 0 & 0 & 1
\end{bmatrix}, \text{Rank } (A) = 3.
\]

1. \(A_{12}(-3), A_{13}(-2)\) 2. \(M_2(-\frac{1}{2})\) 3. \(A_{21}(-2), A_{23}(4)\) 4. \(M_3(3)\) 5. \(A_{32}(-\frac{1}{3}), A_{33}(-\frac{1}{3})\)

Solutions to Section 2.5

True-False Review:

1. \text{FALSE.} This process is known as Gaussian elimination. Gauss-Jordan elimination is the process by which a matrix is brought to reduced row echelon form via elementary row operations.

3. \text{TRUE.} The columns of the row-echelon form that contain leading 1s correspond to leading variables, while columns of the row-echelon form that do not contain leading 1s correspond to free variables.

5. \text{FALSE.} The linear system \(x = 0, y = 0, z = 0\) has a solution in \((0, 0, 0)\) even though none of the variables here is free.

Problems:

For the problems of this section, \(A\) will denote the coefficient matrix of the given system, and \(A^\#\) will denote the augmented matrix of the given system.
1. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

\[
\begin{bmatrix}
1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 5 & -1
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 5 & -1
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1
\end{bmatrix}.
\]

1. \(A_{12}(-3), A_{13}(-2)\)  2. \(M_2(-1)\)  3. \(A_{23}(-2)\)

The last augmented matrix results in the system:

\[
\begin{align*}
 x_1 + 2x_2 + x_3 &= 1, \\
x_2 + 2x_3 &= 0, \\
x_3 &= -1.
\end{align*}
\]

By back substitution we obtain the solution \((-2, 2, -1)\).

3. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

\[
\begin{bmatrix}
3 & 5 & -1 & 14 \\ 1 & 2 & 1 & 3 \\ 2 & 5 & 6 & 2
\end{bmatrix} \sim 
\begin{bmatrix}
3 & 5 & -1 & 14 \\ 0 & -1 & -4 & -5 \\ 0 & 1 & 4 & -4
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1
\end{bmatrix}.
\]

1. \(P_{12}\)  2. \(A_{12}(-3), A_{13}(-2)\)  3. \(M_2(-1)\)  4. \(A_{23}(-1)\)  5. \(M_4(-\frac{1}{3})\)

This system of equations is inconsistent since \(2 = \text{rank}(A) < \text{rank}(A^\#) = 3\).

5. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

\[
\begin{bmatrix}
2 & -1 & 3 & 14 \\ 3 & 1 & -2 & -1 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5
\end{bmatrix} \sim 
\begin{bmatrix}
3 & 1 & -2 & -1 \\ 2 & -1 & 3 & 14 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 2 & -5 & 15 \\ 2 & -1 & 3 & -14 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 2 & -5 & -15 \\ 0 & -5 & 13 & 44 \\ 0 & -5 & 13 & 44 \\ 0 & -11 & 23 & 80
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 2 & -5 & -15 \\ 0 & -1 & 9 & 28 \\ 0 & -5 & 13 & 44 \\ 0 & -11 & 23 & 80
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 2 & -5 & -15 \\ 0 & -1 & 9 & -28 \\ 0 & -5 & 13 & 44 \\ 0 & -11 & 23 & 80
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 2 & -5 & -15 \\ 0 & 1 & -9 & -28 \\ 0 & 0 & 32 & 96 \\ 0 & 0 & 0 & 0
\end{bmatrix}.
\]
The last augmented matrix results in the system of equations:

\[
\begin{align*}
   x_1 - 2x_2 - 5x_3 &= -15, \\
   x_2 - 9x_3 &= -28, \\
   x_3 &= 3.
\end{align*}
\]

Thus, using back substitution, the solution set for our system is given by \(\{(2, -1, 3)\}\).

7. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

\[
\begin{bmatrix}
   1 & 2 & -1 & 1 & 1 \\
   2 & 4 & -2 & 2 & 2 \\
   5 & 10 & -5 & 5 & 5
\end{bmatrix}
\overset{\sim}{\rightarrow}
\begin{bmatrix}
   1 & 2 & -1 & 1 & 1 \\
   0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

The last augmented matrix results in the equation \(x_1 + 2x_3 - x_3 + x_4 = 1\). Now \(x_2, x_3,\) and \(x_4\) are free variables, so we let \(x_2 = r, x_3 = s,\) and \(x_4 = t\). It follows that \(x_1 = 1 - 2r + s - t\). Consequently, the solution set of the system is given by \(\{(1 - 2r + s - t, r, s, t) : r, s, t \text{ and real numbers }\}\).

9. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

\[
\begin{bmatrix}
   1 & 2 & 1 & 1 & -2 & 3 \\
   0 & 0 & 1 & 4 & -3 & 2 \\
   2 & 4 & -1 & -10 & 5 & 0
\end{bmatrix}
\overset{\sim}{\rightarrow}
\begin{bmatrix}
   1 & 2 & 1 & 1 & -2 & 3 \\
   0 & 0 & 1 & 4 & -3 & 2 \\
   0 & 0 & -3 & -12 & 9 & -6
\end{bmatrix}
\overset{\sim}{\rightarrow}
\begin{bmatrix}
   1 & 2 & 1 & 1 & -2 & 3 \\
   0 & 0 & 1 & 4 & -3 & 2 \\
   0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

The last augmented matrix indicates that the first two equations of the initial system completely determine its solution. We see that \(x_4\) and \(x_5\) are free variables, so let \(x_4 = s\) and \(x_5 = t\). Then \(x_3 = 2 - 4x_4 + 3x_5 = 2 - 4s + 3t\). Moreover, \(x_2\) is a free variable, say \(x_2 = r\), so then \(x_1 = 3 - 2r - (2 - 4s + 3t) - s + 2t = 1 - 2r + 3s - t\). Hence, the solution set for the system is

\[\{(1 - 2r + 3s - t, r, 2 - 4s + 3t, s, t) : r, s, t \text{ any real numbers }\}\).

11. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

\[
\begin{bmatrix}
   3 & 1 & 5 & 2 \\
   1 & 1 & -1 & 1 \\
   2 & 1 & 2 & 3
\end{bmatrix}
\overset{\sim}{\rightarrow}
\begin{bmatrix}
   1 & 1 & -1 & 1 \\
   3 & 1 & 5 & 2 \\
   2 & 1 & 2 & 3
\end{bmatrix}
\overset{\sim}{\rightarrow}
\begin{bmatrix}
   1 & 1 & -1 & 1 \\
   0 & 2 & 8 & -1 \\
   0 & 1 & 4 & 1
\end{bmatrix}.
\]

We can stop here, since we see from this last augmented matrix that the system is inconsistent. In particular, \(2 = \text{rank}(A) < \text{rank}(A^\#) = 3\).
13. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

\[
\begin{pmatrix}
2 & -1 & 3 & 1 \\
1 & -3 & 2 & -1 \\
3 & 1 & -2 & -1 \\
5 & -3 & 1 & 2 \\
\end{pmatrix}
\sim
\begin{pmatrix}
1 & -2 & 3 & 1 \\
2 & -1 & 3 & 1 \\
3 & 1 & -2 & -1 \\
5 & -3 & 1 & 2 \\
\end{pmatrix}
\sim
\begin{pmatrix}
1 & -2 & 3 & 1 \\
0 & 1 & -1 & -1 \\
0 & 3 & -3 & -3 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

The last augmented matrix results in the following system of equations:

\[x_1 + x_3 - x_4 = 0 \quad \text{and} \quad x_2 - x_3 - x_4 = -3.\]

Since \(x_3\) and \(x_4\) are free variables, we can let \(x_3 = s\) and \(x_1 = t\), where \(s\) and \(t\) are real numbers. It follows that the solution set of the system is given by \(\{(t - s, s + t - 3, s, t) : s, t \text{ any real numbers}\}\).

15. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

\[
\begin{pmatrix}
2 & -1 & 3 & 1 & 11 \\
1 & -3 & 2 & -1 & 2 \\
3 & 1 & -2 & -1 & 2 \\
5 & -3 & 1 & 2 & 2 \\
\end{pmatrix}
\sim
\begin{pmatrix}
1 & -2 & 3 & 1 & 6 \\
2 & -1 & 3 & 1 & 6 \\
3 & 1 & -2 & -1 & 6 \\
5 & -3 & 1 & 2 & 6 \\
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

1. P_{12}  2. A_{12}(-3), A_{13}(-2)  3. M_{2}(-\frac{1}{2})  4. A_{23}(1)

5. M_{3}(-\frac{1}{10})  6. A_{31}(-\frac{1}{5}), A_{32}(-\frac{1}{10}), A_{33}(4), A_{34}(\frac{49}{5})  7. M_{4}(-\frac{1}{5})

8. A_{41}(\frac{1}{5}), A_{42}(-\frac{1}{5}), A_{43}(-\frac{2}{5}), A_{44}(\frac{3}{2})  9. M_{5}(\frac{1}{2})  10. A_{51}(-\frac{1}{7}), A_{52}(-\frac{1}{7}), A_{53}(\frac{1}{7}), A_{54}(-1)
It follows from the last augmented matrix that the solution to the system is given by $(1, -3, 4, -4, 2)$.

**17.** The equation $Ax = b$ reads

\[
\begin{bmatrix}
1 & 0 & 5 \\
3 & -2 & 11 \\
2 & -2 & 6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
2
\end{bmatrix}.
\]

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

\[
\begin{bmatrix}
1 & 0 & 5 \\
3 & -2 & 11 \\
2 & -2 & 6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 5 \\
0 & 3 & 11 \\
0 & 0 & -4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 5 \\
0 & 3 & 11 \\
0 & 0 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 5 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}.
\]

**1.** $A_{12}(-3)$, $A_{13}(-2)$ **2.** $M_2(-1/2)$ **3.** $A_{23}(2)$

Hence, we have $x_1 + 5x_3 = 0$ and $x_2 + 2x_3 = -1$. Since $x_3$ is a free variable, we can let $x_3 = t$, where $t$ is any real number. It follows that the solution set for the given system is given by \{$(−5t, −2t − 1, t) : t \in \mathbb{R}$\}.

**19.** The equation $Ax = b$ reads

\[
\begin{bmatrix}
1 & -1 & 0 & -1 \\
2 & 1 & 3 & 7 \\
3 & -2 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
2 \\
2 \\
4
\end{bmatrix}.
\]

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

\[
\begin{bmatrix}
1 & -1 & 0 & -1 \\
2 & 1 & 3 & 7 \\
3 & -2 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1 & 2 \\
0 & 3 & 3 & 9 \\
0 & 1 & 3 & -2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1 & 2 \\
0 & 1 & 3 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

**1.** $A_{12}(-2)$, $A_{13}(-3)$ **2.** $P_{23}$ **3.** $A_{21}(1)$, $A_{23}(-3)$

From the last row of the last augmented matrix, it is clear that the given system is inconsistent.

**21.** Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

\[
\begin{bmatrix}
1 & 2 & -1 & 3 \\
2 & 5 & 1 & 7 \\
1 & 1 & -k^2 & -k
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -1 & 3 \\
0 & 1 & 3 & 1 \\
0 & -1 & 1 - k^2 & -3 - k
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -1 & 3 \\
0 & 1 & 3 & 1 \\
0 & 0 & 4 - k^2 & -2 - k
\end{bmatrix}.
\]

**1.** $A_{12}(-2)$, $A_{13}(-1)$ **2.** $A_{23}(-1)$
(a): If \( k = 2 \), then the last row of the last augmented matrix reveals an inconsistency; hence the system has no solutions in this case.

(b): If \( k = -2 \), then the last row of the last augmented matrix consists entirely of zeros, and hence we have only two pivots (first two columns) and a free variable \( x_3 \); hence the system has infinitely many solutions.

(c): If \( k \neq \pm 2 \), then the last augmented matrix above contains a pivot for each variable \( x_1, x_2, \) and \( x_3 \), and can be solved for a unique solution by back-substitution.

23. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

\[
\begin{bmatrix}
1 & 1 & -2 & 4 \\
3 & 5 & -4 & 16 \\
2 & 3 & -a & b
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & -2 & 4 \\
0 & 2 & 2 & 4 \\
0 & 1 & 4 - a & b - 8
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & -2 & 4 \\
0 & 1 & 1 & 2 \\
0 & 1 & 4 - a & b - 8
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -3 & 2 \\
0 & 1 & 1 & 2 \\
0 & 0 & 3 - a & b - 10
\end{bmatrix}.
\]

1. \( A_{12}(-3), A_{13}(-2) \)
2. \( M_2(\frac{1}{2}) \)
3. \( A_{21}(-1), A_{23}(-1) \)

(a): From the last row of the last augmented matrix above, we see that there is no solution if \( a = 3 \) and \( b \neq 10 \).

(b): From the last row of the augmented matrix above, we see that there are infinitely many solutions if \( a = 3 \) and \( b = 10 \), because in that case, there is no pivot in the column of the last augmented matrix corresponding to the third variable \( x_3 \).

(c): From the last row of the augmented matrix above, we see that if \( a \neq 3 \), then regardless of the value of \( b \), there is a pivot corresponding to each variable \( x_1, x_2, \) and \( x_3 \). Therefore, we can uniquely solve the corresponding system by back-substitution.

25. The corresponding augmented matrix for this linear system can be reduced to row-echelon form via

\[
\begin{bmatrix}
1 & 2 & 1 & y_1 \\
2 & 3 & 1 & y_2 \\
3 & 5 & 1 & y_3
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 1 & y_2 - 2y_3 \\
0 & 1 & 1 & y_1 - 2y_3 + y_1 \\
0 & 2 & 1 & y_3 - 3y_1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 1 & y_2 - 2y_1 \\
0 & 1 & 1 & y_3 - 3y_1 \\
0 & 0 & 0 & y_1 - 2y_2 + y_3
\end{bmatrix}.
\]

1. \( A_{12}(-2), A_{13}(-3) \)
2. \( A_{23}(-2) \)

For consistency, we must have \( \text{rank}(A) = \text{rank}(A^\#) \), which requires \( (y_1, y_2, y_3) \) to satisfy \( y_1 - 2y_2 + y_3 = 0 \). If this holds, then the system has an infinite number of solutions, because the column of the augmented matrix corresponding to \( y_3 \) will be unpivoted, indicating that \( y_3 \) is a free variable in the solution set.

27. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

\[
\begin{bmatrix}
1 & 2 & 1 & 1 \\
3 & 5 & 1 & 3 \\
2 & 6 & 7 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
3 & 5 & 1 & 3 \\
1 & 2 & 1 & 1 \\
2 & 6 & 7 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
3 & 5 & 1 & 3 \\
0 & 1/3 & 2/3 & 0 \\
0 & 8/3 & 19/3 & -1
\end{bmatrix}
\sim
\begin{bmatrix}
3 & 5 & 1 & 3 \\
0 & 1/3 & 2/3 & 0 \\
0 & 0 & -1/8 & 1/8
\end{bmatrix}.
\]

1. \( P_{12} \)
2. \( A_{12}(-1/3), A_{13}(-2/3) \)
3. \( P_{23} \)
4. \( A_{23}(-1/8) \)
Using back substitution to solve the equivalent system yields the unique solution \((-2, 2, -1)\).

29. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

\[
\begin{bmatrix}
2 & -1 & -4 & | & 5 \\
3 & 2 & -5 & | & 8 \\
5 & 6 & -6 & | & 20 \\
1 & 1 & -3 & | & -3 \\
\end{bmatrix}
\sim
\begin{bmatrix}
5 & 6 & -6 & | & 20 \\
0 & -17/5 & -8/5 & | & -3 \\
0 & -8/5 & -7/5 & | & -4 \\
0 & -1/5 & -9/5 & | & -7 \\
\end{bmatrix}
\sim
\begin{bmatrix}
5 & 6 & -6 & | & 20 \\
0 & -17/5 & -8/5 & | & -3 \\
0 & -8/5 & -7/5 & | & -4 \\
0 & -1/5 & -9/5 & | & -7 \\
\end{bmatrix}
\sim
\begin{bmatrix}
5 & 6 & -6 & | & 20 \\
0 & -17/5 & -8/5 & | & -3 \\
0 & -29/17 & -116/17 & | & -3 \\
0 & -11/17 & -44/17 & | & 0 \\
\end{bmatrix}.
\]

Using back substitution to solve the equivalent system yields the unique solution \((10, -1, 4)\).

31.

(a): Let

\[
A^# = \begin{bmatrix}
a_{11} & 0 & 0 & \ldots & 0 & b_1 \\
a_{21} & a_{22} & 0 & \ldots & 0 & b_2 \\
a_{31} & a_{32} & a_{33} & \ldots & 0 & b_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} & b_n
\end{bmatrix}
\]

represent the corresponding augmented matrix of the given system. Since \(a_{11}x_1 = b_1\), we can solve for \(x_1\) easily:

\[
x_1 = \frac{b_1}{a_{11}}, \quad (a_{11} \neq 0).
\]

Now since \(a_{21}x_1 + a_{22}x_2 = b_2\), by using the expression for \(x_1\) we just obtained, we can solve for \(x_2\):

\[
x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22}}.
\]

In a similar manner, we can solve for \(x_3, x_4, \ldots, x_n\).

(b): We solve instantly for \(x_1\) from the first equation: \(x_1 = 2\). Substituting this into the middle equation, we obtain \(2 \cdot 2 - 3 \cdot x_2 = 1\), from which it quickly follows that \(x_2 = 1\). Substituting for \(x_1\) and \(x_2\) in the bottom equation yields \(3 \cdot 2 + 1 - x_3 = 8\), from which it quickly follows that \(x_3 = -1\). Consequently, the solution of the given system is \((2, 1, -1)\).

33. Reduce the augmented matrix of the system:

\[
\begin{bmatrix}
3 & 2 & -1 & 0 \\
2 & 1 & 1 & 0 \\
5 & -4 & 1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & -2 & 0 \\
0 & -1 & 5 & 0 \\
0 & -9 & 11 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & -2 & 0 \\
0 & 1 & -5 & 0 \\
0 & -9 & 11 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & -5 & 0 \\
0 & 0 & -34 & 0
\end{bmatrix}
\]
Reduce the augmented matrix of the system:

\[
\begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & -5 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

1. \(A_{21}(-1), A_{12}(-2), A_{13}(-5)\)  2. \(M_2(-1)\)  3. \(A_{21}(-1), A_{23}(9)\)

4. \(M_3(-1/34)\)  5. \(A_{31}(-3), A_{32}(5)\)

Therefore, the unique solution to this system is \(x_1 = x_2 = x_3 = 0\): \((0, 0, 0)\).

35. Reduce the augmented matrix of the system:

\[
\begin{bmatrix}
2 & -1 & -1 & 0 \\
5 & -1 & 2 & 0 \\
1 & 1 & 4 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 4 & 0 \\
5 & -1 & 2 & 0 \\
2 & -1 & 1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

1. \(P_{13}\)  2. \(A_{12}(-5), A_{13}(-2)\)  3. \(M_2(-1/6)\)  4. \(A_{21}(-1), A_{23}(3)\)

It follows that \(x_1 + x_3 = 0\) and \(x_2 + 3x_3 = 0\). Setting \(x_3 = t\), where \(t\) is a free variable, we get \(x_2 = -3t\) and \(x_1 = -t\). Thus we have that the solution set of the system is \(\{(-t, -3t, t) : t \in \mathbb{R}\}\).

37. Reduce the augmented matrix of the system:

\[
\begin{bmatrix}
3 & 2 & 1 & 0 \\
6 & -1 & 2 & 0 \\
12 & 6 & 4 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2/3 & 1/3 & 0 \\
6 & -1 & 2 & 0 \\
12 & 6 & 4 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1/3 & 0 \\
0 & 1 & 0 & 0 \\
0 & -2 & 0 & 0
\end{bmatrix}
\]

1. \(M_{1}(1/3)\)  2. \(A_{12}(-6), A_{13}(-12)\)  3. \(M_2(-1/5)\)  4. \(A_{21}(-2/3), A_{23}(2)\)

From the last augmented matrix, we have \(x_1 + \frac{1}{3}x_3 = 0\) and \(x_2 = 0\). Since \(x_3\) is a free variable, we let \(x_3 = t\), where \(t\) is a real number. It follows that the solution set for the given system is given by \(\{(t, 0, -3t) : t \in \mathbb{R}\}\).

39. Reduce the augmented matrix of the system:

\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 + i & 1 & 0 \\
1 - 2i & -1 + i & 1 - 3i \\
1 & 1 + i & 1 - i & 0 \\
i & 1 & i & 0 \\
i & 1 & i & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 + i & 1 - i & 0 \\
0 & 2 - i & -1 & 0 \\
0 & -4 + 2i & 2 & 0 \\
1 & 1 + i & 1 - i & 0 \\
i & 1 & i & 0 \\
i & 1 & i & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 6/5 & 0 \\
0 & 1 & -3/5 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

1. \(A_{12}(-i), A_{13}(-1 + 2i)\)  2. \(A_{23}(2)\)  3. \(M_2(\frac{1}{\overline{2} - i})\)  4. \(A_{21}(-1 - i)\)
From the last augmented matrix we see that $x_3$ is a free variable. We set $x_3 = 5s$, where $s \in \mathbb{C}$. Then $x_1 = 2(i-3)s$ and $x_2 = (2 + i)s$. Thus, the solution set of the system is $\{(2(i-3)s, (2 + i)s, s) : s \in \mathbb{C}\}$.

41. Reduce the augmented matrix of the system:

\[
\begin{bmatrix}
2 & -4 & 6 & 0 \\
3 & -6 & 9 & 0 \\
1 & -2 & 3 & 0 \\
5 & -10 & 15 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & -2 & 3 & 0 \\
3 & -6 & 9 & 0 \\
2 & -4 & 6 & 0 \\
5 & -10 & 15 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & -2 & 3 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

From the last matrix we have that $x_1 - 2x_3 + 3x_3 = 0$. Since $x_2$ and $x_3$ are free variables, let $x_2 = s$ and let $x_3 = t$, where $s$ and $t$ are real numbers. The solution set of the given system is therefore $\{(2s - 3t, s, t) : s, t \in \mathbb{R}\}$.

43. Reduce the augmented matrix of the system:

\[
\begin{bmatrix}
2 & 1 & -1 & 1 & 0 \\
1 & 1 & 1 & -1 & 0 \\
3 & -1 & 1 & -2 & 0 \\
4 & 2 & -1 & 1 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 1 & -1 & 0 \\
2 & 1 & -1 & 1 & 0 \\
3 & -1 & 1 & -2 & 0 \\
4 & 2 & -1 & 1 & 0
\end{bmatrix} \sim \begin{bmatrix}
0 & -1 & -3 & 3 & 0 \\
0 & -4 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

From the last augmented matrix, it follows that the solution set to the system is given by $\{(0,0,0,0)\}$.

44. The equation $Ax = 0$ is

\[
\begin{bmatrix}
1 -i & 2i \\
1 + i & -2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

Reduce the augmented matrix of the system:

\[
\begin{bmatrix}
1 -i & 2i \\
1 + i & -2
\end{bmatrix} \sim \begin{bmatrix}
1 & -1+i & 0 \\
1 & -1+i & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & -1+i & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

It follows that $x_1 + (1 + i)x_2 = 0$. Since $x_2$ is a free variable, we can let $x_2 = t$, where $t$ is a real number. The solution set to the system is then given by $\{(t(1 + i), t) : t \in \mathbb{R}\}$.
47. The equation $Ax = 0$ is

$$
\begin{bmatrix}
1 & 2 & 3 \\
2 & -1 & 0 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
$$

Reduce the augmented matrix of the system:

$$
\begin{bmatrix}
1 & 2 & 3 & 0 \\
2 & -1 & 0 & 0 \\
1 & 1 & 1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 3 & 0 \\
0 & -5 & -6 & 0 \\
0 & -1 & -2 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 3 & 0 \\
0 & -1 & -2 & 0 \\
0 & -5 & -6 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.
$$

1. $A_{12}(-2), A_{13}(-1)$  
2. $P_{23}$  
3. $M_2(-1)$  
4. $A_{21}(-2), A_{23}(5)$  
5. $M_3(1/4)$  
6. $A_{31}(1), A_{32}(-2)$

From the last augmented matrix, we see that the only solution to the given system is $x_1 = x_2 = x_3 = 0$: $\{(0,0,0)\}$.

49. The equation $Ax = 0$ is

$$
\begin{bmatrix}
2 - 3i & 1 + i & i - 1 & 0 \\
3 + 2i & -1 + i & -1 - i & 0 \\
5 - i & 2i & -2 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \
0
\end{bmatrix}.
$$

Reduce the augmented matrix of this system:

$$
\begin{bmatrix}
2 - 3i & 1 + i & i - 1 & 0 \\
3 + 2i & -1 + i & -1 - i & 0 \\
5 - i & 2i & -2 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & \frac{-1 + 5i}{13} & \frac{-5 - i}{13} & 0 \\
3 + 2i & -1 + i & -1 - i & 0 \\
5 - i & 2i & -2 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & \frac{-1 + 5i}{13} & \frac{-5 - i}{13} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

1. $M_1(\frac{2 i + 3}{13})$  
2. $A_{12}(-3 - 2i), A_{13}(-5 + i)$

From the last augmented matrix, we see that $x_1 + \frac{-1 + 5i}{13}x_2 + \frac{-5 - i}{13}x_3 = 0$. Since $x_2$ and $x_3$ are free variables, we can let $x_2 = 13r$ and $x_3 = 13s$, where $r$ and $s$ are complex numbers. It follows that the solution set of the system is $\{(r(1 - 5i) + s(5 + i), r, s) : r, s \in \mathbb{C}\}$.

51. The equation $Ax = 0$ is

$$
\begin{bmatrix}
1 & 0 & 3 \\
3 & -1 & 7 \\
2 & 1 & 8 \\
1 & 1 & 5 \\
-1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
$$

Reduce the augmented matrix of the system:

$$
\begin{bmatrix}
1 & 0 & 3 & 0 \\
3 & -1 & 7 & 0 \\
2 & 1 & 8 & 0 \\
1 & 1 & 5 & 0 \\
-1 & 1 & -1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & -1 & -2 & 0 \\
0 & 1 & 2 & 0 \\
0 & 1 & 2 & 0 \\
0 & 1 & 2 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 1 & 2 & 0 \\
0 & 1 & 2 & 0 \\
0 & 1 & 2 & 0
\end{bmatrix}.
$$
1. \( A_{12}(-3), A_{13}(-2), A_{14}(-1), A_{15}(1) \)  
2. M_2(-1)  
3. A_{23}(-1), A_{24}(-1), A_{25}(-1)

From the last augmented matrix, we obtain the equations \( x_1 + 3x_3 = 0 \) and \( x_2 + 2x_3 = 0 \). Since \( x_3 \) is a free variable, we let \( x_3 = t \), where \( t \) is a real number. The solution set for the given system is then given by \( \{(-3t, -2t, t) : t \in \mathbb{R}\} \).

53. The equation \( Ax = 0 \) is

\[
\begin{bmatrix}
1 & 0 & -3 & 0 \\
3 & 0 & -9 & 0 \\
-2 & 0 & 6 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
\]

Reduce the augmented matrix of the system:

\[
\begin{bmatrix}
1 & 0 & -3 & 0 & 0 \\
3 & 0 & -9 & 0 & 0 \\
-2 & 0 & 6 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

From the last augmented matrix we obtain \( x_1 - 3x_3 = 0 \). Therefore, \( x_2, x_3, \) and \( x_4 \) are free variables, so we let \( x_2 = r, x_3 = s, \) and \( x_4 = t, \) where \( r, s, t \) are real numbers. The solution set of the given system is therefore \( \{(3s, r, s, t) : r, s, t \in \mathbb{R}\} \).

**Solutions to Section 2.6**

**True-False Review:**

1. **FALSE.** An invertible matrix is also known as a nonsingular matrix.

3. **TRUE.** If \( A \) is invertible, then the unique solution to \( Ax = b \) is \( x = A^{-1}b \).

5. **FALSE.** For instance, if \( A = I_n \) and \( B = -I_n \), then \( A \) and \( B \) are both invertible, but \( A + B = 0_n \) is not invertible.

7. **TRUE.** From \( A^2 = A \), we subtract to obtain \( A(A - I) = 0 \). Left multiplying both sides of this equation by \( A^{-1} \) (since \( A \) is invertible, \( A^{-1} \) exists), we have \( A - I = A^{-1}0 = 0 \). Therefore, \( A = I \), the identity matrix.

9. **TRUE.** Any \( 5 \times 5 \) invertible matrix must have rank 5, not rank 4 (Theorem 2.6.5).

**Problems:**

1. We have

\[
AA^{-1} = \begin{bmatrix}
2 & -1 \\
3 & -1
\end{bmatrix}
\begin{bmatrix}
-1 & 1 \\
-3 & 2
\end{bmatrix} = \begin{bmatrix}
(2)(-1) + (-1)(-3) & (2)(1) + (-1)(2) \\
(3)(-1) + (-1)(-3) & (3)(1) + (-1)(2)
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = I_2.
\]
3. We have
\[
AA^{-1} = \begin{bmatrix}
3 & 5 & 1 \\
1 & 2 & 1 \\
2 & 6 & 7
\end{bmatrix}
\begin{bmatrix}
8 & -29 & 3 \\
-5 & 19 & -2 \\
2 & -8 & 1
\end{bmatrix}
= \begin{bmatrix}
(3)(8) + (5)(-5) + (1)(2) \\
(1)(8) + (2)(-5) + (1)(2) \\
(2)(8) + (6)(-5) + (7)(2)
\end{bmatrix}
\begin{bmatrix}
3 & 5 & 1 \\
1 & 2 & 1 \\
2 & 6 & 7
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = I_3.
\]

5. We have
\[
[A|I_2] = \begin{bmatrix}
1 & 1+i & 1 \\
1 & 0 & 1 \\
1-i & 1 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1+i & 1 \\
0 & -1 & 1+i \\
0 & 1 & 1-i
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1+i & 1 \\
0 & 1 & 1-i \\
-1 & 1+i & -1
\end{bmatrix} = [I_2|A^{-1}].
\]
Thus,
\[
A^{-1} = \begin{bmatrix}
-1 & 1+i \\
1-i & -1
\end{bmatrix}.
\]

1. \(A_{12}(-1+i)\) 2. \(M_2(-1)\) 3. \(A_{21}(-1-i)\)

7. Note that \(AB = 0_2\) for all \(2 \times 2\) matrices \(B\). Therefore, \(A\) is not invertible.

9. We have
\[
[A|I_3] = \begin{bmatrix}
3 & 5 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
2 & 6 & 7 & 0 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 1 & 1 & 0 & 0 \\
3 & 5 & 1 & 1 & 0 & 0 \\
2 & 6 & 7 & 0 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 1 & 1 & 0 & 0 \\
0 & -1 & -2 & 1 & -3 & 0 \\
0 & 2 & 5 & 0 & -2 & 1
\end{bmatrix}
= [I_3|A^{-1}].
\]
Thus,
\[
A^{-1} = \begin{bmatrix}
8 & -29 & 3 \\
-5 & 19 & -2 \\
2 & -8 & 1
\end{bmatrix}.
\]

1. \(P_{12}\) 2. \(A_{12}(-3), A_{13}(-2)\) 3. \(M_2(-1)\) 4. \(A_{21}(-2), A_{23}(-2)\) 5. \(A_{31}(3), A_{32}(-2)\)

11. We have
\[
[A|I_3] = \begin{bmatrix}
4 & 2 & -13 & 1 & 0 & 0 \\
2 & 1 & -7 & 0 & 0 & 1 \\
3 & 2 & 4 & 0 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
3 & 2 & 4 & 0 & 0 & 1 \\
2 & 1 & -7 & 0 & 0 & 1 \\
4 & 2 & -13 & 1 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 11 & 0 & -1 & 1 \\
2 & 1 & -7 & 0 & 0 & 1 \\
4 & 2 & -13 & 1 & 0 & 0
\end{bmatrix}
\[
\begin{bmatrix}
1 & 1 & 11 & 0 & -1 & 1 \\
0 & -1 & -29 & 0 & 3 & -2 \\
0 & -2 & -57 & 1 & 4 & -4
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 11 & 0 & -1 & 1 \\
0 & 1 & 29 & 0 & -3 & 2 \\
0 & -2 & -57 & 1 & 4 & -4
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -18 & 0 & 2 & -1 \\
0 & 1 & 29 & 0 & -3 & 2 \\
0 & 0 & 1 & 1 & -2 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 & 18 & -34 & -1 \\
0 & 1 & 0 & -29 & 55 & 2 \\
0 & 0 & 1 & 1 & -2 & 0
\end{bmatrix}
= [I_3|A^{-1}].
\]

Thus,
\[
A^{-1} = \begin{bmatrix}
18 & -34 & -1 \\
-29 & 55 & 2 \\
1 & -2 & 0
\end{bmatrix}.
\]

13. We have
\[
[A|I_3] = \begin{bmatrix}
1 & i & 2 \\
1+i & -1 & 2i \\
2 & 2i & 5
\end{bmatrix}
\sim
\begin{bmatrix}
1 & i & 2 \\
0 & -i & -2 \\
0 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
-i & 1 & 0 \\
0 & 2 & 1 \\
0 & 0 & 1
\end{bmatrix}
= [I_3|A^{-1}].
\]

Thus,
\[
A^{-1} = \begin{bmatrix}
-i & 1 & 0 \\
1 & 2 & 1 \\
-2 & 0 & 1
\end{bmatrix}.
\]

15. We have
\[
[A|I_4] = \begin{bmatrix}
1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\
2 & 0 & 3 & -4 & 0 & 1 & 0 & 0 \\
3 & -1 & 7 & 8 & 0 & 0 & 1 & 0 \\
1 & 0 & 3 & 5 & 0 & 0 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\
0 & 2 & -1 & -10 & 0 & 2 & 1 & 0 \\
0 & 2 & -1 & -10 & 0 & 2 & 1 & 0 \\
0 & 1 & 1 & 2 & -1 & 0 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 3 & 5 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \\
0 & 0 & -3 & -14 & 0 & 1 & 0 & 2 \\
0 & 0 & -3 & -14 & 0 & 1 & 0 & 2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & -10 & -3 & 0 & 3 & -5 \\
0 & 1 & 0 & -3 & -2 & 0 & 1 & -1 \\
0 & 0 & 1 & 5 & 1 & 0 & -1 & 2 \\
0 & 0 & -3 & -14 & 0 & 1 & 0 & -2
\end{bmatrix}
= [I_4|A^{-1}].
\]

Thus,
\[
A^{-1} = \begin{bmatrix}
1 & 0 & 0 & -10 & -3 & 0 & 3 & -5 \\
0 & 1 & 0 & -3 & -2 & 0 & 1 & -1 \\
0 & 0 & 1 & 5 & 1 & 0 & -1 & 2 \\
0 & 0 & -3 & -14 & 0 & 1 & 0 & -2
\end{bmatrix}.
\]
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
27 & 10 & -27 & 35 \\
7 & 3 & -8 & 11 \\
-14 & -5 & 14 & -18 \\
3 & 1 & -3 & 4
\end{bmatrix}
= [I_4|A^{-1}].
\]

Thus,
\[
A^{-1} =
\begin{bmatrix}
27 & 10 & -27 & 35 \\
7 & 3 & -8 & 11 \\
-14 & -5 & 14 & -18 \\
3 & 1 & -3 & 4
\end{bmatrix}.
\]

1. \(A_{12}(-2), A_{13}(-3), A_{14}(-1)\)
2. \(P_{13}\)
3. \(A_{21}(1), A_{23}(-2), A_{24}(-2)\)
4. \(M_3(-1)\)
5. \(A_{31}(-3), A_{32}(-1), A_{34}(3)\)
6. \(A_{41}(10), A_{42}(3), A_{43}(5)\)

17. To determine the second column vector of \(A^{-1}\) without determining the whole inverse, we solve the linear system
\[
\begin{bmatrix}
2 & -1 & 4 \\
5 & 1 & 2 \\
1 & -1 & 3
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]

19. We have \(A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 2 & 4 & -3 \end{bmatrix}\), \(b = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\), and the Gauss-Jordan method yields \(A^{-1} = \begin{bmatrix} 7 & 5 & -3 \\ -2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}\).

Therefore, we have
\[
x = A^{-1}b = \begin{bmatrix} 7 & 5 & -3 \\ -2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}
\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}
= \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.
\]

Hence, we have \(x_1 = -2, x_2 = 2,\) and \(x_3 = 1\).

21. We have \(A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 10 & 1 \\ 4 & 1 & 8 \end{bmatrix}\), \(b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\), and the Gauss-Jordan method yields \(A^{-1} = \begin{bmatrix} -79 & 27 & 46 \\ 12 & -4 & -7 \\ 38 & -13 & -22 \end{bmatrix}\).

Therefore, we have
\[
x = A^{-1}b = \begin{bmatrix} -79 & 27 & 46 \\ 12 & -4 & -7 \\ 38 & -13 & -22 \end{bmatrix}
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
= \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix}.
\]

Hence, we have \(x_1 = -6, x_2 = 1,\) and \(x_3 = 3\).

23. We have
\[
AA^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}
= \begin{bmatrix} (0)(0) + (1)(1) & (0)(-1) + (1)(0) \\ (-1)(0) + (0)(1) & (-1)(-1) + (0)(0) \end{bmatrix}
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2,
\]

so \(A^T = A^{-1}\).

25. We have
\[
AA^T = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}
\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}
= \begin{bmatrix} \cos^2 \alpha + \sin^2 \alpha & (\cos \alpha)(-\sin \alpha) + (\sin \alpha)(\cos \alpha) \\ (-\sin \alpha)(\cos \alpha) + (\cos \alpha)(\sin \alpha) & (-\sin \alpha)^2 + \cos^2 \alpha \end{bmatrix}
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2,
\]
so $A^T = A^{-1}$.

27. For part 2, we have

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n,$$

and for part 3, we have

$$(A^{-1})^T A^T = (AA^{-1})^T = I_n^T = I_n.$$

29. Since $A$ is symmetric, we know that $A^T = A$. We wish to show that $(A^{-1})^T = A^{-1}$. We have

$$(A^{-1})^T = (A^T)^{-1} = A^{-1},$$

which shows that $A^{-1}$ is symmetric. The first equality follows from part 3 of Theorem 2.6.9, and the second equality results from the assumption that $A$ is symmetric.

31. We have

$$(I_n - A)(I_n + A + A^2 + A^3) = I_n(I_n + A + A^2 + A^3) - A(I_n + A + A^2 + A^3)$$

$$= I_n + A + A^2 + A^3 - A - A^2 - A^3 - A^4 = I_n - A^4 = I_n,$$

where the last equality uses the assumption that $A^4 = 0$. This calculation shows that $I_n - A$ and $I_n + A + A^2 + A^3$ are inverses of one another.

33. YES. Since $BA = I_n$, we know that $A^{-1} = B$ (see Theorem 2.6.11). Likewise, since $CA = I_n$, $A^{-1} = C$. Since the inverse of $A$ is unique, it must follow that $B = C$.

35. Assume that $A$ is an invertible matrix and that $Ax_i = b_i$ for $i = 1, 2, \ldots, p$ (where each $b_i$ is given). Use elementary row operations on the augmented matrix of the system to obtain the equivalence

$$[A|b_1 \ b_2 \ b_3 \ \ldots \ b_p] \sim [I_n|c_1 \ c_2 \ c_3 \ \ldots \ c_p].$$

The solutions to the system can be read from the last matrix: $x_i = c_i$ for each $i = 1, 2, \ldots, p$.

37.

(a) Let $e_i$ denote the $i$th column vector of the identity matrix $I_m$, and consider the $m$ linear systems of equations

$$Ax_i = e_i$$

for $i = 1, 2, \ldots, m$. Since $\text{rank}(A) = m$ and each $e_i$ is a column $m$-vector, it follows that

$$\text{rank}(A^R) = m = \text{rank}(A)$$

and so each of the systems $Ax_i = e_i$ above has a solution (Note that if $m < n$, then there will be an infinite number of solutions). If we let $B = [x_1, x_2, \ldots, x_m]$, then

$$AB = A[x_1, x_2, \ldots, x_m] = [Ax_1, Ax_2, \ldots, Ax_m] = [e_1, e_2, \ldots, e_m] = I_n.$$
Thus, we must have
\[ a + 3b + c = 1, \quad d + 3e + f = 0, \quad 2a + 7b + 4c = 0, \quad 2d + 7e + 4f = 1. \]
The first and third equation comprise a linear system with augmented matrix
\[
\begin{bmatrix}
1 & 3 & 1 & 1 \\
2 & 7 & 4 & 0
\end{bmatrix}
\]
for \(a, b,\) and \(c\). The row-echelon form of this augmented matrix is
\[
\begin{bmatrix}
1 & 3 & 1 & 1 \\
0 & 1 & 2 & 0
\end{bmatrix}
\]. Setting
\( c = t \), we have
\( b = -2 - 2t \)
and
\( a = 7 + 5t \).
Next, the second and fourth equation above comprise a linear system with augmented matrix
\[
\begin{bmatrix}
1 & 3 & 1 & 1 \\
0 & 1 & 2 & 1
\end{bmatrix}
\]. Setting
\( f = s \), we have
\( e = 1 - 2s \)
and
\( d = -3 + 5s \). Thus, right inverses of \( A \) are precisely the matrices of the form
\[
\begin{bmatrix}
7 + 5t & -3 + 5s & -2 - 2t & 1 - 2s \\
0 & 0 & 1 & t
\end{bmatrix}
\].

**Solutions to Section 2.7**

**True-False Review:**

1. **TRUE.** Since every elementary matrix corresponds to a (reversible) elementary row operation, the reverse elementary row operation will correspond to an elementary matrix that is the inverse of the original elementary matrix.

3. **FALSE.** Every *invertible* matrix can be expressed as a product of elementary matrices. Since every elementary matrix is invertible and products of invertible matrices are invertible, any product of elementary matrices must be an invertible matrix.

5. **FALSE.** If \( P_{ij} \) is a permutation matrix, then \( P^2_{ij} = I_n \), since permuting the \(i\)th and \(j\)th rows of \( I_n \) twice yields \( I_n \). Alternatively, we can observe that \( P^2_{ij} = I_n \) from the fact that \( P_{ij}^{-1} = P_{ij} \).

7. **FALSE.** For example, consider the elementary matrices
\[
E_1 = \begin{bmatrix}
1 & 3 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
and
\[
E_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\].

Then we have
\[
E_1 E_2 = \begin{bmatrix}
1 & 3 & 6 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\]
and
\[
E_2 E_1 = \begin{bmatrix}
1 & 3 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\].

9. **FALSE.** The matrix \( U \) need not be a *unit* upper triangular matrix.

**Problems:**

1. **Permutation Matrices:**
\[
P_{12} = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad P_{13} = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad P_{23} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]

**Scaling Matrices:**
\[
M_1(k) = \begin{bmatrix}
k & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad M_2(k) = \begin{bmatrix}
1 & 0 & 0 \\
0 & k & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad M_3(k) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & k
\end{bmatrix}
\]

**Row Combinations:**
\[
A_{12}(k) = \begin{bmatrix}
1 & 0 & 0 \\
k & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad A_{13}(k) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
k & 0 & 1
\end{bmatrix}, \quad A_{23}(k) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & k
\end{bmatrix}
\]
\[ A_{21}(k) = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{31}(k) = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{32}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}. \]

3. We have

\[ \begin{bmatrix} 5 & 8 & 2 \\ 1 & 3 & -1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 5 & 8 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -7 & 7 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix}. \]

Elementary Matrices: \( M_2(\frac{-1}{7}) \), \( A_{12}(-5) \), \( P_{12} \).

5. We have

\[ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

Elementary Matrices: \( A_{23}(2) \), \( M_2(-1) \), \( A_{13}(-3) \), \( A_{12}(-2) \).

7. We reduce \( A \) to the identity matrix:

\[ \begin{bmatrix} -2 & -3 \\ 5 & 7 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

The elementary matrices corresponding to these row operations are

\[ E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]

We have \( E_3E_4E_3E_2E_1A = I_2 \), so

\[ A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \]

which is the desired expression since each \( E_i^{-1} \) is an elementary matrix.

9. We reduce \( A \) to the identity matrix:

\[ \begin{bmatrix} 4 & -5 \\ 1 & 4 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 4 \\ 4 & -5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 4 \\ 0 & -21 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

The elementary matrices corresponding to these row operations are

\[ E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{27} & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}. \]
We have $E_4E_3E_2E_1A = I_2$, so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -21 \end{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix},$$

which is the desired expression since each $E_i^{-1}$ is an elementary matrix.

11. We reduce $A$ to the identity matrix:

$$\begin{bmatrix} 0 & -4 & -2 \\ 1 & -1 & 3 \\ -2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & -2 \\ -2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & 8 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$
15. We have
\[
\begin{bmatrix}
2 & 3 \\
5 & 1 
\end{bmatrix} \sim \begin{bmatrix}
2 & -3/5 \\
0 & -12/5 
\end{bmatrix} = U \implies m_{21} = 5/2 \implies L = \begin{bmatrix}
1/2 & 0 \\
5 & 1 
\end{bmatrix}.
\]
Then
\[
LU = \begin{bmatrix}
1/2 & 0 \\
5 & 1 
\end{bmatrix} \begin{bmatrix}
2 & -3/5 \\
0 & -12/5 
\end{bmatrix} = \begin{bmatrix}
2 & 3 \\
5 & 1 
\end{bmatrix} = A.
\]

1. $A_{12}(-\frac{3}{2})$

17. We have
\[
\begin{bmatrix}
3 & -1 & 2 \\
6 & -1 & 1 \\
-3 & 5 & 2 
\end{bmatrix} \sim \begin{bmatrix}
3 & -1 & 2 \\
0 & 1 & -3 \\
0 & 4 & 4 
\end{bmatrix} \sim \begin{bmatrix}
3 & -1 & 2 \\
0 & 1 & -3 \\
0 & 0 & 16 
\end{bmatrix} = U \implies m_{21} = 2, m_{31} = -1, m_{32} = 4.
\]
Hence,
\[
L = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
-1 & 4 & 1 
\end{bmatrix} \quad \text{and} \quad LU = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
-1 & 4 & 1 
\end{bmatrix} \begin{bmatrix}
3 & -1 & 2 \\
0 & 1 & -3 \\
0 & 0 & 16 
\end{bmatrix} = \begin{bmatrix}
3 & -1 & 2 \\
6 & -1 & 1 \\
-3 & 5 & 2 
\end{bmatrix} = A.
\]

1. $A_{12}(-2), A_{13}(1)$  
2. $A_{23}(-4)$

19. We have
\[
\begin{bmatrix}
1 & -1 & 2 & 3 \\
2 & 0 & 3 & -4 \\
3 & -1 & 7 & 8 \\
1 & 3 & 4 & 5 
\end{bmatrix} \sim \begin{bmatrix}
1 & -1 & 2 & 3 \\
0 & 2 & -1 & -10 \\
0 & 2 & 1 & -1 \\
0 & 4 & 2 & 2 
\end{bmatrix} \sim \begin{bmatrix}
1 & -1 & 2 & 3 \\
0 & 2 & -1 & -10 \\
0 & 0 & 2 & 9 \\
0 & 0 & 4 & 22 
\end{bmatrix} \sim \begin{bmatrix}
1 & -1 & 2 & 3 \\
0 & 2 & -1 & -10 \\
0 & 0 & 2 & 9 \\
0 & 0 & 0 & 4 
\end{bmatrix} = U.
\]

1. $A_{12}(-2), A_{13}(-3), A_{14}(-1)$  
2. $A_{23}(-1), A_{24}(-2)$  
3. $A_{34}(-2)$

Hence,
\[
m_{21} = 2, \quad m_{31} = 3, m_{41} = 1, m_{32} = 1, m_{42} = 2, m_{43} = 2.
\]
Hence,
\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
3 & 1 & 1 & 0 \\
1 & 2 & 2 & 1 
\end{bmatrix} \quad \text{and} \quad LU = \begin{bmatrix}
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
3 & 1 & 1 & 0 \\
1 & 2 & 2 & 1 
\end{bmatrix} \begin{bmatrix}
1 & -1 & 2 & 3 \\
0 & 2 & -1 & -10 \\
0 & 0 & 2 & 9 \\
0 & 0 & 0 & 4 
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 2 & 3 \\
2 & 0 & 3 & -4 \\
3 & -1 & 7 & 8 \\
1 & 3 & 4 & 5 
\end{bmatrix} = A.
\]

21. We have
\[
\begin{bmatrix}
1 & 2 \\
2 & 3 
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 \\
0 & -1 
\end{bmatrix} = U \implies m_{21} = 2 \implies L = \begin{bmatrix}
1 & 0 \\
2 & 1 
\end{bmatrix}.
\]

1. $A_{12}(-2)$
We now solve the triangular systems $Ly = b$ and $Ux = y$. From $Ly = b$, we obtain $y = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$. Then $Ux = y$ yields $x = \begin{bmatrix} -11 \\ 7 \end{bmatrix}$.

23. We have

$$\begin{bmatrix} 2 & 2 & 1 \\ 6 & 3 & -1 \\ -4 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & -4 \end{bmatrix} = U \implies m_{21} = 3, m_{31} = -2, m_{32} = -2.

1. $A_{12}(-3), A_{13}(2)$  2. $A_{23}(2)$

Hence, $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$. We now solve the triangular systems $Ly = b$ and $Ux = y$. From $Ly = b$, we obtain $y = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$. Then $Ux = y$ yields $x = \begin{bmatrix} -1/12 \\ 1/3 \\ 1/2 \end{bmatrix}$.

25. We have

$$\begin{bmatrix} 2 & -1 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} = U \implies m_{21} = -4 \implies L = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}.

1. $A_{12}(4)$

We now solve the triangular systems

$$Ly_i = b_i, \quad Ux_i = y_i$$

for $i = 1, 2, 3$. We have

$$Ly_1 = b_1 \implies y_1 = \begin{bmatrix} 3 \\ 11 \end{bmatrix}. \text{ Then } Ux_1 = y_1 \implies x_1 = \begin{bmatrix} -4 \\ -1 \end{bmatrix};$$

$$Ly_2 = b_2 \implies y_2 = \begin{bmatrix} 2 \\ 15 \end{bmatrix}. \text{ Then } Ux_2 = y_2 \implies x_2 = \begin{bmatrix} -6.5 \\ -15 \end{bmatrix};$$

$$Ly_3 = b_3 \implies y_3 = \begin{bmatrix} 5 \\ 11 \end{bmatrix}. \text{ Then } Ux_3 = y_3 \implies x_3 = \begin{bmatrix} -3 \\ -11 \end{bmatrix}.$$  

27. Observe that if $P_i$ is an elementary permutation matrix, then $P_i^{-1} = P_i = P_i^T$. Therefore, we have

$$P^{-1} = (P_1 P_2 \ldots P_k)^{-1} = P_k^{-1} P_{k-1}^{-1} \ldots P_2^{-1} P_1^{-1} = P_k^T P_{k-1}^T \ldots P_2^T P_1^T = (P_1 P_2 \ldots P_k)^T = P^T.$$

29.

(a): Since $A$ is invertible, Corollary 2.6.12 implies that both $L_2$ and $U_1$ are invertible. Since $L_1 U_1 = L_2 U_2$, we can left-multiply by $L_2^{-1}$ and right-multiply by $U_1^{-1}$ to obtain $L_2^{-1} L_1 = U_2 U_1^{-1}$.

(b): By Problem 28, we know that $L_2^{-1}$ is a unit lower triangular matrix and $U_1^{-1}$ is an upper triangular matrix. Therefore, $L_2^{-1} L_1$ is a unit lower triangular matrix and $U_2 U_1^{-1}$ is an upper triangular matrix. Since these two matrices are equal, we must have $L_2^{-1} L_1 = I_n$ and $U_2 U_1^{-1} = I_n$. Therefore, $L_1 = L_2$ and $U_1 = U_2$.

Solutions to Section 2.8
True-False Review:

1. **FALSE.** According to the given information, part (c) of the Invertible Matrix Theorem fails, while part (e) holds. This is impossible.

2. **FALSE.** Part (d) of the Invertible Matrix Theorem fails according to the given information, and therefore part (b) also fails. Hence, the equation $A\mathbf{x} = \mathbf{b}$ does not have a unique solution. But it is not valid to conclude that the equation has infinitely many solutions; it could have no solutions. For instance, if

$$
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

and $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, there are no solutions to $A\mathbf{x} = \mathbf{b}$, although $\text{rank}(A) = 2$.

Problems:

1. Since $A$ is an invertible matrix, the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. However, if we assume that $AB = AC$, then $A(B - C) = 0$. If $x_i$ denotes the $i$th column of $B - C$, then $x_i = \mathbf{0}$ for each $i$. That is, $B - C = \mathbf{0}$, or $B = C$, as required.

3. Since $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, $\text{REF}(A)$ contains a pivot in every column. Therefore, the linear system $A\mathbf{x} = \mathbf{b}$ can be solved by back-substitution for every $\mathbf{b}$ in $\mathbb{R}^n$. Therefore, $A\mathbf{x} = \mathbf{b}$ does have a solution.

Now suppose there are two solutions $\mathbf{y}$ and $\mathbf{z}$ to the system $A\mathbf{x} = \mathbf{b}$. That is, $A\mathbf{y} = \mathbf{b}$ and $A\mathbf{z} = \mathbf{b}$. Subtracting, we find

$$
A(\mathbf{y} - \mathbf{z}) = \mathbf{0},
$$

and so by assumption, $\mathbf{y} - \mathbf{z} = \mathbf{0}$. That is, $\mathbf{y} = \mathbf{z}$. Therefore, there is only one solution to the linear system $A\mathbf{x} = \mathbf{b}$.

5. We are assuming that the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ each have only the trivial solution $\mathbf{x} = \mathbf{0}$. Now consider the linear system

$$
(AB)\mathbf{x} = \mathbf{0}.
$$

Viewing this equation as

$$
A(B\mathbf{x}) = \mathbf{0},
$$

we conclude that $B\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{x} = \mathbf{0}$. Hence, the linear equation $(AB)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Solutions to Section 2.9

Problems:

1. We have

$$
(-4)A - B^T = \begin{bmatrix}
8 & -16 & -8 & -24 \\
4 & 4 & -20 & 0
\end{bmatrix} - \begin{bmatrix}
-3 & 2 & 1 & 0 \\
0 & 2 & -3 & 1
\end{bmatrix} = \begin{bmatrix}
11 & -18 & -9 & -24 \\
4 & 2 & -17 & -1
\end{bmatrix}.
$$

3. We have

$$
(AC)(AC)^T = \begin{bmatrix}
-2 \\ 26
\end{bmatrix} \begin{bmatrix}
-2 & 26 \\
26 & 676
\end{bmatrix} = \begin{bmatrix}
4 & -52 \\
-52 & 676
\end{bmatrix}.
$$

5. Using Problem 2, we find that

$$
(AB)^{-1} = \begin{bmatrix}
16 & 8 \\
6 & -17
\end{bmatrix}^{-1} = \frac{1}{320} \begin{bmatrix}
-17 & -8 \\
-6 & 16
\end{bmatrix}.
$$
7. (a): We have
\[
AB = \begin{bmatrix}
1 & 2 & 3 \\
2 & 5 & 7 \\
\end{bmatrix}
\begin{bmatrix}
3 & b \\
-4 & a \\
\end{bmatrix}
= \begin{bmatrix}
3a - 5 & 2a + 4b \\
7a - 14 & 5a + 9b \\
\end{bmatrix}.
\]
In order for this product to equal \(I_2\), we require
\[
3a - 5 = 1, \quad 2a + 4b = 0, \quad 7a - 14 = 0, \quad 5a + 9b = 1.
\]
We quickly solve this for the unique solution: \(a = 2\) and \(b = -1\).

(b): We have
\[
BA = \begin{bmatrix}
3 & -1 \\
-4 & 2 \\
2 & -1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
2 & 5 & 7 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 2 \\
0 & 2 & 2 \\
0 & -1 & -1 \\
\end{bmatrix}.
\]

9. (a): The \((i,j)\)-entry of \(A^2\) is
\[
\sum_{k=1}^{n} a_{ik} a_{kj}.
\]

(b): Assume that \(A\) is symmetric. That means that \(A^T = A\). We claim that \(A^2\) is symmetric. To see this, note that
\[
\]
Thus, \((A^2)^T = A^2\), and so \(A^2\) is symmetric.

11. We have
\[
A^2 = \begin{bmatrix}
3 & 9 \\
-1 & -3 \\
\end{bmatrix}^2 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix},
\]
so \(A\) is nilpotent.

13. We have
\[
A'(t) = 
\begin{bmatrix}
-3e^{-3t} & -2 \sec^2 t \tan t \\
6t^2 & -\sin t \\
6/t & -5 \\
\end{bmatrix}.
\]

15. Since \(A(t)\) is \(3 \times 2\) and \(B(t)\) is \(4 \times 2\), it is impossible to perform the indicated subtraction.

17. From the last equation, we see that \(x_3 = 0\). Substituting this into the middle equation, we find that \(x_2 = 0.5\). Finally, putting the values of \(x_2\) and \(x_3\) into the first equation, we find \(x_1 = -6 - 2.5 = -8.5\). Thus, there is a unique solution to the linear system, and the solution set is \([-8.5, 0.5, 0]\).

19. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us
\[
\begin{bmatrix}
1 & 2 & -1 & 1 \\
1 & 0 & 1 & 5 \\
4 & 4 & 0 & 12 \\
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 2 & -1 & 1 \\
0 & -2 & 2 & 4 \\
0 & -4 & 4 & 8 \\
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 2 & -1 & 1 \\
0 & 1 & -1 & -2 \\
0 & -4 & 4 & 8 \\
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 2 & -1 & 1 \\
0 & 1 & -1 & -2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]
From this row-echelon form, we see that $z$ is a free variable. Set $z = t$. Then from the middle row of the matrix, $y = t - 2$, and from the top row, $x + 2(t - 2) - t = 1$ or $x = -t + 5$. So the solution set is

$$\{(t + 5, t - 2, t) \in \mathbb{R} : t \in \mathbb{R}\} = \{(5, -2, 0) + t(-1, 1, 1) : t \in \mathbb{R}\}.$$ 

| $\mathbf{1.}$ | $A_{12}(-1), A_{13}(-4)$ | $\mathbf{2.}$ | $M_2(-1/2)$ | $\mathbf{3.}$ | $A_{23}(4)$ |

21. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\begin{bmatrix} 3 & 0 & -1 & 2 & -1 & 1 \\ 1 & 3 & 1 & -3 & 2 & -1 \\ 4 & -2 & -3 & 6 & -1 & 5 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & -27 & -12 & 33 & -21 & 12 \\ 0 & 28 & 14 & -36 & 18 & -18 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{bmatrix}.$$ 

We see that $x_5 = t$ is the only free variable. Back substitution yields the remaining values:

$$x_5 = t, \quad x_4 = -4t - 2, \quad x_3 = -\frac{41}{7} - \frac{15}{7}t, \quad x_2 = -\frac{2}{7} - \frac{33}{7}t, \quad x_1 = -\frac{2}{7} + \frac{16}{7}t.$$ 

So the solution set is

$$\left\{ \left( -\frac{2}{7} + \frac{16}{7}t, -\frac{2}{7} - \frac{33}{7}t, -\frac{41}{7} - \frac{15}{7}t, -4t - 2, t \right) : t \in \mathbb{R} \right\} = \left\{ t \left( -\frac{2}{7} + \frac{16}{7}, -\frac{33}{7}, -\frac{41}{7}, -4, 1 \right) + \left( -\frac{2}{7}, -\frac{2}{7}, -\frac{41}{7}, -2, 0 \right) : t \in \mathbb{R} \right\}.$$ 

| $\mathbf{1.}$ | $P_{12}$ | $\mathbf{2.}$ | $A_{12}(-3), A_{13}(-4)$ | $\mathbf{3.}$ | $M_2(3), M_3(-2)$ | $\mathbf{4.}$ | $A_{23}(1)$ | $\mathbf{5.}$ | $P_{23}$ | $\mathbf{6.}$ | $A_{23}(27)$ | $\mathbf{7.}$ | $M_3(1/42)$ |

23. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\begin{bmatrix} 1 & -3 & 2i & 1 \\ -2i & 6 & 2 & -2 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 1 & -3 & 2i & 1 \\ 0 & 6 - 6i & -2 & -2 + 2i \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 1 & -3 & 2i & 1 \\ 0 & 0 & 1 - \frac{2i}{3}(1 + i) & -\frac{1}{3} \end{bmatrix}.$$ 

| $\mathbf{1.}$ | $A_{12}(2i)$ | $\mathbf{2.}$ | $M_2\left(\frac{1}{6-6i}\right)$ |
From the last augmented matrix above, we see that \( x_3 \) is a free variable. Let us set \( x_3 = t \), where \( t \) is a complex number. Then we can solve for \( x_2 \) using the equation corresponding to the second row of the row-echelon form: \( x_2 = -\frac{1}{3} + \frac{1}{6}(1+i)t \). Finally, using the first row of the row-echelon form, we can determine that \( x_1 = \frac{1}{2}t(1-3i) \). Therefore, the solution set for this linear system of equations is
\[
\{(\frac{1}{2}t(1-3i), -\frac{1}{3} + \frac{1}{6}(1+i)t, t) : t \in \mathbb{C}\}.
\]

25. First observe that if \( k = 0 \), then the second equation requires that \( x_3 = 2 \), and then the first equation requires \( x_2 = 2 \). However, \( x_1 \) is a free variable in this case, so there are infinitely many solutions.

Now suppose that \( k \neq 0 \). Then multiplying each row of the corresponding augmented matrix for the linear system by \( \frac{1}{k} \) yields a row-echelon form with pivots in the first two columns only. Therefore, the third variable, \( x_3 \), is free in this case. So once again, there are infinitely many solutions to the system.

We conclude that the system has infinitely many solutions for all values of \( k \).

27. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us
\[
\begin{bmatrix}
1 & -k & k^2 & 0 \\
1 & 0 & k & 0 \\
0 & 1 & -1 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & -k & k^2 & 0 \\
0 & k & k-k^2 & 0 \\
0 & 1 & -1 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & -k & k^2 & 0 \\
0 & 1 & -1 & k-k^2 & 0 \\
0 & 0 & 1 & 0 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & -k & k^2 & 0 \\
0 & 1 & -1 & k-k^2 & 0 \\
0 & 0 & 2k-k^2 & -k & 0
\end{bmatrix}.
\]

Now provided that \( 2k-k^2 \neq 0 \), the system can be solved without free variables via back-substitution, and therefore, there is a unique solution. Consider now what happens if \( 2k-k^2 = 0 \). If \( k = 0 \), then only the first two columns of the last augmented matrix above are pivoted, and we have a free variable corresponding to \( x_3 \). Therefore, there are infinitely many solutions in this case. On the other hand, if \( k = 2 \), then the last row of the last matrix above reflects an inconsistency in the linear system, and there are no solutions.

To summarize, the system has no solutions if \( k = 2 \), a unique solution if \( k \neq 0 \) and \( k \neq 2 \), and infinitely many solutions if \( k = 0 \).

29.
(a): We have
\[
\begin{bmatrix}
4 & 7 \\
-2 & 5
\end{bmatrix} \sim \begin{bmatrix}
1 & 7/4 \\
-2 & 5
\end{bmatrix} \sim \begin{bmatrix}
1 & 7/4 \\
0 & 17/2
\end{bmatrix} \sim \begin{bmatrix}
1 & 7/4 \\
0 & 1
\end{bmatrix}.
\]

(b): We have: \( \text{rank}(A) = 2 \), since the row-echelon form of \( A \) in (a) consists two nonzero rows.

(c): We have
\[
\begin{bmatrix}
4 & 7 & 1 & 0 \\
-2 & 5 & 0 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & 7/4 & 1/4 & 0 \\
-2 & 5 & 0 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & 7/4 & 1/4 & 0 \\
0 & 17/2 & 1/2 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & 7/4 & 1/4 & 0 \\
0 & 1 & 1/17 & 2/17
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 5/34 & -7/34 \\
0 & 1 & 1/17 & 2/17
\end{bmatrix}.
\]
Thus,

\[ A^{-1} = \begin{bmatrix} \frac{5}{17} & -\frac{7}{17} \\ \frac{5}{17} & \frac{7}{17} \end{bmatrix}. \]

31.
(a): We have

\[
\begin{bmatrix}
3 & -1 & 6 \\
0 & 2 & 3 \\
3 & -5 & 0
\end{bmatrix} \sim \\
\begin{bmatrix}
1 & -1/3 & 2 \\
0 & 2 & 3 \\
1 & -5/3 & 0
\end{bmatrix} \sim \\
\begin{bmatrix}
1 & -1/3 & 2 \\
0 & 2 & 3 \\
0 & -4/3 & -2
\end{bmatrix} \sim \\
\begin{bmatrix}
1 & -1/3 & 2 \\
0 & 2 & 3 \\
0 & 0 & 0
\end{bmatrix} \sim \\
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 3/2 \\
0 & 0 & 0
\end{bmatrix}.
\]

(b): We have: \( \text{rank}(A) = 2 \), since the row-echelon form of \( A \) in (a) consists of two nonzero rows.
(c): Since \( \text{rank}(A) < 3 \), \( A \) is not invertible.

33.
(a): We have

\[
\begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & -1 \\
1 & -1 & 2
\end{bmatrix} \sim \\
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & -1 \\
1 & -1 & 2
\end{bmatrix} \sim \\
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & -1 \\
0 & 0 & 1
\end{bmatrix} \sim \\
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \sim \\
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

(b): We have: \( \text{rank}(A) = 3 \), since the row-echelon form of \( A \) in (a) has 3 nonzero rows.
(c): We have
Hence,

\[
A^{-1} = \begin{bmatrix}
\frac{1}{3} & 0 & 0 \\
-\frac{1}{9} & \frac{2}{3} & \frac{1}{3} \\
-\frac{2}{9} & \frac{1}{3} & \frac{2}{3}
\end{bmatrix}.
\]

35. We use the Gauss-Jordan method to find \(A^{-1}\):

\[
\begin{bmatrix}
1 & -1 & 3 & 1 & 0 & 0 \\
4 & -3 & 13 & 0 & 1 & 0 \\
1 & 1 & 4 & 0 & 0 & 1
\end{bmatrix}
\xrightarrow{\frac{1}{4}}
\begin{bmatrix}
1 & -1 & 3 & 1 & 0 & 0 \\
0 & 1 & 1 & -4 & 1 & 0 \\
0 & 2 & 1 & -1 & 0 & 1
\end{bmatrix}
\xrightarrow{\frac{1}{2}}
\begin{bmatrix}
1 & -1 & 3 & 1 & 0 & 0 \\
0 & 1 & 1 & -4 & 1 & 0 \\
0 & 0 & -1 & 7 & -2 & 1
\end{bmatrix}
\]

Thus,

\[
A^{-1} = \begin{bmatrix}
25 & -7 & 4 \\
3 & -1 & 1 \\
-7 & 2 & -1
\end{bmatrix}.
\]

Now \(x_i = A^{-1}e_i\) for each \(i\). So

\[
x_1 = A^{-1}e_1 = \begin{bmatrix} 25 \\ 3 \\ -7 \end{bmatrix}, \quad x_2 = A^{-1}e_2 = \begin{bmatrix} -7 \\ -1 \\ 2 \end{bmatrix}, \quad x_3 = A^{-1}e_3 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}.
\]

37.

(a): We have

\[(A^{-1}B)(B^{-1}A) = A^{-1}(BB^{-1})A = A^{-1}I_nA = A^{-1}A = I_n\]

and

\[(B^{-1}A)(A^{-1}B) = B^{-1}(AA^{-1})B = B^{-1}I_nB = B^{-1}B = I_n.\]

Therefore,

\[(B^{-1}A)^{-1} = A^{-1}B.\]

(b): We have

\[(A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A,\]

as required.

39.

(a): We reduce \(A\) to the identity matrix:

\[
\begin{bmatrix}
4 & 7 \\
-2 & 5
\end{bmatrix}
\xrightarrow{\frac{1}{4}}
\begin{bmatrix}
1 & \frac{7}{4} \\
-2 & \frac{5}{2}
\end{bmatrix}
\xrightarrow{\frac{1}{2}}
\begin{bmatrix}
1 & \frac{7}{4} \\
0 & \frac{1}{2}
\end{bmatrix}
\xrightarrow{\frac{1}{2}}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

Thus, \(A_1, A_2, A_3, A_4, A_5\) are reduced to the identity matrix.
The elementary matrices corresponding to these row operations are

\[
E_1 = \begin{bmatrix}
\frac{1}{3} & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad E_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 1 & 0
\end{bmatrix}, \quad E_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & -\frac{7}{2} & 1 \\
0 & 1 & 0
\end{bmatrix}, \quad E_4 = \begin{bmatrix}
1 & 0 & -\frac{7}{4} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

We have \(E_4 E_3 E_2 E_1 A = I_2\), so that

\[
A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} = \begin{bmatrix}
4 & 0 & 0 \\
0 & 1 & 0 \\
-2 & 1 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2 & 1 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2 & 1 & 1
\end{bmatrix},
\]

which is the desired expression since \(E_i^{-1}\) is an elementary matrix for each \(i\).

(b): We can reduce \(A\) to upper triangular form by the following elementary row operation:

\[
\begin{bmatrix}
4 & 7 \\
-2 & 5
\end{bmatrix} \sim \begin{bmatrix}
4 & 7 \\
0 & 17
\end{bmatrix}.
\]

Therefore we have the multiplier \(m_{12} = -\frac{1}{2}\). Hence, setting

\[
L = \begin{bmatrix}
1 & 0 \\
-\frac{1}{2} & 1
\end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix}
4 & 7 \\
0 & 17
\end{bmatrix},
\]

we have the LU factorization \(A = LU\), which can be easily verified by direct multiplication.

(a): We reduce \(A\) to the identity matrix:

\[
\begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & -1 \\
1 & -1 & 2
\end{bmatrix} \sim \begin{bmatrix}
1 & -1 & 2 \\
0 & 2 & -1 \\
3 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & -1 & 2 \\
0 & 2 & -1 \\
0 & 3 & -6
\end{bmatrix} \sim \begin{bmatrix}
1 & -1 & 2 \\
0 & 1 & -\frac{1}{2} \\
0 & 3 & -6
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

The elementary matrices corresponding to these row operations are

\[
E_1 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad E_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-3 & 0 & 1
\end{bmatrix}, \quad E_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad E_4 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -3 & 1
\end{bmatrix}
\]

\[
E_5 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -\frac{7}{9}
\end{bmatrix}, \quad E_6 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad E_7 = \begin{bmatrix}
1 & 0 & -\frac{3}{2} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad E_8 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & \frac{1}{2} \\
0 & 0 & 1
\end{bmatrix}.
\]

We have

\[
E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I_3
\]
so that
\[ A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}E_8^{-1} \]
\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
3 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix} \ldots
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -\frac{3}{2}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
3 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

which is the desired expression since \( E_i^{-1} \) is an elementary matrix for each \( i \).

(b): We can reduce \( A \) to upper triangular form by the following elementary row operations:
\[
\begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & -1 \\
1 & -1 & 2
\end{bmatrix} \sim \begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & -1 \\
0 & 0 & 3
\end{bmatrix}
\]

Therefore, the nonzero multipliers are \( m_{13} = \frac{1}{3} \) and \( m_{23} = -\frac{1}{2} \). Hence, setting
\[
L = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{3} & 1 & 1
\end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix}
3 & 0 & 0 \\
0 & 2 & -1 \\
0 & 0 & \frac{3}{2}
\end{bmatrix}
\]

we have the LU factorization \( A = LU \), which can be verified by direct multiplication.

43.
(a): Note that
\[
\]

(b): We have
\[
\]

(c): The answer is \( 2^k \), because each term in the expansion of \( (A+B)^k \) consists of a string of \( k \) matrices, each of which is either \( A \) or \( B \) (2 possibilities for each matrix in the string). Multiplying the possibilities for each position in the string of length \( k \), we get \( 2^k \) different strings, and hence \( 2^k \) different terms in the expansion of \( (A+B)^k \).

45. For a \( 2 \times 4 \) matrix, the leading ones can occur in 6 different positions:
\[
\begin{bmatrix}
1 & * & * & * \\
0 & 1 & * & *
\end{bmatrix}, \begin{bmatrix}
1 & * & * & * \\
0 & 0 & 1 & *
\end{bmatrix}, \begin{bmatrix}
1 & * & * & * \\
0 & 0 & 0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 1 & * & * \\
0 & 0 & 1 & *
\end{bmatrix}, \begin{bmatrix}
0 & 1 & * & * \\
0 & 0 & 0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 1 & * \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

For a \( 3 \times 4 \) matrix, the leading ones can occur in 4 different positions:
\[
\begin{bmatrix}
1 & * & * & * \\
0 & 1 & * & *
\end{bmatrix}, \begin{bmatrix}
1 & * & * & * \\
0 & 0 & 1 & *
\end{bmatrix}, \begin{bmatrix}
1 & * & * & * \\
0 & 0 & 0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 1 & * & * \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
For a $4 \times 6$ matrix, the leading ones can occur in 15 different positions:

\[
\begin{bmatrix}
1 & * & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 1 & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 1 & * & *
\end{bmatrix},
\begin{bmatrix}
1 & * & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 1 & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 1 & * & *
\end{bmatrix},
\begin{bmatrix}
1 & * & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 1 & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 1 & * & *
\end{bmatrix},
\begin{bmatrix}
1 & * & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 1 & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 1 & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 1 & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & * & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 1 & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 1 & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & * & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 1 & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 1 & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 1 & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & * & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 1 & * & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & * & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 1 & * & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & *
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

For an $m \times n$ matrix with $m \leq n$, the answer is the binomial coefficient

\[ C(n, m) = \binom{n}{m} = \frac{n!}{m!(n-m)!}. \]

This represents $n$ “choose” $m$, which is the number of ways to choose $m$ columns from the $n$ columns of the matrix in which to put the leading ones. This choice then determines the structure of the matrix.

**Solutions to Section 3.1**

**True-False Review:**

1. **TRUE.** Let $A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$. Then

\[ \det(A) = ac - b0 = ac, \]

which is the product of the elements on the main diagonal of $A$.

3. **FALSE.** The volume of this parallelepiped is determined by the absolute value of $\det(A)$, since $\det(A)$ could very well be negative.

5. **FALSE.** Many examples are possible here. If we take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $\det(A) = \det(B) = 0$, but $A + B = I_2$, and $\det(I_2) = 1 \neq 0$.

7. **TRUE.** In the summation that defines the determinant, each term in the sum is a product consisting of one element from each row and each column of the matrix. But that means one of the factors in each term will be zero, since it must come from the row containing all zeros. Hence, each term is zero, and the summation is zero.

**Problems:**
1. \( \sigma(2, 1, 3, 4) = (-1)^{N(2,1,3,4)} = (-1)^1 = -1 \), odd.

2. \( \sigma(1, 4, 3, 5, 2) = (-1)^{N(1,4,3,5,2)} = (-1)^4 = 1 \), even.

3. \( \sigma(1, 5, 2, 4, 3) = (-1)^{N(1,5,2,4,3)} = (-1)^4 = 1 \), even.

4. \( \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sigma(1, 2)a_{11}a_{22} + \sigma(2, 1)a_{12}a_{21} = a_{11}a_{22} - a_{12}a_{21}. \)

5. \( \det(A) = \begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix} = 2(-3) - (-1)6 = 0. \)

6. \( \det(A) = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 6 \\ 0 & 2 & -1 \end{vmatrix} = 1 \cdot 3(-1) + (-1)6 \cdot 0 + 0 \cdot 2 \cdot 2 - 0 \cdot 3 \cdot 0 - 6 \cdot 2 \cdot 1 - (-1)(-1)2 = -17. \)

7. \( \det(A) = \begin{vmatrix} 0 & 0 & 2 \\ 0 & -4 & 1 \\ -1 & 5 & -7 \end{vmatrix} = 0(-4)(-7) + 0 \cdot 1(-1) + 2 \cdot 0 \cdot 5 - 2(-4)(-1) - 1 \cdot 5 \cdot 0 - (-7)0 \cdot 0 = -8. \)

8. \( \det(A) = \begin{vmatrix} 0 & 0 & 2 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \end{vmatrix} = -60, \) since of the 24 terms in the expression (3.1.3) for the determinant, only the term \( \sigma(3, 1, 4, 2)a_{13}a_{21}a_{34}a_{42} \) contains all nonzero entries, and since \( \sigma(3, 1, 4, 2) = -1 \), we obtain \( \sigma(3, 1, 4, 2)a_{13}a_{21}a_{34}a_{42} = (-1) \cdot 2 \cdot 5 \cdot 3 \cdot 2 = -60. \)

9. \( \begin{vmatrix} 2 & 3 & -1 \\ 1 & 4 & 1 \\ 3 & 1 & 6 \end{vmatrix} = (2)(4)(6) + (3)(1)(3) + (-1)(1)(1) - (3)(4)(-1) - (1)(1)(2) - (6)(1)(3) = 48. \)

10. \( \begin{vmatrix} 2 & 3 & 6 \\ 0 & 1 & 2 \\ 1 & 5 & 0 \end{vmatrix} = (2)(1)(0) + (3)(2)(1) + (6)(0)(5) - (1)(1)(6) - (5)(2)(2) - (0)(0)(3) = -20. \)

11. \( \begin{vmatrix} e^{2t} & e^{3t} & e^{-4t} \\ 2e^{2t} & 3e^{3t} & -4e^{-4t} \\ 4e^{2t} & 9e^{3t} & 16e^{-4t} \end{vmatrix} = (e^{2t})(3e^{3t})(16e^{-4t}) + (e^{3t})(-4e^{-4t})(4e^{2t}) + (e^{-4t})(2e^{2t})(9e^{3t}) - (4e^{2t})(3e^{3t})(e^{-4t}) - (9e^{3t})(-4e^{-4t})(e^{2t}) - (16e^{-4t})(2e^{2t})(e^{3t}) = 42e^t. \)

12. \( \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x & e^x \\ -2 \sin 2x & 2 \cos 2x & e^x \\ -4 \cos 2x & -4 \sin 2x & e^x \end{vmatrix} = 2e^x \cos^2 2x - 4e^x \sin 2x \cos 2x + 8e^x \sin^2 2x + 8e^x \cos^2 2x + 4e^x \sin 2x \cos 2x + 2e^x \sin^2 2x = 10e^x. \)

13. \( y'''_1 - y''_1 - y'_1 + y_1 = e^x - e^x - e^x + e^x = 0, \)

14. \( y'''_2 - y''_2 - y'_2 + y_2 = \sinh x - \cosh x - \sinh x + \cosh x = 0, \)

15. \( y'''_3 - y''_3 - y'_3 + y_3 = \cosh x - \sinh x - \cosh x + \sinh x = 0. \)

16. \( \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^x \cosh x & \sinh x \\ e^x \sinh x & \cosh x \\ e^x \cosh x & \sinh x \end{vmatrix} \)
= e^x \sinh^2 x + e^x \cosh^2 x + e^x \sinh x \cosh x - e^x \sinh^2 x - e^x \cosh^2 x - e^x \sinh x \cosh x = 0.

(b): The formulas we need are

\[
\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}.
\]

Adding the two equations, we find that \( \cosh x + \sinh x = e^x \), so that \( -e^x + \cosh x + \sinh x = 0 \). Therefore, we may take \( d_1 = -1, d_2 = 1, \) and \( d_3 = 1. \)

25.

\[
det(A) = \begin{vmatrix} 1 & -1 & 0 & 1 \\ 3 & 0 & 2 & 5 \\ 2 & 1 & 0 & 3 \\ 9 & -1 & 2 & 1 \end{vmatrix}
\]

\[
= 1 \cdot 0 \cdot 0 \cdot 1 - 1 \cdot 0 \cdot 2 \cdot 3 - 1 \cdot 1 \cdot 2 \cdot 5 + 1(-1)2 \cdot 3 - 1(-1)0 \cdot 5
\]

\[-3(-1)0 \cdot 1 + 3(-1)2 \cdot 3 + 3 \cdot 1 \cdot 0 \cdot 1 - 3 \cdot 1 \cdot 2 \cdot 1 - 3(-1)0 \cdot 5 + 1(-1)0 \cdot 1
\]

\[+2(-1)2 \cdot 1 - 2(-1)2 \cdot 5 - 2 \cdot 0 \cdot 0 \cdot 1 + 2 \cdot 0 \cdot 2 \cdot 1 + 2(-1)0 \cdot 5 - 2(-1)2 \cdot 1
\]

\[-9(-1)2 \cdot 3 + 9(-1)0 \cdot 5 + 9 \cdot 0 \cdot 0 \cdot 3 - 9 \cdot 0 \cdot 0 \cdot 1 - 9 \cdot 1 \cdot 0 \cdot 5 + 9 \cdot 1 \cdot 2 \cdot 1
\]

\[= 70.
\]

27.

\[
det(A) = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 3 & 4 \\ 3 & 4 & 0 & 5 \\ 4 & 5 & 6 & 0 \end{vmatrix}
\]

\[
= 0 \cdot 0 \cdot 0 \cdot 0 - 0 \cdot 0 \cdot 6 \cdot 5 - 0 \cdot 4 \cdot 3 \cdot 0 + 0 \cdot 4 \cdot 6 \cdot 4 + 0 \cdot 5 \cdot 3 \cdot 5 - 0 \cdot 5 \cdot 0 \cdot 4
\]

\[-2 \cdot 1 \cdot 0 \cdot 0 + 2 \cdot 1 \cdot 6 \cdot 5 + 2 \cdot 4 \cdot 2 \cdot 0 - 2 \cdot 4 \cdot 6 \cdot 3 - 2 \cdot 5 \cdot 2 \cdot 5 + 2 \cdot 5 \cdot 0 \cdot 3
\]

\[+3 \cdot 1 \cdot 3 \cdot 0 - 3 \cdot 1 \cdot 6 \cdot 4 - 3 \cdot 0 \cdot 2 \cdot 0 + 3 \cdot 0 \cdot 6 \cdot 3 + 3 \cdot 5 \cdot 2 \cdot 4 - 3 \cdot 5 \cdot 3 \cdot 3
\]

\[-4 \cdot 1 \cdot 3 \cdot 5 + 4 \cdot 1 \cdot 0 \cdot 4 + 4 \cdot 0 \cdot 2 \cdot 5 - 4 \cdot 0 \cdot 0 \cdot 3 - 4 \cdot 4 \cdot 2 \cdot 4 + 4 \cdot 4 \cdot 3 \cdot 3
\]

\[= -315.
\]

29.

(a):

\[
det(cA) = \begin{vmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{vmatrix} = (ca_{11})(ca_{22}) - (ca_{12})(ca_{21})
\]

\[= c^2a_{11}a_{22} - c^2a_{12}a_{21} = c^2(a_{11}a_{22} - a_{12}a_{21}) = c^2 \det(A).
\]

(b):

\[
det(cA) = \sum \sigma(p_1,p_2,p_3,\ldots,p_n)ca_{1p_1}ca_{2p_2}ca_{3p_3}\cdots ca_{np_n}
\]

\[= c^n \sum \sigma(p_1,p_2,p_3,\ldots,p_n)a_{1p_1}a_{2p_2}a_{3p_3}\cdots a_{np_n}
\]

\[= c^n \det(A),
\]

where each summation above runs over all permutations \( \sigma \) of \( \{1,2,3,\ldots,n\} \).
31. $a_{11}a_{23}a_{34}a_{43}a_{52}$. This is not a possible term of an order 5 determinant, since the column indices are not distinct.

33. $a_{11}a_{23}a_{42}a_{34}a_{53}$. This is a possible term of an order 5 determinant.

35. $a_{p2}a_{q3}a_{34}a_{43}$. We must choose $p = 1$ and $q = 4$.

37. $a_{pq}a_{34}a_{13}a_{42}$. We must choose $p = 2$ and $q = 1$.

39. From the given term, we have

$$N(n, n-1, n-2, \ldots, 1) = 1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2},$$

because the series of $(n-1)$ terms is just an arithmetic series which has a first term of one, common difference of one, and last term $(n-1)$. Thus, $\sigma(n, n-1, n-2, \ldots, 1) = (-1)^{n(n-1)/2}$.

**Solutions to Section 3.2**

True-False Review:

1. FALSE. The determinant of the matrix will increase by a factor of $2^n$. For instance, if $A = I_2$, then $\det(A) = 1$. However, $\det(2A) = \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4$, so the determinant in this case increases by a factor of four.

3. TRUE. This follows by repeated application of Property (P8):

$$\det(A^n) = \det(AAAAA) = (\det(A))(\det(A))(\det(A))(\det(A))(\det(A)) = (\det(A))^5.$$  

5. FALSE. The matrix is not invertible if and only if its determinant, $x^2y - xy^2 = xy(x - y)$, is zero. For example, if $x = y = 1$, the matrix is not invertible since $x = y$; however, neither $x$ nor $y$ is zero in this case. We conclude that the statement is false.

Problems:

From this point on, $P2$ will denote an application of the property of determinants that states that if every element in any row (column) of a matrix $A$ is multiplied by a number $c$, then the determinant of the resulting matrix is $c \det(A)$.

1. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & -5 & 2 \end{vmatrix}$  2. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 0 & -11 & -7 \end{vmatrix}$  3. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 4 & 1 & 12 \end{vmatrix}$

1. $A_{12}(-2)$, $A_{13}(-3)$  2. $P2$  3. $A_{23}(11)$

3. $\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 6 \\ -1 & 2 & 6 \end{vmatrix}$  $\begin{vmatrix} 2 & 1 & 3 \\ 0 & 5 & 15 \\ 0 & 9 & 36 \end{vmatrix}$  $\begin{vmatrix} -1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{vmatrix}$
\[ \begin{pmatrix} 4 & -1 & 2 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = (-45)(-1) = 45. \]

1. \( P_{12} \)  2. \( \text{A}_{12}(2), \text{A}_{13}(4) \)  3. \( P_{2} \)  4. \( \text{A}_{23}(-1) \)

\[
\begin{pmatrix} 3 & 7 & 1 \\ 5 & 9 & -6 \\ 2 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 6 & -2 \\ 5 & 9 & -6 \\ 2 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 6 & -2 \\ 0 & -21 & 4 \\ 0 & -11 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 6 & -2 \\ 0 & 1 & -10 \\ 0 & -11 & 7 \end{pmatrix}
\]

\[ \frac{4}{\begin{pmatrix} 1 & 6 & -2 \\ 0 & 1 & -10 \\ 0 & 0 & -103 \end{pmatrix} = -103.} \]

1. \( \text{A}_{31}(-1) \)  2. \( \text{A}_{12}(-5), \text{A}_{13}(-2) \)  3. \( \text{A}_{32}(-2) \)  4. \( \text{A}_{24}(11) \)

7. Note that in the first step below, we extract a factor of 13 from the second row of the matrix, and we also extract a factor of 8 from the second column of the matrix.

\[
\begin{pmatrix} 2 & 32 & 1 & 4 \\ 26 & 104 & 26 & -13 \\ 2 & 56 & 2 & 7 \\ 1 & 40 & 1 & 5 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \cdot 8 \\ 1 & 5 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -6 & -1 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 & 5 \\ 2 & 1 & 2 & -1 \\ 2 & 7 & 2 & 7 \\ 2 & 4 & 1 & 4 \end{pmatrix}
\]

\[ \frac{3}{-104} \begin{pmatrix} 1 & 5 & 1 \\ 0 & -9 & 0 & -11 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 1 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 & 5 \\ 2 & 1 & 2 & -1 \\ 2 & 7 & 2 & 7 \\ 2 & 4 & 1 & 4 \end{pmatrix}
\]

\[ \frac{5}{312} \begin{pmatrix} 1 & 5 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

\[ = 312(-1)(-2) = 624. \]

1. \( P_{2} \)  2. \( P_{14} \)  3. \( \text{A}_{12}(-2), \text{A}_{13}(-2), \text{A}_{14}(-2) \)  4. \( P_{2}, P_{23} \)  5. \( \text{A}_{23}(9), \text{A}_{24}(6) \)  6. \( P_{34} \)

9.

\[
\begin{pmatrix} 2 & 1 & 3 & 5 \\ 3 & 0 & 1 & 2 \\ 4 & 1 & 4 & 3 \\ 5 & 2 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 3 & 1 & 2 \\ 1 & 4 & 4 & 3 \\ 2 & 5 & 5 & 3 \end{pmatrix}
\]

\[ \frac{3}{-1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & -7 \\ 0 & 2 & 1 & -2 \\ 0 & 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & -7 \\ 0 & 2 & 1 & -2 \\ 0 & 3 & 1 & 2 \end{pmatrix}
\]

\[ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & -7 \\ 0 & 2 & 1 & -2 \\ 0 & 3 & 1 & 2 \end{pmatrix}
\]
\[ \frac{5}{3} = 3 \quad \frac{6}{3} = 2 \]

\[ 1 \quad 2 \quad 3 \quad 4 \quad \begin{array}{cccc}
0 & 1 & -1 & -7 \\
0 & 0 & 1 & 4 \\
0 & 0 & 4 & 23 \\
\end{array} = 3 \cdot 1 \cdot 1 \cdot 1 \cdot 7 = 21. \]

1. CP
2. \(A_{13}(-1), A_{14}(-2)\)
3. P
4. \(A_{23}(-2), A_{24}(-3)\)
5. P
6. \(A_{34}(-4)\)

11.
\[
\begin{array}{cccc}
7 & -1 & 3 & 4 \\
14 & 2 & 4 & 6 \\
21 & 1 & 3 & 4 \\
-7 & 4 & 5 & 8 \\
\end{array} = 1 \cdot 7 \cdot 2 \\
\begin{array}{cccc}
1 & -1 & 3 & 2 \\
0 & 4 & -2 & -1 \\
0 & 0 & -4 & -3 \\
0 & 3 & 8 & 6 \\
\end{array} = 14 \\
\begin{array}{cccc}
1 & -1 & 3 & 2 \\
0 & 1 & -10 & -7 \\
0 & 0 & 1 & 3/4 \\
0 & 0 & 38 & 27 \\
\end{array} = 56 \cdot (-3/2) = 84. \]

13. \(\begin{vmatrix}
2 & 1 \\
3 & 2 \\
\end{vmatrix} = 1;\) invertible.

15. \(\begin{vmatrix}
3 & 5 & -1 \\
2 & 0 & 1 \\
\end{vmatrix} = 14;\) invertible.

17.
\[
\begin{array}{cccc}
1 & 0 & 2 & -1 \\
3 & -2 & 1 & 4 \\
2 & 1 & 6 & 2 \\
1 & -3 & 4 & 0 \\
\end{array} = \begin{array}{cccc}
1 & 0 & 2 & -1 \\
0 & -2 & -5 & 7 \\
0 & 1 & 2 & 4 \\
0 & -3 & 2 & 1 \\
\end{array} = 133;\) invertible.

19.
\[
\begin{array}{cccc}
1 & 2 & -3 & 5 \\
-1 & 2 & -3 & 6 \\
2 & 3 & -1 & 4 \\
1 & -2 & 3 & -6 \\
\end{array} = \begin{array}{cccc}
1 & 2 & -3 & 5 \\
1 & -2 & 3 & -6 \\
2 & 3 & -1 & 4 \\
1 & -2 & 3 & -6 \\
\end{array} = 0;\) not invertible.

21. \(\det(A) = \begin{vmatrix}
2 & -k & 1 \\
3 & 6 & 1 \\
\end{vmatrix} = (3k-1)(k+4).\) Consequently, the system has an infinite number of solutions if and only if \(k = -4\) or \(k = 1/3\) (Corollary 3.2.5).
23. \[ \det(A) = \begin{vmatrix} 1 & k & 0 \\ k & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1 + k - 1 - k^2 = k(1 - k). \] Consequently, the system has a unique solution if and only if \( k \neq 0, 1. \)

25. \[ A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}. \] \( \det(A) \det(B) = |3 \cdot 1 - (-1)2| \cdot |1 \cdot 4 - 2(-2)| = 5 \cdot 8 = 40. \)

\[ \det(AB) = \begin{vmatrix} 3 & -2 \\ -4 & 16 \end{vmatrix} = 3 \cdot 16 - (-2)(-4) = 40. \] Hence, \( \det(AB) = \det(A) \det(B). \)

27. \[ \begin{vmatrix} 3 & 2 & 1 \\ 6 & 4 & -1 \\ 9 & 6 & 2 \end{vmatrix} = \frac{1}{3} \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \\ 3 & 6 & 2 \end{vmatrix} = \frac{1}{3} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 3 & 3 & 2 \end{vmatrix} = 0. \] \( 1. \ P2 \quad 2. \ P2 \quad 3. \ P7 \)

29. \[ \begin{vmatrix} 1 + 3a & 1 & 3 \\ 1 + 2a & 1 & 2 \\ 2 & 2 & 0 \end{vmatrix} = \frac{1}{3} \begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 3a & 1 & 3 \\ 2a & 1 & 2 \\ 0 & 2 & 0 \end{vmatrix} = 0 + a \begin{vmatrix} 3 & 1 & 3 \\ 2 & 1 & 2 \\ 0 & 2 & 0 \end{vmatrix} = 3 \cdot a - 0 = 0. \] \( 1. \ P5 \quad 2. \ P2, P7 \quad 3. \ P7 \)

31. \( B \) is obtained from \( A^T \) by the following elementary row operations: (1) \( A_{13}(3), \) (2) \( M_1(-2). \) Since \( \det(A^T) = \det(A) = 1, \) and (1) leaves the determinant unchanged, we have \( \det(B) = -2. \)

33. \( B \) is obtained from \( A \) by the following row operations: (1) \( A_{13}(5), \) (2) \( M_2(-4), \) (3) \( P_{12}, \) (4) \( P_{23}. \) Thus, \( \det(B) = \det(A) \cdot (-4) \cdot (-1) \cdot (-1) = (-6) \cdot (-4) = 24. \)

35. \( B \) is obtained from \( A^T \) by the following row operations: (1) \( M_1(2), \) (2) \( A_{32}(-1), \) (3) \( A_{13}(-1). \) Thus, \( \det(B) = \det(A^T) \cdot 2 = (-6) \cdot (2) = 12. \)

37. We have \( \det(A^2B^5) = (\det(A))^2(\det(B))^5 = 5^2 \cdot 3^5 = 6075. \)

39. We have

\[ \det((2B)^{-1}(AB)^T) = (\det((2B)^{-1}))(\det(AB)^T) = \left( \frac{\det(A)\det(B)}{\det(2B)} \right) = \left( \frac{5 \cdot 3}{3 \cdot 2^4} \right) = \frac{5}{16}. \]

41.

(a): The volume of the parallelepiped is given by \( |\det(A)|. \) In this case, we have

\[ |\det(A)| = |2 + 12k + 36 - 4k - 18 - 12| = |8 + 8k|. \]

(b): NO. The answer does not change because the determinants of \( A \) and \( A^T \) are the same, so the volume determined by the columns of \( A, \) which is the same as the volume determined by the rows of \( A^T, \) is \( |\det(A^T)| = |\det(A)|. \)
$$\begin{vmatrix} \alpha x - \beta y & \beta x - \alpha y \\ \beta x + \alpha y & \alpha x + \beta y \end{vmatrix} = \begin{vmatrix} \alpha x & \beta x - \alpha y \\ \beta x & \alpha x + \beta y \end{vmatrix} + \begin{vmatrix} -\beta y & \beta x - \alpha y \\ \alpha y & \alpha x + \beta y \end{vmatrix}$$

$$= x^2 \begin{vmatrix} \alpha & \beta \\ \beta & \alpha \end{vmatrix} + xy \begin{vmatrix} \alpha & -\alpha \\ \beta & \alpha \end{vmatrix} - xy \begin{vmatrix} \beta & \beta \\ \alpha & \beta \end{vmatrix} + y^2 \begin{vmatrix} \alpha & \beta \\ \beta & \alpha \end{vmatrix}$$

$$= (x^2 + y^2) \begin{vmatrix} \alpha & \beta \\ \beta & \alpha \end{vmatrix} + xy \begin{vmatrix} \alpha & -\alpha \\ \beta & \beta \end{vmatrix} - xy \begin{vmatrix} \beta & \beta \\ \alpha & \beta \end{vmatrix}$$

$$= (x^2 + y^2) \begin{vmatrix} \alpha & \beta \\ \beta & \alpha \end{vmatrix} + xy \begin{vmatrix} \alpha & -\alpha \\ \beta & \beta \end{vmatrix} - xy \begin{vmatrix} \beta & \beta \\ \alpha & \beta \end{vmatrix}$$

$$= (x^2 + y^2) \begin{vmatrix} \alpha & \beta \\ \beta & \alpha \end{vmatrix} .$$

45. Suppose $A$ is a matrix with a row of zeros. We will use (P3) and (P7) to justify the fact that $\det(A) = 0$. If $A$ has more than one row of zeros, then by (P7), since two rows of $A$ are the same, $\det(A) = 0$. Assume instead that only one row of $A$ consists entirely of zeros. Adding a nonzero row of $A$ to the row of zeros yields a new matrix $B$ with two equal rows. Thus, by (P7), $\det(B) = 0$. However, $B$ was obtained from $A$ by adding a multiple of one row to another row, and by (P3), $\det(B) = \det(A)$. Hence, $\det(A) = 0$, as required.

47. (a): From the definition of determinant we have:

$$\det(A) = \sum_{n!} \sigma(p_1, p_2, p_3, \ldots, p_n)a_{1p_1}a_{2p_2}a_{3p_3}\cdots a_{np_n}. \quad (47.1)$$

If $A$ is lower triangular, then $a_{ij} = 0$ whenever $i < j$, and therefore the only nonzero terms in (47.1) are those with $p_i \leq i$ for all $i$. Since all the $p_i$ must be distinct, the only possibility is $p_i = i$ for all $i$ with $1 \leq i \leq n$, and so (47.1) reduces to the single term:

$$\det(A) = \sigma(1, 2, 3, \ldots, n)a_{11}a_{22}a_{33}\cdots a_{nn}.$$ (b):

$$\det(A) = \begin{vmatrix} 2 & -1 & 3 & 5 \\ 1 & 2 & 2 & 1 \\ 3 & 0 & 1 & 4 \\ 1 & 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -3 & -11 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & -8 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 13 & 0 & 0 \\ 2 & 16 & 0 & 0 \\ -1 & -8 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{vmatrix} = -26.$$ 1. $A_{41}(-5)$, $A_{42}(-1)$, $A_{43}(-4)$ 2. $A_{31}(-3)$, $A_{32}(-2)$ 3. $P_{12}$ 4. $A_{21}(-16/13)$

49. $\det(S^{-1}AS) = \det(S^{-1})\det(A)\det(S) = \det(S^{-1})\det(S)\det(A) = \det(S^{-1}S)\det(A) = \det(I_n)\det(A) = \det(A).$
51. Let $E$ be an elementary matrix. There are three different possibilities for $E$.

(a): $E$ permutes two rows: Then $E$ is obtained from $I_n$ by interchanging two rows of $I_n$. Since $\det(I_n) = 1$ and using Property P1, we obtain $\det(E) = -1$.

(b): $E$ adds a multiple of one row to another: Then $E$ is obtained from $I_n$ by adding a multiple of one row of $I_n$ to another. Since $\det(I_n) = 1$ and using Property P3, we obtain $\det(E) = +1$.

(c): $E$ scales a row by $k$: Then $E$ is obtained from $I_n$ by multiplying a row of $I_n$ by $k$. Since $\det(I_n) = 1$ and using Property P2, we obtain $\det(E) = k$.

53.

\[
\begin{vmatrix}
1 & x & x^2 \\
1 & y & y^2 \\
1 & z & z^2 \\
\end{vmatrix} = \frac{1}{2} \begin{vmatrix}
1 & x & x^2 \\
0 & y - x & (y - x)(y + x) \\
0 & z - x & (z + x)(z - x) \\
\end{vmatrix} = \frac{2}{3} (y - x)(z - x) \\
\begin{vmatrix}
1 & x & x^2 \\
0 & 1 & y + x \\
0 & 0 & z + x \\
\end{vmatrix} = (y - x)(z - x)(z - y) = (y - z)(z - x)(x - y).
\]

\[
\begin{array}{llll}
1 & A_{12}(-1)A_{13}(-1) & 2 & P2 \\
3 & A_{23}(-1)
\end{array}
\]

55. Solving $b = c_1a_1 + c_2a_2 + \cdots + c_na_n$ for $c_1a_1 = b - c_2a_2 - \cdots - c_na_n$.

Consequently, $\det(B_k)$ can be written as

\[
\det(B_k) = \det([a_1, a_2, \ldots, a_{k-1}, b, a_{k+1}, \ldots, a_n])
\]

\[
= \det([a_1, a_2, \ldots, a_{k-1}, (c_1a_1 + c_2a_2 + \cdots + c_na_n), a_{k+1}, \ldots, a_n])
\]

\[
= c_1\det([a_1, \ldots, a_{k-1}, a_1, a_{k+1}, \ldots, a_n]) + c_2\det([a_1, \ldots, a_{k-1}, a_2, a_{k+1}, \ldots, a_n]) + \cdots + c_n\det([a_1, \ldots, a_{k-1}, a_n, a_{k+1}, \ldots, a_n]).
\]

Now by P7, all except the $k^{th}$ determinant are zero since they have two equal columns, so that we are left with $\det(B_k) = c_k\det(A)$.

59. Using technology we find that:

\[
\begin{vmatrix}
1 - k & 4 & 1 \\
3 & 2 - k & 1 \\
3 & 4 & -1 - k \\
\end{vmatrix} = -(k - 6)(k + 2)^2.
\]

Consequently, the system has an infinite number of solutions if and only if $k = 6, -2$.

$k = 6$: In this case, the system is $Bx = 0$, where $B =$ \[
\begin{bmatrix}
-5 & 4 & 1 \\
3 & -4 & 1 \\
3 & 4 & -7 \\
\end{bmatrix}. \text{ This system has solution set } \{t(1,1,1) : t \in \mathbb{R}\}.
\]

$k = -2$: In this case, the system is $Cx = 0$, where $C =$ \[
\begin{bmatrix}
3 & 4 & 1 \\
3 & 4 & 1 \\
3 & 4 & 1 \\
\end{bmatrix}. \text{ This system has solution set } \{(r, s, -3r - 4s) : r, s \in \mathbb{R}\}.
\]

Solutions to Section 3.3
True-False Review:

1. **FALSE.** Because $2 + 3 = 5$ is odd, the $(2, 3)$-cofactor is the negative of the $(2, 3)$-minor of the matrix.

3. **TRUE.** The Cofactor Expansion Theorem allows for expansion along any row or any column of the matrix, and in all cases, the result is the determinant of the matrix.

5. **FALSE.** For example, let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 8 & 7 \\ 3 & 2 & 1 \end{bmatrix}$. Then $A + B = \begin{bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \end{bmatrix}$. The $(1, 1)$-entry of $\text{adj}(A + B)$ is therefore 0. However, the $(1, 1)$-entry of $\text{adj}(A)$ is $-3$, and the $(1, 1)$-entry of $\text{adj}(B)$ is also $-3$. But $(-3) + (-3) \neq 0$. Many other examples abound, of course.

7. **TRUE.** This can be immediately deduced by substituting $I_n$ for $A$ in Theorem 3.3.16.

Problems:

*From this point on, CET(col#n) will mean that the Cofactor Expansion Theorem has been applied to column n of the determinant, and CET(row#n) will mean that the Cofactor Expansion Theorem has been applied to row n of the determinant.*

1. Minors: $M_{11} = 4, M_{21} = -3, M_{12} = 2, M_{22} = 1$; Cofactors: $C_{11} = 4, C_{21} = 3, C_{12} = -2, C_{22} = 1$.

3. Minors: $M_{11} = -5, M_{21} = 47, M_{31} = 3, M_{12} = 0, M_{22} = -2, M_{32} = 0, M_{13} = 4, M_{23} = -38, M_{33} = -2$; Cofactors: $C_{11} = -5, C_{21} = -47, C_{31} = 3, C_{12} = 0, C_{22} = -2, C_{32} = 0, C_{13} = 4, C_{23} = 38, C_{33} = -2$.

5. \[ \begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix} = 1 \cdot |3| + 2 \cdot |1| = 5. \]

7. \[ \begin{vmatrix} 2 & 1 & -4 \\ 7 & 1 & 3 \\ 1 & 5 & -2 \end{vmatrix} = -7 \cdot \begin{vmatrix} 1 & -4 \\ 1 & -2 \end{vmatrix} + 2 \cdot 4 = -7 \cdot 18 + 0 - 3 \cdot 9 = -153. \]

9. \[ \begin{vmatrix} -2 & 0 \\ 0 & 5 \\ 3 & -5 \end{vmatrix} = 3 \cdot 10 + 5(-6) + 0 \cdot 4 = 0. \]

11. \[ \begin{vmatrix} 1 & 0 & -2 \\ 3 & 1 & -1 \\ 7 & 2 & 5 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & -1 \\ 7 & 2 \end{vmatrix} - \frac{1}{2} \begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} = 7 - 2(-1) = 9. \]

13. \[ \begin{vmatrix} 5 & 2 & 1 \\ 3 & -3 & 7 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 3 \\ -3 & 7 \end{vmatrix} - 5 \cdot \begin{vmatrix} -1 & 3 \\ -3 & 7 \end{vmatrix} + 3 \cdot \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} = 2 \cdot 17 - 5 \cdot 2 + 3(-7) = 3. \]

15. \[ \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = -2 - 2 = -4. \]
17.

\[
\begin{array}{ccc}
3 & 5 & 2 \\
2 & 3 & 5 \\
7 & 5 & -3 \\
9 & -6 & 27
\end{array}
\begin{array}{ccc}
1 & 2 & -3 \\
2 & 3 & 5 \\
7 & 5 & -3 \\
9 & -6 & 27
\end{array}
\begin{array}{c}
1 \\
2 \\
\frac{1}{2} \\
\frac{1}{2}
\end{array}
\begin{array}{ccc}
1 & 2 & -3 \\
0 & -1 & 11 \\
0 & -9 & 18 \\
0 & -24 & 54
\end{array}
\begin{array}{c}
11 \\
27 \\
-93 \\
-111
\end{array}
\begin{array}{ccc}
-1 & 11 & -27 \\
-9 & 18 & -93 \\
-24 & 54 & -111
\end{array}
\begin{array}{c}
4 \\
5 \\
3
\end{array}
\begin{array}{c}
41 \\
11997
\end{array}
\]

\[
\begin{array}{c}
\text{1. } A_{24}(-1) \\
\text{2. } A_{12}(-2), A_{13}(-7), A_{14}(-9) \\
\text{3. CET(col#1)} \\
\text{4. } A_{12}(-9), A_{13}(-24) \\
\text{5. CET(col#1)}
\end{array}
\]

19.

\[
\begin{array}{ccc}
2 & 0 & -1 \\
0 & 3 & 0 \\
0 & 1 & 3
\end{array}
\begin{array}{ccc}
0 & 0 & -3 \\
0 & 3 & 0 \\
0 & 1 & 3
\end{array}
\begin{array}{c}
1 \\
2 \\
\frac{1}{2}
\end{array}
\begin{array}{ccc}
0 & -1 & 3 \\
0 & 0 & 1 \\
1 & 3 & 0
\end{array}
\begin{array}{c}
5 \\
9 \\
0
\end{array}
\begin{array}{ccc}
0 & -9 & 1 \\
0 & 1 & 3 \\
0 & 3 & 0
\end{array}
\begin{array}{c}
-10 \\
-5 \\
4
\end{array}
\begin{array}{c}
5 \\
0 \\
0
\end{array}
\]

\[
\begin{array}{c}
\text{1. } A_{24}(-1), A_{45}(-3) \\
\text{2. CET(col#1)} \\
\text{3. } A_{12}(-3) \\
\text{4. CET(col#1)} \\
\text{5. P2} \\
\text{6. } A_{21}(-5), A_{23}(3) \\
\text{7. CET(col#2)}
\end{array}
\]

21.

(a):

\[
V(r_1, r_2, r_3) = \frac{1}{r_1 r_2 r_3} \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{bmatrix} = \frac{1}{r_1 r_2 - r_3 - r_1} \begin{bmatrix} 1 & 0 & 0 \\ r_1 & r_2 - r_3 & r_2 - r_3 \\ r_1^2 & r_2^2 - r_3^2 & r_2^2 - r_3^2 \end{bmatrix} = \frac{1}{r_2 - r_1} \begin{bmatrix} r_2 - r_1 & r_3 - r_1 \\ r_3 - r_1 & r_2 - r_3 \end{bmatrix} = (r_2 - r_1)(r_3 - r_1) - (r_3 - r_1)(r_2 - r_3) = (r_2 - r_1)(r_3 + r_1) - (r_2 + r_1) = (r_2 - r_1)(r_3 - r_1)(r_3 - r_2).
\]

\[
\begin{array}{c}
\text{1. CA}_{12}(-1), CA_{13}(-1) \\
\text{2. CET(row#1)}
\end{array}
\]

(b): We use mathematical induction. The result is vacuously true when \( n = 1 \) and quickly verified when \( n = 2 \). Suppose that the result is true when \( n = k - 1 \), for \( k \geq 3 \), and consider
The determinant vanishes when \( r_k = r_1, r_2, \ldots, r_{k-1} \), so we can write

\[
V(r_1, r_2, \ldots, r_k) = a(r_1, r_2, \ldots, r_k) \prod_{i=1}^{n-1} (r_k - r_i),
\]

where \( a(r_1, r_2, \ldots, r_k) \) is the coefficient of \( r_k^{k-1} \) in the expansion of \( V(r_1, r_2, \ldots, r_k) \). However, using the Cofactor Expansion Theorem along column \( k \), we see that this coefficient is just \( V(r_1, r_2, \ldots, r_{k-1}) \), so by hypothesis,

\[
a(r_1, r_2, \ldots, r_k) = V(r_1, r_2, \ldots, r_{k-1}) = \prod_{1 \leq i < m \leq n-1} (r_m - r_i).
\]

Thus,

\[
V(r_1, r_2, \ldots, r_k) = \prod_{1 \leq i < m \leq n-1} (r_m - r_i) \prod_{i=1}^{n-1} (r_k - r_i) = \prod_{1 \leq i < m \leq n} (r_m - r_i).
\]

Hence the result is true for \( n = k \), so, by induction, is true for all non-negative integers \( n \).

23.
(a): \( \det(A) = 7 \);

(b): \( M_C = \begin{bmatrix} 1 & -4 \\ 2 & -1 \end{bmatrix} \);

(c): \( \text{adj}(A) = \begin{bmatrix} 1 & 4 \\ -4 & -1 \end{bmatrix} \);

(d): \( A^{-1} = \frac{1}{7} \begin{bmatrix} 1 & 2 \\ -4 & -1 \end{bmatrix} \).

25.
(a): \( \det(A) = \begin{vmatrix} 2 & -3 & 0 \\ 2 & 1 & 5 \\ 0 & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} -3 & 0 \\ 1 & 5 \end{vmatrix} = 2 \cdot 13 = 26; \)

(b): \( M_C = \begin{bmatrix} 7 & -4 & -2 \\ 6 & 4 & 2 \\ -15 & -10 & 8 \end{bmatrix} \);

(c): \( \text{adj}(A) = \begin{bmatrix} 7 & 6 & -15 \\ -4 & 4 & -10 \\ -2 & 2 & 8 \end{bmatrix} \);
(d): \[ A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{26} \begin{bmatrix} 7 & 6 & -15 \\ -4 & 4 & -10 \\ -2 & 2 & 8 \end{bmatrix}. \]

27.
(a): \[ \det(A) = \begin{vmatrix} 1 & -1 & 2 \\ 3 & -1 & 4 \\ 5 & 1 & 7 \end{vmatrix} = 6 \]
\[ \begin{vmatrix} 8 & 0 & 9 \\ 8 & 0 & 11 \end{vmatrix} = 6; \]
(b): \[ M_C = \begin{bmatrix} -11 & -1 & 8 \\ 9 & -3 & -6 \\ -2 & 2 & 2 \end{bmatrix}; \]
(c): \[ \text{adj}(A) = \begin{bmatrix} -11 & 9 & -2 \\ -1 & -3 & 2 \\ 8 & -6 & 2 \end{bmatrix}; \]
(d): \[ A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{6} \begin{bmatrix} -11 & 9 & -2 \\ -1 & -3 & 2 \\ 8 & -6 & 2 \end{bmatrix} = \begin{bmatrix} -11/6 & 3/2 & -1/3 \\ -1/6 & -1/2 & 1/3 \\ 4/3 & -1 & 1/3 \end{bmatrix}. \]

29.
(a): \[ \det(A) = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 2 & 1 \\ 0 & 7 & -1 \end{vmatrix} = 14; \]
(b): \[ M_C = \begin{bmatrix} -9 & 1 & 7 \\ 32 & -2 & -14 \\ -13 & 3 & 7 \end{bmatrix}; \]
(c): \[ \text{adj}(A) = \begin{bmatrix} -9 & 32 & -13 \\ 1 & -2 & 3 \\ 7 & -14 & 7 \end{bmatrix}; \]
(d): \[ A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{6} \begin{bmatrix} -9 & 32 & -13 \\ 1 & -2 & 3 \\ 7 & -14 & 7 \end{bmatrix} = \begin{bmatrix} -9/14 & 16/7 & -13/14 \\ 1/2 & -1 & 3/4 \\ 1/2 & -1 & 3/4 \end{bmatrix}. \]

31.
(a): \[ \det(A) = \begin{vmatrix} 1 & 0 & 3 & 5 \\ -2 & 1 & 1 & 3 \\ 3 & 9 & 0 & 2 \\ 2 & 0 & 3 & -1 \end{vmatrix} = 0 \]
\[ \begin{vmatrix} 11 & 0 & 18 & 5 \\ 4 & 1 & 10 & 3 \\ 7 & 9 & 6 & 2 \\ 7 & 9 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 11 & 0 & 18 & 5 \\ 4 & 1 & 10 & 3 \\ 7 & 9 & 6 & 2 \\ 7 & 9 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 11 & 0 & 18 \\ 4 & 1 & 10 \\ 7 & 9 & 6 \\ 7 & 9 & 6 \end{vmatrix} = 402; \]
(b): \[ M_C = \begin{bmatrix} 84 & -46 & -29 & 81 \\ -162 & 60 & 99 & -27 \\ -30 & 26 & 130 & -72 \end{bmatrix}; \]
(c): \[ \text{adj}(A) = \begin{bmatrix} 84 & -162 & 18 & -30 \\ -46 & 60 & 38 & 26 \\ -29 & 99 & -11 & 130 \\ 81 & -27 & 3 & -72 \end{bmatrix}; \]
(d):

\[
A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{402} \begin{bmatrix}
84 & -162 & 18 & -30 \\
-46 & 60 & 38 & 26 \\
-29 & 99 & -11 & 130 \\
81 & -27 & 3 & -72 \\
\end{bmatrix}
= \begin{bmatrix}
-23/201 & 10/67 & 19/201 & 13/201 \\
-29/402 & 33/134 & -11/402 & 65/201 \\
27/134 & -9/134 & 1/134 & -12/67 \\
\end{bmatrix}.
\]

33. \(\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1,\)

and

\[C_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -(2 - 1) = -1.\]

Thus,

\[(A^{-1})_{32} = \frac{(\text{adj}(A))_{32}}{\det(A)} = \frac{C_{23}}{\det(A)} = -1.\]

35.

\[
\det(A) = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ -1 & 1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & 1 \\ -1 & 1 \\ 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} = 4 + 12 = 16,
\]

and

\[C_{32} = -\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ -1 & 2 & 0 \end{vmatrix} = -(3) \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 3(2 + 1) = 9.\]

Thus,

\[(A^{-1})_{23} = \frac{(\text{adj}(A))_{23}}{\det(A)} = \frac{C_{32}}{\det(A)} = \frac{9}{16}.\]

37. \(M_C = \begin{bmatrix} e^{-t} \sin(2t) & -e^t \cos(2t) \\ -e^t \cos(2t) & e^t \sin(2t) \end{bmatrix}, \text{adj}(A) = \begin{bmatrix} e^{-t} \sin(2t) & e^t \cos(2t) \\ -e^t \cos(2t) & e^t \sin(2t) \end{bmatrix},\) and

\[
\det(A) = \sin^2(2t) + \cos^2(2t) = 1,
\]

so

\[A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \begin{bmatrix} e^{-t} \sin(2t) & e^t \cos(2t) \\ -e^t \cos(2t) & e^t \sin(2t) \end{bmatrix}.\]
39. \( M_C = \begin{bmatrix} -1 & 2 & -1 \\ 3 & -6 & 3 \\ -2 & 4 & -2 \end{bmatrix} \), \( \text{adj}(A) = \begin{bmatrix} -1 & 3 & -2 \\ 2 & -6 & 4 \\ -1 & 3 & -2 \end{bmatrix} \). Hence,

\[
A \cdot \text{adj}(A) = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 3 & -2 \\ 2 & -6 & 4 \\ -1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0_3.
\]

From Equation (3.3.4) of the text we have that, in general, \( A \cdot \text{adj}(A) = \det(A) \cdot I_n \). Since, for the given matrix, \( A \cdot \text{adj}(A) = 0_3 \), we must have \( \det(A) = 0 \).

41.

\[
\det(A) = \begin{vmatrix} 3 & -2 & 1 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 7,
\]

\[
\det(B_1) = \begin{vmatrix} 4 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 3 \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 9,
\]

\[
\det(B_2) = \begin{vmatrix} 3 & 4 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{vmatrix} = 4 \begin{vmatrix} 6 & 0 \\ 3 & 0 \end{vmatrix} = 1 \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} = -6,
\]

\[
\det(B_3) = \begin{vmatrix} 3 & -2 & 4 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -5 \begin{vmatrix} -2 & 4 \\ 0 & 1 \end{vmatrix} = -3 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -11.
\]

Thus, \( x_1 = \frac{\det(B_1)}{\det(A)} = \frac{9}{7} \), \( x_2 = \frac{\det(B_2)}{\det(A)} = \frac{-6}{7} \), and \( x_3 = \frac{\det(B_3)}{\det(A)} = \frac{-11}{7} \).

Solution: \((9/7, -6/7, -11/7)\).
43.

\[
\begin{align*}
\det(A) &= \begin{vmatrix} 1 & -2 & 3 & -1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -5 & -2 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 4 & 1 & -2 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & -1 & 0 \\ 5 & 2 & 0 \\ 4 & 1 & 1 \end{vmatrix} = -3, \\
\det(B_1) &= \begin{vmatrix} 1 & -2 & 3 & -1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 3 & 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 3 & -1 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 3 & 2 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -5 & -3 & 3 \\ 2 & 0 & 1 \\ 3 & 2 & -2 \end{vmatrix} = -11, \\
\det(B_2) &= \begin{vmatrix} 1 & 1 & 3 & -1 \\ 2 & 2 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 3 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = 17, \\
\det(B_3) &= \begin{vmatrix} 1 & -2 & 1 & -1 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -2 & 1 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & -1 \\ -3 & 1 & 3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 0 & -2 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ -3 & 1 & 1 \end{vmatrix} = -2(-8) = 16, \\
\det(B_4) &= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 3 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 3 & 1 \\ 2 & 1 & 2 \\ -1 & -2 & 3 \end{vmatrix} = -3 \begin{vmatrix} 3 & -5 \\ 3 & 7 \end{vmatrix} = 6.
\end{align*}
\]

Therefore,

\[x_1 = \frac{11}{3}, \ x_2 = -\frac{17}{3}, \ x_3 = -\frac{16}{3}, \text{ and } x_4 = -2.\]

Solution: \((11/3, -17/3, -16/3, -2)\).
45. $\det(A) = \begin{vmatrix} 1 & 4 & -2 & 1 \\ 2 & 9 & -3 & -2 \\ 1 & 5 & 0 & -1 \\ 3 & 14 & 7 & -2 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 2 \\ -1 & -1 & 0 \\ -1 & 7 & 1 \\ -1 & 7 & 1 \end{vmatrix} = -19$, $\det(B_2) = \begin{vmatrix} 1 & 2 & -2 & 1 \\ 2 & 5 & -3 & -2 \\ 1 & 3 & 0 & -1 \\ 3 & 6 & 7 & -2 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 2 \\ -1 & -1 & 0 \\ -3 & 7 & 1 \\ -3 & 7 & 1 \end{vmatrix} = 5$, $\therefore x_2 = \frac{\det(B_2)}{\det(A)} = \frac{31}{19}$.

47. Let $B$ be the matrix obtained from $A$ by adding column $i$ to column $j$ $(i \neq j)$ in the matrix $A$. By the property for columns corresponding to Property P3, we have $\det(B) = \det(A)$. Cofactor expansion of $B$ along column $j$ gives

$$\det(A) = \det(B) = \sum_{k=1}^{n} (a_{kj} + a_{ki})C_{kj} = \sum_{k=1}^{n} a_{kj}C_{kj} + \sum_{k=1}^{n} a_{ki}C_{kj}.$$ 

That is,

$$\det(A) = \det(A) + \sum_{k=1}^{n} a_{ki}C_{kj},$$

since by the Cofactor Expansion Theorem the first summation on the right-hand side is simply $\det(A)$. It follows immediately that

$$\sum_{k=1}^{n} a_{ki}C_{kj} = 0, \quad i \neq j.$$

51. 

$$A = \begin{bmatrix} 1.21 & 3.42 & 2.15 \\ 5.41 & 2.32 & 7.15 \\ 21.63 & 3.51 & 9.22 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 3.25 & 3.42 & 2.15 \\ 4.61 & 2.32 & 7.15 \\ 9.93 & 3.51 & 9.22 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1.21 & 3.25 & 2.15 \\ 5.41 & 4.61 & 7.15 \\ 21.63 & 9.93 & 9.22 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1.21 & 3.42 & 3.25 \\ 5.41 & 2.32 & 4.61 \\ 21.63 & 3.51 & 9.93 \end{bmatrix}.$$ 

From Cramer’s Rule,

$$x_1 = \frac{\det(B_1)}{\det(A)} \approx 0.25, \quad x_2 = \frac{\det(B_2)}{\det(A)} \approx 0.72, \quad x_3 = \frac{\det(B_3)}{\det(A)} \approx 0.22.$$ 

52. $\det(A) = 32$, $\det(B_1) = -3218$, $\det(B_2) = 3207$, $\det(B_3) = 2896$, $\det(B_4) = -9682$, $\det(B_5) = 2414$. So,

$$x_1 = -\frac{1609}{16}, \quad x_2 = \frac{3207}{32}, \quad x_3 = \frac{181}{2}, \quad x_4 = -\frac{4841}{32}, \quad x_5 = \frac{1207}{16}.$$
53. We have

\[(BA)_{ji} = \sum_{k=1}^{n} b_{jk} a_{ki} = \sum_{k=1}^{n} \frac{1}{\det(A)} \cdot \text{adj}(A)_{jk} \cdot a_{ki} = \frac{1}{\det(A)} \sum_{k=1}^{n} C_{kj} a_{ki} = \delta_{ij},\]

where we have used Equation (3.3.4) in the last step.

### Solutions to Section 3.4

#### Problems:

1. \[\begin{vmatrix} 5 & -1 \\ 3 & 7 \end{vmatrix} = 5 \cdot 7 - 3(-1) = 38.\]

3. \[\begin{vmatrix} 6 & 1 & 3 \\ 14 & 2 & 7 \end{vmatrix} = -3.\]

5. \[
\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a \begin{vmatrix} c & a \\ b & c \end{vmatrix} - b \begin{vmatrix} a & b \\ c & a \end{vmatrix} + c \begin{vmatrix} b & a \\ c & b \end{vmatrix} = a(bc - a^2) - b(b^2 - ac) + c(ab - c^2) = 3abc - a^3 - b^3 - c^3.
\]

7. \[
\begin{vmatrix} 7 & 1 & 2 & 3 \\ 2 & -2 & 4 & 6 \\ 3 & -1 & 5 & 4 \\ 18 & 9 & 27 & 54 \end{vmatrix} = 9 \begin{vmatrix} 7 & 1 & 2 & 3 \\ 2 & -2 & 4 & 6 \\ 3 & -1 & 5 & 4 \\ 2 & 1 & 3 & 6 \end{vmatrix} = 9 \begin{vmatrix} 16 & 0 & 8 & 12 \\ 10 & 0 & 7 & 7 \\ -5 & 0 & 1 & 3 \end{vmatrix} = -9 \begin{vmatrix} 10 & 7 & 7 \\ -5 & 1 & 3 \end{vmatrix} = -36 \begin{vmatrix} 14 & 0 & -3 \\ -5 & 1 & 3 \end{vmatrix} = -36 \begin{vmatrix} 14 & -3 \\ 45 & -14 \end{vmatrix} = -2196.
\]

9. \[
\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 0 -1 -5 = -18.
\]

\[M_C = \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix} \Rightarrow \text{adj}(A) = \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}, \text{ so that}
\]
\[
A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \begin{bmatrix} -5/18 & 1/18 & 7/18 \\ 1/18 & 7/18 & -5/18 \\ 7/18 & -5/18 & 1/18 \end{bmatrix}.
\]

11. \[
\det(A) = \begin{vmatrix} 2 & 5 & 7 \\ 4 & -3 & 2 \\ 6 & 9 & 11 \end{vmatrix} = 116.
\]
\[
M_C = \begin{bmatrix}
-51 & -32 & 54 \\
8 & -20 & 12 \\
31 & 24 & -26 \\
\end{bmatrix}
\implies \text{adj}(A) = \begin{bmatrix}
-51 & 8 & 31 \\
-32 & -20 & 24 \\
54 & 12 & -26 \\
\end{bmatrix}, \text{ so that }
A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \begin{bmatrix}
-51/116 & 2/29 & 31/116 \\
-8/29 & -5/29 & 6/29 \\
27/58 & 3/29 & -13/58 \\
\end{bmatrix}.
\]

13. \( A = \begin{bmatrix} 3 & 5 \\ 6 & 2 \end{bmatrix} \), \( B_1 = \begin{bmatrix} 4 & 5 \\ 9 & 2 \end{bmatrix} \), \( B_2 = \begin{bmatrix} 3 & 4 \\ 6 & 9 \end{bmatrix} \), so that
\[x_1 = \frac{\det(B_1)}{\det(A)} = \frac{37}{24}, \quad x_2 = \frac{\det(B_2)}{\det(A)} = -\frac{1}{8}.
\]

15. \( A = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 5 \\ 2 & 3 & 1 \end{bmatrix} \), \( B_1 = \begin{bmatrix} 5 & 1 & 3 \\ 7 & -1 & 5 \\ 2 & 3 & 1 \end{bmatrix} \), \( B_2 = \begin{bmatrix} 4 & 5 & 3 \\ 2 & 7 & 5 \\ 2 & 2 & 1 \end{bmatrix} \), \( B_3 = \begin{bmatrix} 4 & 1 & 5 \\ 2 & -1 & 7 \\ 2 & 3 & 2 \end{bmatrix} \), so that
\[x_1 = \frac{\det(B_1)}{\det(A)} = \frac{1}{4}, \quad x_2 = \frac{\det(B_2)}{\det(A)} = \frac{1}{16}, \quad x_3 = \frac{\det(B_3)}{\det(A)} = \frac{21}{16}.
\]

17. \( A = \begin{bmatrix} 3.1 & 3.5 & 7.1 \\ 2.2 & 5.2 & 6.3 \\ 1.4 & 8.1 & 0.9 \end{bmatrix} \), \( B_1 = \begin{bmatrix} 3.6 & 3.5 & 7.1 \\ 2.5 & 5.2 & 6.3 \\ 9.3 & 8.1 & 0.9 \end{bmatrix} \), \( B_2 = \begin{bmatrix} 3.1 & 3.6 & 7.1 \\ 2.2 & 2.5 & 6.3 \\ 1.4 & 9.3 & 0.9 \end{bmatrix} \), \( B_3 = \begin{bmatrix} 3.1 & 3.5 & 3.6 \\ 2.2 & 5.2 & 2.5 \\ 1.4 & 8.1 & 9.3 \end{bmatrix} \),
so that
\[x_1 = \frac{\det(B_1)}{\det(A)} = 3.77, \quad x_2 = \frac{\det(B_2)}{\det(A)} = 0.66, \quad x_3 = \frac{\det(B_3)}{\det(A)} = -1.46.
\]

19. \( \det(2A) = 2^3 \det(A) = 24. \)
From the result of the preceding problem, \( \det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{3}. \)
\( \det(A^T B) = \det(A^T) \det(B) = \det(A) \det(B) = -12. \)
\( \det(B^{10}) = [\det(B)]^{10} = -1024. \)
\( \det(B^{-1} A B) = \det(B^{-1}) \det(A) \det(B) = \frac{1}{\det(B)} \det(A) \det(B) = \det(A) = 3. \)

**Solutions to Section 3.5**

**Additional Problems:**

1. (a): We have \( \det(A) = (-7)(-5) - (1)(-2) = 37. \)
(b): We have \( A^{-1} \simeq \begin{bmatrix} 1 & -5 \\ -7 & -2 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 0 & -37 \end{bmatrix} = B. \)
Now, \( \det(A) = -\det(B) = -(37) = 37. \)

\( (c): \) Let us use the Cofactor Expansion Theorem along the first row:

\[
\det(A) = a_{11} C_{11} + a_{12} C_{12} = (-7)(-5) + (-2)(-1) = 37.
\]

3.

\( (a): \) Using Equation (3.1.2), we have

\[
\det(A) = (-1)(2)(-3) + (4)(2)(2) + (1)(0)(2) - (-1)(2)(-4)(0)(-3) - (1)(2)(2) = 6 + 16 + 0 - (4) - 0 - 4 = 22.
\]

\( (b): \) We have

\[
A \sim \begin{bmatrix}
-1 & 4 & 1 \\
0 & 2 & 2 \\
0 & 10 & -1
\end{bmatrix} \sim \begin{bmatrix}
-1 & 4 & 1 \\
0 & 2 & 2 \\
0 & 0 & -11
\end{bmatrix} = B.
\]

1. \( A_{13}(2) \)
2. \( A_{23}(-5) \)

Now, \( \det(A) = \det(B) = (-1)(2)(-11) = 22. \)

\( (c): \) Let us use the Cofactor Expansion Theorem along the first column:

\[
\det(A) = a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31} = (-1)(-10) + (0)(14) + (2)(6) = 22.
\]

5.

\( (a): \) Of the 24 terms appearing in the determinant expression (3.1.3), only terms containing the factors \( a_{11} \) and \( a_{44} \) will be nonzero (all other entries in the first column and fourth row of \( A \) are zero). Looking at entries in the second and third rows and columns of \( A \), we see that only the product \( a_{23}a_{32} \) is nonzero. Therefore, the only nonzero term in the summation (3.1.3) is \( a_{11}a_{23}a_{32}a_{44} = (3)(1)(2)(-4) = -24. \) The permutation associated with this term is \( (1, 3, 2, 4) \) which contains one inversion. Therefore, \( \sigma(1, 3, 2, 4) = -1 \), and so the determinant is \( (-24)(-1) = 24. \)

\( (b): \) We have

\[
A \sim \begin{bmatrix}
3 & -1 & -2 & 1 \\
0 & 2 & 1 & -1 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & -4
\end{bmatrix} \sim \begin{bmatrix}
0 & 1 & 4 \\
2 & 1 & -1 \\
0 & 0 & -4
\end{bmatrix} = B.
\]

1. \( P_{23} \)

Now, \( \det(A) = -\det(B) = -(3)(2)(1)(-4) = 24. \)

\( (c): \) Cofactor expansion along the first column yields: \( \det(A) = 3 \cdot \begin{vmatrix}
0 & 1 & 4 \\
2 & 1 & -1 \\
0 & 0 & -4
\end{vmatrix}. \) This latter determinant can be found by cofactor expansion along the last column: \( (-4)[(0)(1) - (2)(1)] = 8. \) Thus, \( \det(A) = 3 \cdot 8 = 24. \)

7. To obtain the given matrix from \( A \), we perform two row permutations, multiply a row through by \( -4 \), and multiply a row through by \( 2 \). The combined effect of these operations is to multiply the determinant of \( A \) by \( (-1)^2 \cdot (-4) \cdot (2) = -8. \) Hence, the given matrix has determinant \( \det(A) \cdot (-8) = 4 \cdot (-8) = -32. \)
9. To obtain the given matrix from $A^T$, we do two row permutations, multiply a row by $-1$, multiply a row by 3, and add 2 times one row to another. The combined effect of these operations is to multiply the determinant of $A$ by $(-1)(-1)(-1)(3)(1) = -3$. Hence, the given matrix has determinant $\det(A) \cdot (-3) = (4) \cdot (-3) = -12$.

11. We have $\det(AB) = \det(A)\det(B) = (-2) \cdot 3 = -6$.

13. We have $\det((A^{-1}B^T)(2B^{-1})) = \det(A^{-1}B) \cdot \det(2B^{-1}) = -\frac{1}{2} \cdot 3 \cdot 2^4 \cdot \frac{1}{3} = -8$.

15. Since $A$ and $B$ are not square matrices, $\det(A)$ and $\det(B)$ are not possible to compute. We have $\det(C) = -18$, and therefore, $\det(C^T) = -18$. Now, $AB = \begin{bmatrix} 8 & -10 \\ 25 & 28 \end{bmatrix}$, and so $\det(AB) = 474$. Since $BA = \begin{bmatrix} 4 & 5 & 2 \\ 1 & 8 & -13 \\ 18 & 15 & 24 \end{bmatrix}$, we have $\det(BA) = 0$. Next, $\det(B^T A^T) = \det((AB)^T) = \det(AB) = 474$. Next, $\det(BAC) = \det(BA)\det(C) = 0 \cdot (-18) = 0$. Finally, we have $\det(ACB) = \det \begin{bmatrix} 38 & 54 \\ 133 & 297 \end{bmatrix} = 4104$.

17. 

$$M_C = \begin{bmatrix} 16 & -1 & -5 \\ 4 & 5 & -3 \\ -4 & 2 & 10 \end{bmatrix};$$

$$\text{adj}(A) = \begin{bmatrix} 16 & 4 & -4 \\ -1 & 5 & 2 \\ -5 & -3 & 10 \end{bmatrix};$$

$$\det(A) = 28;$$

$$A^{-1} = \begin{bmatrix} 4/7 & 1/7 & -1/7 \\ -1/28 & 5/28 & 1/14 \\ -5/28 & -3/28 & 5/14 \end{bmatrix}.$$ 

19. 

$$M_C = \begin{bmatrix} 88 & -24 & -40 & -48 \\ 32 & 12 & -20 & 0 \\ 16 & 12 & -4 & 0 \\ -4 & 6 & -2 & 0 \end{bmatrix};$$

$$\text{adj}(A) = \begin{bmatrix} 88 & 32 & 16 & -4 \\ -24 & 12 & 12 & 6 \\ -40 & -20 & -4 & -2 \\ -48 & 0 & 0 & 0 \end{bmatrix};$$

$$\det(A) = -48;$$

$$A^{-1} = \frac{1}{48} \begin{bmatrix} 88 & 32 & 16 & -4 \\ -24 & 12 & 12 & 6 \\ -40 & -20 & -4 & -2 \\ -48 & 0 & 0 & 0 \end{bmatrix}.$$ 

23. FALSE. For instance, if one entry of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ is changed, two of the rows of $A$ will still be identical, and therefore, the determinant of the resulting matrix must be zero. It is not possible to force the determinant to equal $r$.

24. Note that $\det(A) = 2 + 12k + 36 - 4k - 18 - 12 = 8k + 8 = 8(k + 1)$. 
25. Note that
\[ \det(A) = 3(k + 1) + 2k + 0 - 3 - k(k + 1) - 0 = -k^2 + 4k = k(4 - k). \]

(a): Based on the calculation above, we see that \( A \) fails to be invertible if and only if \( k = 0 \) or \( k = 4 \).
(b): The volume of the parallelepiped determined by the row vectors of \( A \) is precisely \( |\det(A)| = |−k^2 + 4k| \). The volume of the parallelepiped determined by the column vectors of \( A \) is the same as the volume of the parallelepiped determined by the row vectors of \( A^T \), which is \( |\det(A^T)| = |\det(A)| = |−k^2 + 4k| \). Thus, the volume is the same.

27. From the assumption that \( AB = −BA \), we can take the determinant of each side: \( \det(AB) = \det(−BA) \). Hence, \( \det(A)\det(B) = (−1)^n\det(B)\det(A) = −\det(A)\det(B) \). From this, it follows that \( \det(A)\det(B) = 0 \), and therefore, either \( \det(A) = 0 \) or \( \det(B) = 0 \). Thus, either \( A \) or \( B \) (or both) fails to be invertible.

29. The coefficient matrix of this linear system is \( A = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} \). We have
\[ \det(A) = \begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} = -7, \quad \det(B_1) = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5, \quad \text{and} \quad \det(B_2) = \begin{vmatrix} -3 & 3 \\ 1 & 1 \end{vmatrix} = -6. \]
Thus,
\[ x_1 = \frac{\det(B_1)}{\det(A)} = \frac{-5}{7} \quad \text{and} \quad x_2 = \frac{\det(B_2)}{\det(A)} = \frac{6}{7}. \]
Solution: \((-5/7, 6/7)\).

31. The coefficient matrix of this linear system is \( A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -1 & 1 \\ 0 & 5 & 5 \end{bmatrix} \). We have
\[ \det(A) = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -1 & 1 \\ 0 & 5 & 5 \end{vmatrix} = -20, \quad \det(B_1) = \begin{vmatrix} -1 & 1 & 2 \\ -1 & -1 & 1 \\ -5 & 5 & 5 \end{vmatrix} = -10, \]
\[ \det(B_2) = \begin{vmatrix} 3 & -1 & 2 \\ 2 & -1 & 1 \\ 0 & -5 & 5 \end{vmatrix} = -10, \quad \det(B_3) = \begin{vmatrix} 3 & 1 & -1 \\ 2 & -1 & -1 \\ 0 & 5 & -5 \end{vmatrix} = 30. \]
Thus,
\[ x_1 = \frac{\det(B_1)}{\det(A)} = \frac{1}{2}, \quad x_2 = \frac{\det(B_2)}{\det(A)} = \frac{1}{2}, \quad \text{and} \quad x_3 = \frac{\det(B_3)}{\det(A)} = \frac{3}{2}. \]
Solution: \((1/2, 1/2, -3/2)\).

Solutions to Section 4.1

True-False Review:

1. FALSE. The vectors \((x, y)\) and \((x, y, 0)\) do not belong to the same set, so they are not even comparable, let alone equal to one another.
3. TRUE. The solution set refers to collections of the unknowns that solve the linear system. Since this system has 6 unknowns, the solution set will consist of vectors belonging to \( \mathbb{R}^6 \).

5. FALSE. There is no such name for a vector whose components are all positive.

7. TRUE. When the vector \( \mathbf{x} \) is scalar multiplied by zero, each component becomes zero: \( 0 \mathbf{x} = \mathbf{0} \). This is the zero vector in \( \mathbb{R}^n \).

9. FALSE. If \( k < 0 \), then \( k \mathbf{x} \) is a vector in the third quadrant. For instance, \((1, 1)\) lies in the first quadrant, but \((-2)(1, 1) = (-2, -2)\) lies in the third quadrant.

11. FALSE. If the three vectors lie on the same line or the same plane, the resulting object may determine a one-dimensional segment or two-dimensional area. For instance, if \( \mathbf{x} = \mathbf{y} = \mathbf{z} = (1, 0, 0) \), then the vectors \( x, y, \) and \( z \) rest on the segment from \((0, 0, 0)\) to \((1, 0, 0)\), and do not determine a three-dimensional solid region.

Problems:

1. \( \mathbf{v}_1 = (6, 2), \mathbf{v}_2 = (-3, 6), \mathbf{v}_3 = (6, 2) + (-3, 6) = (3, 8) \).

3. \( \mathbf{v} = 5(3, -1, 2, 5) - 7(-1, 2, 9, -2) = (15, -5, 10, 25) - (-7, 14, 63, -14) = (22, -19, -53, 39) \). Additive inverse: \(-\mathbf{v} = (-1)\mathbf{v} = (-22, 19, 53, -39)\).

5. Let \( \mathbf{x} = (x_1, x_2, x_3, x_4), \mathbf{y} = (y_1, y_2, y_3, y_4) \) be arbitrary vectors in \( \mathbb{R}^4 \). Then
\[
\mathbf{x} + \mathbf{y} = (x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)
\]

7. Let \( \mathbf{x} = (x_1, x_2, x_3) \) and \( \mathbf{y} = (y_1, y_2, y_3) \) be arbitrary vectors in \( \mathbb{R}^3 \), and let \( r, s, t \) be arbitrary real numbers. Then:
\[
1\mathbf{x} = 1(x_1, x_2, x_3) = (1x_1, 1x_2, 1x_3) = (x_1, x_2, x_3) = \mathbf{x}.
\]
\[
(st)\mathbf{x} = (st)(x_1, x_2, x_3) = ((st)x_1, (st)x_2, (st)x_3) = (s(tx_1), s(tx_2), s(tx_3)) = s(tx_1, tx_2, tx_3) = s(\mathbf{x}).
\]
\[
r(\mathbf{x} + \mathbf{y}) = r(x_1 + y_1, x_2 + y_2, x_3 + y_3) = (r(x_1 + y_1), r(x_2 + y_2), r(x_3 + y_3))
\]
\[
= (rx_1 + ry_1, rx_2 + ry_2, rx_3 + ry_3) = (rx_1, rx_2, rx_3) + (ry_1, ry_2, ry_3) = r\mathbf{x} + r\mathbf{y}.
\]
(s + t)x = (s + t)(x_1, x_2, x_3) = ((s + t)x_1, (s + t)x_2, (s + t)x_3) \\
= (sx_1 + tx_1, sx_2 + tx_2, sx_3 + tx_3) = (sx_1, sx_2, sx_3) + (tx_1, tx_2, tx_3) \\
= sx + tx.

**Solutions to Section 4.2**

**True-False Review:**

1. **TRUE.** This is part 1 of Theorem 4.2.6.

2. **FALSE.** This set is not closed under addition, since \(1 + 1 \notin \mathbb{N}\). Therefore, (A1) fails, and hence, this set does not form a vector space. (It is worth noting that the set is also not closed under scalar multiplication.)

3. **TRUE.** This is part 1 of Theorem 4.2.6.

4. **FALSE.** This set is not closed under scalar multiplication. In particular, if \(k\) is an irrational number such as \(k = \pi\) and \(v\) is an integer, then \(kv\) is not an integer.

5. **TRUE.** This is part 1 of Theorem 4.2.6.

6. **FALSE.** This set is not closed under addition, since \(1 + 1 \notin \{0,1\}\). Therefore, (A1) fails, and hence, this set does not form a vector space. (It is worth noting that the set is also not closed under scalar multiplication.)

**Problems:**

1. If \(x = p/q\) and \(y = r/s\), where \(p, q, r, s\) are integers \((q \neq 0, s \neq 0)\), then \(x + y = (ps + qr)/(qs)\), which is a rational number. Consequently, the set of all rational numbers is closed under addition. The set is not closed under scalar multiplication since, if we multiply a rational number by an irrational number, the result is an irrational number.

2. \(V = \{y : y^2 + 9y = 4x^2\}\) is not a vector space because it is not closed under vector addition. Let \(u, v \in V\). Then \(u^2 + 9u = 4x^2\) and \(v^2 + 9v = 4x^2\). It follows that \((u + v)^2 + 9(u + v) = (u^2 + v^2) + 9(u + v) = u^2 + 9u + v^2 + 9v = 4x^2 + 4x^2 = 8x^2 \neq 4x^2\). Thus, \(u + v \notin V\). Likewise, \(V\) is not closed under scalar multiplication.

3. \(V = \{x \mid Ax = 0\}\), where \(A\) is a fixed matrix} is closed under addition and scalar multiplication, as we now show:

Let \(u, v \in V\) and \(k \in \mathbb{R}\).

A1: Addition: \(A(u + v) = Au + Av = 0 + 0 = 0\), so \(u + v \in V\).

A2: Scalar Multiplication: \(A(ku) = kAu = k0 = 0\), thus \(ku \in V\).

7. (1) \(\mathbb{N}\) is not closed under scalar multiplication, since multiplication of a positive integer by a real number does not, in general, result in a positive integer.

2. There is no zero vector in \(\mathbb{N}\).

3. No element of \(\mathbb{N}\) has an additive inverse in \(\mathbb{N}\).

9. Let \(A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in M_2x3(\mathbb{R})\). Then we see that the zero vector is \(0_{2x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\), since \(A + 0_{2x3} = A\). Further, \(A\) has additive inverse \(-A = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \end{bmatrix}\) since \(A + (-A) = 0_{2x3}\).

11. \(V = \{p : p\) is a polynomial in \(x\) of degree 2\}\). \(V\) is not a vector space because it is not closed under addition. For example, \(x^2 \in V\) and \(-x^2 \in V\), yet \(x^2 + (-x^2) = 0 \notin V\).

13. Axioms A1 and A2 clearly hold under the given operations.

A3: \(u + v = (u_1, u_2) + (v_1, v_2) = (u_1 - v_1, u_2 - v_2) = (v_1 - u_1, -v_2 - u_2) \neq (v_1 - u_1, v_2 - u_2) = v + u\). Consequently, A3 does not hold.
A4: \((u + v) + w = (u_1-v_1, u_2-v_2, (u_1,v_2) - w_2) = (u_1-(v_1+w_1), u_2-(v_2+w_2)) = (u_1,v_2) + (v_1+w_1) \neq (u_1,v_2) + (v_1-w_1, v_2-w_2) = u + (v+w).\) Consequently, A4 does not hold.

A5: \(0 = (0,0)\) since \(u + 0 = (u_1, u_2) + (0,0) = (u_1 - 0, u_2 - 0) = (u_1, u_2) = u.\)

A6: If \(u = (u_1, u_2)\), then \(-u = (u_1, u_2)\) since \(u + (-u) = (u_1, u_2) + (u_1, u_2) = (u_1 - u_1, u_2 - u_2) = (0,0) = 0.\)

Each of the remaining axioms do not hold.

15. Let \(A, B, C \in M_2(\mathbb{R})\) and \(r, s, t \in \mathbb{R}.
A3: The addition operation is not commutative since

\[ A + B = AB \neq BA = B + A. \]

A4: Addition is associative since

\[(A + B) + C = AB + C = (AB)C = A(BC) = A(B + C) = A + (B + C).\]

A5: \(I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) is the zero vector in \(M_2(\mathbb{R})\) because \(A + I_2 = AI_2 = A\) for all \(A \in M_2(\mathbb{R}).\)

A6: We wish to determine whether for each matrix \(A \in M_2(\mathbb{R})\) we can find a matrix \(B \in M_2(\mathbb{R})\) such that \(A + B = I_2\) (remember that we have shown in A5 that the zero vector is \(I_2\), equivalently, such that \(AB = I_2.\)

However, this equation can be satisfied only if \(A\) is nonsingular, therefore the axiom fails.

A7: \(1 \cdot A = A\) is true for all \(A \in M_2(\mathbb{R}).\)

A8: \((st)A = s(tA)\) is true for all \(A \in M_2(\mathbb{R})\) and \(s, t \in \mathbb{R}.
A9: \(sA + tA = (sA) + (tA) = (s+t)A\) for all \(s, t \in \mathbb{R}\) and \(A \in M_2(\mathbb{R}).\) Consequently, the axiom fails.

A10: \(rA + rB = (rA) + (rB) = r^2A + r^2B = (A + B)\). Thus, \(rA + rB \neq rA + rB\) for all \(r \in \mathbb{R},\) so the axiom fails.

17. Let \(C^2 = \{(z_1, z_2) : z_i \in \mathbb{C}\}\) under the usual operations of addition and scalar multiplication.

A3 and A4: Follow from the properties of addition in \(C^2.\)

A5: \((0, 0)\) is the zero vector in \(C^2\) since \((z_1, z_2) + (0,0) = (z_1 + 0, z_2 + 0) = (z_1, z_2)\) for all \((z_1, z_2) \in C^2.\)

A6: The additive inverse of the vector \((z_1, z_2) \in C^2\) is the vector \((-z_1, -z_2)\) for all \((z_1, z_2) \in C^2.\)

A7-A10: Follows from properties in \(C^2.\)

Thus, \(C^2\) together with its defined operations, is a complex vector space.

19. Let \(u = (u_1, u_2, u_3)\) and \(v = (v_1, v_2, v_3)\) be vectors in \(C^3,\) and let \(k \in \mathbb{R}.
A1: \(u + v = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \in C^3.\)

A2: \(ku = k(u_1, u_2, u_3) = (ku_1, ku_2, ku_3) \in C^3.\)

A3 and A4: Satisfied by the properties of addition in \(C^3.\)

A5: \((0, 0, 0)\) is the zero vector in \(C^3\) since \((0, 0, 0) + (z_1, z_2, z_3) = (0 + z_1, 0 + z_2, 0 + z_3) = (z_1, z_2, z_3)\) for all \((z_1, z_2, z_3) \in C^3.\)

A6: \((-z_1, -z_2, -z_3)\) is the additive inverse of \((z_1, z_2, z_3)\) because \((z_1, z_2, z_3) + (-z_1, -z_2, -z_3) = (0,0,0)\) for all \((z_1, z_2, z_3) \in C^3.\)

Let \(r, s, t \in \mathbb{R}.
A7: \(1 \cdot u = 1 \cdot (u_1, u_2, u_3) = (u_1u_1, 1u_2, 1u_3) = (u_1, u_2, u_3) = u.\)

A8: \((st)u = (st)(u_1, u_2, u_3) = (su_1, su_2, su_3)\) for all \((u_1, u_2, u_3) \in C^3.\)

A9: \(r(u + v) = r(u_1 + v_1, u_2 + v_2, u_3 + v_3) = (ru_1 + rv_1, ru_2 + rv_2, ru_3 + rv_3) = (ru_1, ru_2, ru_3) + (rv_1, rv_2, rv_3) = r(u_1, v_1, u_2) + r(v_1, v_2, v_3) = ru + rv.\)

A10: \((s + t)u = (s + t)(u_1, u_2, u_3) = ((s + t)u_1, (s + t)u_2, (s + t)u_3) = (su_1 + tu_1, su_2 + tu_2, su_3 + tu_3) = su_1 + su_2 + su_3 + tu_1 + tu_2 + tu_3 = su + tu.\)

Thus, \(C^3\) is a real vector space.
21. Let \( k \) be an arbitrary scalar, and let \( \mathbf{u} \) be an arbitrary vector in \( V \). Then, using property 2 of Theorem 4.2.6, we have
\[
k \mathbf{0} = k(0 \mathbf{u}) = (k0)\mathbf{u} = 0 \mathbf{u} = \mathbf{0}.
\]

23. We verify the axioms A1-A10 for a vector space.
A1: If \( a_0 + a_1x + \cdots + a_nx^n \) and \( b_0 + b_1x + \cdots + b_nx^n \) belong to \( P_n \), then
\[
(a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n,
\]
which again belongs to \( P_n \). Therefore, \( P_n \) is closed under addition.
A2: If \( a_0 + a_1x + \cdots + a_nx^n \) and \( r \) is a scalar, then
\[
r \cdot (a_0 + a_1x + \cdots + a_nx^n) = (ra_0) + (ra_1)x + \cdots + (ra_n)x^n,
\]
which again belongs to \( P_n \). Therefore, \( P_n \) is closed under scalar multiplication.
A3: Let \( p(x) = a_0 + a_1x + \cdots + a_nx^n \) and \( q(x) = b_0 + b_1x + \cdots + b_nx^n \) belong to \( P_n \). Then
\[
p(x) + q(x) = (a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n
\]
and it is readily verified that this satisfies commutativity under addition.
A4: Let \( p(x) = a_0 + a_1x + \cdots + a_nx^n \), \( q(x) = b_0 + b_1x + \cdots + b_nx^n \), and \( r(x) = c_0 + c_1x + \cdots + c_nx^n \) belong to \( P_n \). Then
\[
[p(x) + q(x)] + r(x) = [(a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n)] + (c_0 + c_1x + \cdots + c_nx^n) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n + (c_0 + c_1x + \cdots + c_nx^n)
\]
so \( P_n \) satisfies associativity under addition.
A5: The zero vector is the zero polynomial \( z(x) = 0 + 0 \cdot x + \cdots + 0 \cdot x^n \), and it is readily verified that this polynomial satisfies \( z(x) + p(x) = p(x) + z(x) \) for all \( p(x) \in P_n \).
A6: The additive inverse of \( p(x) = a_0 + a_1x + \cdots + a_nx^n \) is
\[
-p(x) = (-a_0) + (-a_1)x + \cdots + (-a_n)x^n.
\]
It is readily verified that \( p(x) + (-p(x)) = z(x) \), where \( z(x) \) is defined in A5.
A7: We have
\[
1 \cdot (a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1x + \cdots + a_nx^n,
\]
which demonstrates the unit property in $P_n$.

A8: Let $r, s \in \mathbb{R}$, and $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in P_n$. Then

$$(rs) \cdot p(x) = (rs) \cdot (a_0 + a_1 x + \cdots + a_n x^n)$$

$$= [(rs)a_0] + [(rs)a_1] x + \cdots + [(rs)a_n] x^n$$

$$= r[(sa_0) + (sa_1) x + \cdots + (sa_n) x^n]$$

$$= r[s(a_0 + a_1 x + \cdots + a_n x^n)]$$

$$= r \cdot (s \cdot p(x)),$$

which verifies the associativity of scalar multiplication.

A9: Let $r \in \mathbb{R}$, let $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in P_n$, and let $q(x) = b_0 + b_1 x + \cdots + b_n x^n \in P_n$. Then

$$r \cdot (p(x) + q(x)) = r \cdot ((a_0 + a_1 x + \cdots + a_n x^n) + (b_0 + b_1 x + \cdots + b_n x^n))$$

$$= r \cdot [(a_0 + b_0) + (a_1 + b_1) x + \cdots + (a_n + b_n) x^n]$$

$$= [r(a_0 + b_0)] + [r(a_1 + b_1)] x + \cdots + [r(a_n + b_n)] x^n$$

$$= [(ra_0) + (ra_1) x + \cdots + (ra_n) x^n] + [(rb_0) + (rb_1) x + \cdots + (rb_n) x^n]$$

$$= [r(a_0 + a_1 x + \cdots + a_n x^n)] + [r(b_0 + b_1 x + \cdots + b_n x^n)]$$

$$= r \cdot p(x) + r \cdot q(x),$$

which verifies the distributivity of scalar multiplication over vector addition.

A10: Let $r, s \in \mathbb{R}$ and let $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in P_n$. Then

$$(r + s) \cdot p(x) = (r + s) \cdot (a_0 + a_1 x + \cdots + a_n x^n)$$

$$= [(r + s)a_0] + [(r + s)a_1] x + \cdots + [(r + s)a_n] x^n$$

$$= [ra_0 + ra_1 x + \cdots + ra_n x^n] + [sa_0 + sa_1 x + \cdots + sa_n x^n]$$

$$= r(a_0 + a_1 x + \cdots + a_n x^n) + s(a_0 + a_1 x + \cdots + a_n x^n)$$

$$= r \cdot p(x) + s \cdot p(x),$$

which verifies the distributivity of scalar multiplication over scalar addition.

The above verification of axioms A1-A10 shows that $P_n$ is a vector space.

**Solutions to Section 4.3**

**True-False Review:**

1. **FALSE.** The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$, not $\mathbb{R}^m$.

2. **TRUE.** If $b = 0$, then the line is $y = mx$, which is a line through the origin of $\mathbb{R}^2$, a one-dimensional subspace of $\mathbb{R}^2$. On the other hand, if $b \neq 0$, then the origin does not lie on the given line, and therefore since the line does not contain the zero vector, it cannot form a subspace of $\mathbb{R}^2$ in this case.

3. **TRUE.** Choosing any vector $v$ in $S$, the scalar multiple $0v = 0$ still belongs to $S$.

4. **FALSE.** This set is not closed under addition. For instance, the point $(1,1,0)$ lies in the $xy$-plane, the point $(0,1,1)$ lies in the $yz$-plane, but

$$(1,1,0) + (0,1,1) = (1,2,1)$$

does not belong to $S$. Therefore, $S$ is not a subspace of $V$. 

Problems:

1. \( S = \{ \mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = (2k, -3k), k \in \mathbb{R} \} \).

(a) \( S \) is certainly nonempty. Let \( \mathbf{x}, \mathbf{y} \in S \). Then for some \( r, s \in \mathbb{R} \),

\[ \mathbf{x} = (2r, -3r) \quad \text{and} \quad \mathbf{y} = (2s, -3s). \]

Hence,

\[ \mathbf{x} + \mathbf{y} = (2r, -3r) + (2s, -3s) = (2(r + s), -3(r + s)) = (2k, -3k), \]

where \( k = r + s \). Consequently, \( S \) is closed under addition. Further, if \( c \in \mathbb{R} \), then

\[ c\mathbf{x} = c(2r, -3r) = (2cr, -3cr) = (2t, -3t), \]

where \( t = cr \). Therefore \( S \) is also closed under scalar multiplication. It follows from Theorem 4.3.2 that \( S \) is a subspace of \( \mathbb{R}^2 \).

(b) The subspace \( S \) consists of all points lying along the line in the accompanying figure.

\[ y = 3x/2 \]

Figure 0.0.32: Figure for Exercise 1(b)

3. \( S = \{ (x, y) \in \mathbb{R}^2 : 3x + 2y = 0 \} \). \( S \neq \emptyset \) since \((0, 0) \in S\).

Closure under Addition: Let \((x_1, x_2), (y_1, y_2) \in S \). Then \( 3x_1 + 2x_2 = 0 \) and \( 3y_1 + 2y_2 = 0 \), so \( 3(x_1 + y_1) + 2(x_2 + y_2) = 0 \), which implies that \((x_1 + y_1, x_2 + y_2) \in S \).

Closure under Scalar Multiplication: Let \( a \in \mathbb{R} \) and \((x_1, x_2) \in S \). Then \( 3x_1 + 2x_2 = 0 \implies a(3x_1 + 2x_2) = a \cdot 0 \implies 3(ax_1) + 2(ax_2) = 0 \), which shows that \((ax_1, ax_2) \in S \).

Thus, \( S \) is a subspace of \( \mathbb{R}^2 \) by Theorem 4.3.2.

5. \( S = \{ (x, y, z) \in \mathbb{R}^3 : x + y + z = 1 \} \) is not a subspace of \( \mathbb{R}^3 \) because \((0, 0, 0) \notin S \) since \( 0 + 0 + 0 \neq 1 \).

7. \( S = \{ (x, y) \in \mathbb{R}^2 : x^2 - y^2 = 0 \} \) is not a subspace of \( \mathbb{R}^2 \), since it is not closed under addition, as we now observe: If \((x_1, y_1), (x_2, y_2) \in S \), then \((x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \).

\[
\begin{align*}
(x_1 + x_2)^2 - (y_1 + y_2)^2 &= x_1^2 + 2x_1x_2 + x_2^2 - (y_1^2 + 2y_1y_2 + y_2^2) \\
&= (x_1^2 - y_1^2) + (x_2^2 - y_2^2) + 2(x_1x_2 - y_1y_2) \\
&= 0 + 0 + 2(x_1x_2 - y_1y_2) \\
&= 0 + 0 + 2 \neq 0, \text{ in general.}
\end{align*}
\]

Thus, \((x_1, y_1) + (x_2, y_2) \notin S \).
9. \( S = \{ A = [a_{ij}] \in M_n(\mathbb{R}) : a_{ij} = 0 \text{ whenever } i < j \} \). Note that \( S \neq \emptyset \) since \( 0_n \in S \). Now let \( A = [a_{ij}] \) and \( B = [b_{ij}] \) be lower triangular matrices. Then \( a_{ij} = 0 \) and \( b_{ij} = 0 \) whenever \( i < j \). Then \( a_{ij} + b_{ij} = 0 \) and \( ca_{ij} = 0 \) whenever \( i < j \). Hence \( A + B = [a_{ij} + b_{ij}] \) and \( cA = [ca_{ij}] \) are also lower triangular matrices. Therefore \( S \) is closed under addition and scalar multiplication. Consequently, \( S \) is a subspace of \( M_2(\mathbb{R}) \) by Theorem 4.3.2.

11. \( S = \{ A \in M_2(\mathbb{R}) : A^T = A \} \). \( S \neq \emptyset \) since \( 0_2 \in S \).

**Closure under Addition:** If \( A, B \in S \), then \( (A + B)^T = A^T + B^T = A + B \), which shows that \( A + B \in S \).

**Closure under Scalar Multiplication:** If \( r \in \mathbb{R} \) and \( A \in S \), then \( (rA)^T = rA^T = rA \), which shows that \( rA \in S \). Consequently, \( S \) is a subspace of \( M_2(\mathbb{R}) \) by Theorem 4.3.2.

13. \( S = \{ f \in V : f(a) = f(b) \} \), where \( V \) is the vector space of all real-valued functions defined on \( [a, b] \). Note that \( S \neq \emptyset \) since the zero function \( O(x) = 0 \) for all \( x \) belongs to \( S \).

**Closure under Addition:** If \( f, g \in S \), then \( (f + g)(a) = f(a) + g(a) = f(b) + g(b) = (f + g)(b) \), which shows that \( f + g \in S \).

**Closure under Scalar Multiplication:** If \( k \in \mathbb{R} \) and \( f \in S \), then \( (kf)(a) = kf(a) = kf(b) = (kf)(b) \), which shows that \( kf \in S \).

Therefore \( S \) is a subspace of \( V \) by Theorem 4.3.2.

15. \( S = \{ f \in V : f(-x) = f(x) \text{ for all } x \in \mathbb{R} \} \). Note that \( S \neq \emptyset \) since the zero function \( O(x) = 0 \) for all \( x \) belongs to \( S \). Let \( f, g \in S \). Then

\[
(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)
\]

and if \( c \in \mathbb{R} \), then

\[
(cf)(-x) = cf(-x) = cf(x) = (cf)(x),
\]

so \( f + g \) and \( c \cdot f \) belong to \( S \). Therefore, \( S \) is closed under addition and scalar multiplication. Therefore, \( S \) is a subspace of \( V \) by Theorem 4.3.2.

17. \( S = \{ p \in P_2 : p(x) = ax^2 + 1, a \in \mathbb{R} \} \). We claim that \( S \) is not closed under addition:

**Not Closed under Addition:** Let \( p, q \in S \). Then for some \( a_1, a_2 \in \mathbb{R} \),

\[
p(x) = a_1x^2 + 1 \quad \text{and} \quad q(x) = a_2x^2 + 1.
\]

Hence,

\[
(p + q)(x) = p(x) + q(x) = (a_1 + a_2)x^2 + 2 = ax^2 + 2
\]

where \( a = a_1 + a_2 \). Consequently, \( p + q \notin S \), and therefore \( S \) is not closed under addition. It follows that \( S \) is not a subspace of \( P_2 \).

19. \( S = \{ y \in C^2(\mathbb{I}) : y'' + 2y' - y = 1 \} \). \( S \) is not a subspace of \( V \).

We show that \( S \) fails to be closed under addition (one can also verify that it is not closed under scalar multiplication, but this is unnecessary if one shows the failure of closure under addition):

**Not Closed under Addition:** Let \( y_1, y_2 \in S \).

\[
(y_1 + y_2)'' + 2(y_1 + y_2)' - (y_1 + y_2) = y_1'' + y_2'' + 2(y_1' + y_2') - y_1 - y_2
\]

\[
= (y_1'' + 2y_1' - y_1) + (y_2'' + 2y_2' - y_2) \quad \text{Thus, } y_1 + y_2 \notin S.
\]

Or alternatively:

**Not Closed under Scalar Multiplication:** Let \( k \in \mathbb{R} \) and \( y_1 \in S \).
\[(ky_1)'' + 2(ky_1)' - (ky_1) = ky_1'' + 2ky_1' - ky_1 = k(y_1'' + 2y_1' - y_1) = k \cdot 1 = k \neq 1, \text{ unless } k = 1. \] 
Therefore, \( ky_1 \not\in S \) unless \( k = 1 \).

21. \( A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 3 & 10 & -4 & 6 \\ 2 & 5 & -6 & -1 \end{bmatrix} \). \( \text{nullspace}(A) = \{ x \in \mathbb{R}^3 : Ax = 0 \} \). The RREF of the augmented matrix of the system \( Ax = 0 \) is \( \begin{bmatrix} 1 & 0 & -8 & -8 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \), so that \( \text{nullspace}(A) = \{ (8r + 8s, -2r - 3s, r, s) : r, s \in \mathbb{R} \} \).

23. Since the zero function \( y(x) = 0 \) for all \( x \in I \) is not a solution to the differential equation, the set of all solutions does not contain the zero vector from \( C^2(I) \), hence it is not a vector space at all and cannot be a subspace.

### Solutions to Section 4.4

**True-False Review:**

1. **TRUE.** By its very definition, when a linear span of a set of vectors is formed, that span becomes closed under addition and under scalar multiplication. Therefore, it is a subspace of \( V \).

3. **TRUE.** Every vector in \( V \) can be expressed as a linear combination of the vectors in \( S \), and therefore, it is also true that every vector in \( W \) can be expressed as a linear combination of the vectors in \( S \). Therefore, \( S \) spans \( W \), and \( S \) is a spanning set for \( W \).

5. **TRUE.** To say that a set \( S \) of vectors in \( V \) spans \( V \) is to say that every vector in \( V \) belongs to \( \text{span}(S) \). So \( V \) is a subset of \( \text{span}(S) \). But of course, every vector in \( \text{span}(S) \) belongs to the vector space \( V \), and so \( \text{span}(S) \) is a subset of \( V \). Therefore, \( \text{span}(S) = V \).

7. **FALSE.** There are vector spaces that do not contain finite spanning sets. For instance, if \( V \) is the vector space consisting of all polynomials with coefficients in \( \mathbb{R} \), then since a finite spanning set could not contain polynomials of arbitrarily large degree, no finite spanning set is possible for \( V \).

9. **TRUE.** The general matrix \( \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \) in this vector space can be written as \( aE_{11} + bE_{12} + cE_{13} + dE_{22} + eE_{23} + fE_{33} \), and therefore the matrices in the set \{ \( E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33} \) \} span the vector space.

11. **FALSE.** For instance, consider \( m = 2 \) and \( n = 3 \). Then one spanning set for \( \mathbb{R}^2 \) is \{ \( (1, 0), (0, 1), (1, 1), (2, 2) \) \}, which consists of four vectors. On the other hand, one spanning set for \( \mathbb{R}^3 \) is \{ \( (1, 0, 0), (0, 1, 0), (0, 0, 1) \) \}, which consists of only three vectors.

**Problems:**

1. \{ \( (1, -1), (2, -2), (2, 3) \) \}. Since \( \mathbf{v}_1 = (1, -1) \), and \( \mathbf{v}_2 = (2, 3) \) are noncolinear, the given set of vectors does span \( \mathbb{R}^2 \). (See the comment preceding Example 4.4.3 in the text.)

3. The three vectors in the given set are all collinear. Consequently, the set of vectors does not span \( \mathbb{R}^2 \).

5. Since \( \begin{vmatrix} 1 & 2 & 4 \\ -2 & 3 & -1 \\ 1 & 1 & 2 \end{vmatrix} = -7 \neq 0 \), the given vectors are not coplanar, and therefore span \( \mathbb{R}^3 \). Note that we can simply ignore the zero vector \( (0, 0, 0) \).
7. Since \[
\begin{vmatrix}
1 & 3 & 4 \\
2 & 4 & 5 \\
3 & 5 & 6
\end{vmatrix}
= 0,
\] the vectors are coplanar, and therefore the given set does not span \( \mathbb{R}^3 \). The linear span of the vectors is those points \((x, y, z)\) for which the system
\[
c_1(1, 2, 3) + c_2(3, 4, 5) + c_3(4, 5, 6) = (x, y, z)
\]
is consistent. Reducing the augmented matrix
\[
\begin{bmatrix}
1 & 3 & 4 & x \\
2 & 4 & 5 & y \\
3 & 5 & 6 & z
\end{bmatrix}
\]
of this system yields
\[
\begin{bmatrix}
1 & 3 & 4 & x \\
0 & 2 & 3 & 2x - y \\
0 & 0 & 0 & x - 2y + z
\end{bmatrix}.
\]
This system is consistent if and only if \( x - 2y + z = 0 \). Consequently, the linear span of the given set of vectors consists of all points lying on the plane with the equation \( x - 2y + z = 0 \).

9. Let \((x, y, z) \in \mathbb{R}^3\) and \(a, b, c \in \mathbb{R}\).
\((x, y, z) = av_1 + bv_2 + cv_3 = a(-1, 3, 2) + b(1, -2, 1) + c(2,1,1)
\]
\[= (-a, 3a, 2a) + (b, -2b, b) + (2c, c, c)
\]
\[= (-a + b + 2c, 3a - 2b + c, 2a + b + c).
\]
These equalities result in the system:
\[
\begin{align*}
a + b + 2c &= x \\
3a - 2b + c &= y \\
2a + b + c &= z
\end{align*}
\]
Upon solving for \(a, b\), and \(c\) we obtain
\[
a = \frac{-3x + y + 5z}{16}, \quad b = \frac{-x - 5y + 7z}{16}, \quad \text{and} \quad c = \frac{7x + 3y - z}{16}.
\]
Consequently, \(\{v_1, v_2, v_3\}\) spans \(\mathbb{R}^3\), and
\[
(x, y, z) = \left(\frac{-3x + y + 5z}{16}\right)v_1 + \left(\frac{-x - 5y + 7z}{16}\right)v_2 + \left(\frac{7x + 3y - z}{16}\right)v_3.
\]

11. \(\mathbf{x} = (c_1, c_2, c_2 - 2c_1) = (c_1, 0, -2c_1) + (0, c_2, c_2) = c_1(1, 0, -2) + c_2(0,1,1) = c_1v_1 + c_2v_2\). Thus, \(\{(1,0,-2), (0,1,1)\}\) spans \(S\).

13. \(x - 2y - z = 0 \implies x = 2y + z\), so \(v \in \mathbb{R}^3\).
\[
\implies v = (2y + z, y, z) = (2y, y, 0) + (z, 0, z) = y(2,1,0) + z(1,0,1).
\]
Therefore \(S = \{v \in \mathbb{R}^3 : v = a(2, 1, 0) + b(1, 0, 1), a, b \in \mathbb{R}\}\), hence \(\{(2,1,0), (1,0,1)\}\) spans \(S\).

15. \(A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 3 & 4 & 2 \\ 2 & 4 & 6 & -1 \end{bmatrix}\). nullspace\(A = \{x \in \mathbb{R}^4 : Ax = 0\}\). The RREF of the augmented matrix of this system is
\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]
Consequently, nullspace\(A = \{r(-1, -1, 1, 0) : r \in \mathbb{R}\} = \text{span}\{-(-1, -1, 1, 0)\}\).

17. \(\mathcal{A} = \{A \in M_2(\mathbb{R}) : A = \begin{bmatrix} -\alpha & 0 \\ \alpha & 0 \end{bmatrix}, \alpha \in \mathbb{R}\} = \{A \in M_2(\mathbb{R}) : A = \alpha \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \} = \text{span}\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}\).

19. Let \(v \in \text{span}\{v_1, v_2\}\) and \(a, b \in \mathbb{R}\).
\[
v = av_1 + bv_2 = a(1, -1, 2) + b(2, -1, 3) = (a - a, 2a) + (2b, -b, 3b) = (a + 2b, -a - b, 2a + 3b).
\]
Thus, \(\text{span}\{v_1, v_2\} = \{v \in \mathbb{R}^3 : v = (a + 2b, -a - b, 2a + 3b), a, b \in \mathbb{R}\} \).
Geometrically, span\(\{v_1, v_2\}\) is the plane through the origin determined by the two given vectors. The plane has parametric equations \(x = a + 2b, y = -a - b, z = 2a + 3b\). If \(a, b, c\) are eliminated from the equations, then the resulting Cartesian equation is given by \(x - y - z = 0\).

21. Let \(v \in \text{span}\{v_1, v_2, v_3\}\) and \(a, b, c \in \mathbb{R}\).
\[
v = av_1 + bv_2 + cv_3 = (a(1, 1, -1) + b(2, 1, 3) + c(-2, -2, 2) = (a, -a) + (2b, b, 3b) + (-2c, -2c, 2c).
\]

Assuming that \(v = (x, y, z)\) and using the last ordered triple, we obtain the system:
\[
\begin{align*}
  a + 2b - 2c = & x \\
  a + b - 2c = & y \\
  -a + 3b + 2c = & z
\end{align*}
\]

Performing Gauss-Jordan elimination on the augmented matrix of the system, we obtain:
\[
\begin{bmatrix}
  1 & 2 & -2 & | & x \\
  1 & 1 & -2 & | & y \\
  -1 & 3 & 2 & | & z
\end{bmatrix}
\sim
\begin{bmatrix}
  1 & 2 & -2 & | & x \\
  0 & -1 & 0 & | & y - x \\
  0 & 5 & 0 & | & x + z
\end{bmatrix}
\sim
\begin{bmatrix}
  1 & 0 & -2 & | & 2y - x \\
  0 & 1 & 0 & | & x - y \\
  0 & 0 & 1 & | & 5y - 4x + z
\end{bmatrix}
\]

It is clear from the last matrix that the subspace, \(S\), of \(\mathbb{R}^3\) is a plane through \((0, 0, 0)\) with Cartesian equation \(4x - 5y - z = 0\). Moreover, \(\{v_1, v_2\}\) also spans the subspace \(S\) since
\[
v = av_1 + bv_2 + cv_3 = a(1, 1, -1) + b(2, 1, 3) + c(-2, -2, 2) = a(1, 1, -1) + b(2, 1, 3) - 2c(1, 1, -1)
\]
\[
= (a - 2c)(1, 1, -1) + b(2, 1, 3) = dv_1 + bv_2 \text{ where } d = a - 2c \in \mathbb{R}.
\]

23. If \(v \in \text{span}\{v_1, v_2\}\) then there exist \(a, b \in \mathbb{R}\) such that
\[
v = av_1 + bv_2 = a(-1, 1, 2) + b(3, 1, -4) = (-a, 2a) + (3b, b, -4b) = (-a + 3b, a + b, 2a - 4b).
\]

Hence \(v = (5, 3, -6)\) is in \(\text{span}\{v_1, v_2\}\) provided there exists \(a, b \in \mathbb{R}\) satisfying the system:
\[
\begin{align*}
  -a + 3b &= 5 \\
  a + b &= 3 \\
  2a - 4b &= -6
\end{align*}
\]

Solving this system we find that \(a = 1\) and \(b = 2\).
Consequently, \(v = v_1 + 2v_2\) so that \((5, 3, -6) \in \text{span}\{v_1, v_2\}\).

25. If \(p \in \text{span}\{p_1, p_2\}\) then there exist \(a, b \in \mathbb{R}\) such that \(p(x) = ap_1(x) + bp_2(x)\), so \(p(x) = 2x^2 - x + 2\) is in \(\text{span}\{p_1, p_2\}\) provided there exist \(a, b \in \mathbb{R}\) such that
\[
2x^2 - x + 2 = a(x - 4) + b(x^2 - x + 3)
\]
\[
= ax - 4a + bx^2 - bx + 3b
\]
\[
= bx^2 + (a - b)x + (3b - 4a).
\]

Equating like coefficients and solving, we find that \(a = 1\) and \(b = 2\).
Thus, \(2x^2 - x + 2 = 1 \cdot (x - 4) + 2 \cdot (x^2 - x + 3) = p_1(x) + 2p_2(x)\) so \(p \in \text{span}\{p_1, p_2\}\).

27. Let \(A \in \text{span}\{A_1, A_2\}\) and \(a, b \in \mathbb{R}\).
\[
A = aA_1 + bA_2 = a \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + b \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a & -a \\ -a & 3a \end{bmatrix} + \begin{bmatrix} -2b & 2b \\ b & -b \end{bmatrix} = \begin{bmatrix} a - 2b & 2a + b \\ -a + b & 3a - b \end{bmatrix}.
\]

So \(\text{span}\{A_1, A_2\} = \{A \in M_2(\mathbb{R}) : A = \begin{bmatrix} a - 2b & 2a + b \\ -a + b & 3a - b \end{bmatrix}\}\). Now, to determine whether \(B \in \text{span}\{A_1, A_2\}\), let \(\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}\). This implies that \(a = 1\) and \(b = -1\), thus \(B \in \text{span}\{A_1, A_2\}\).

29. The origin in \(\mathbb{R}^3\).
31. All points lying on the plane through the origin containing \(v_1\) and \(v_2\).

33. Suppose that \(S\) is a subset of \(S'\). We must show that every vector in \(\text{span}(S)\) also belongs to \(\text{span}(S')\). Every vector \(v\) that lies in \(\text{span}(S)\) can be expressed as \(v = c_1v_1 + c_2v_2 + \cdots + c_kv_k\), where \(v_1, v_2, \ldots, v_k\) belong to \(S\). However, since \(S\) is a subset of \(S'\), \(v_1, v_2, \ldots, v_k\) also belong to \(S'\), and therefore, \(v\) belongs to \(\text{span}(S')\). Thus, we have shown that every vector in \(\text{span}(S)\) also lies in \(\text{span}(S')\).

Solutions to Section 4.5

1. FALSE. For instance, consider the vector space \(V = \mathbb{R}^2\). Here are two different minimal spanning sets for \(V\):

\[\{(1, 0), (0, 1)\} \quad \text{and} \quad \{(1, 0), (1, 1)\}\]

Many other examples of this abound.

3. FALSE. For instance, the \(7 \times 5\) zero matrix, \(0_{7 \times 5}\), does not have linearly independent columns.

5. TRUE. This is stated in Theorem 4.5.21.

7. TRUE. This is a rephrasing of the statement in True-False Review Question 5 above.

9. FALSE. The illustration given in part (c) of Example 4.5.22 gives an excellent case-in-point here.

Problems:

1. \(\{(1, -1), (1, 1)\}\). These vectors are elements of \(\mathbb{R}^2\). Since there are two vectors, and the dimension of \(\mathbb{R}^2\) is two, Corollary 4.5.15 states that the vectors will be linearly dependent if and only if \[\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 0\]. Now \[\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \neq 0\]. Consequently, the given vectors are linearly independent.

3. \(\{(1, -1, 0), (0, 1, -1), (1, 1, 1)\}\). These vectors are elements of \(\mathbb{R}^3\).

\[\begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 3 \neq 0\], so by Corollary 4.5.15, the vectors are linearly independent.

5. Given \(\{(-2, 4, -6), (3, -6, 9)\}\). The vectors are linearly dependent because \(3(-2, 4, -6) + 2(3, -6, 9) = (0, 0, 0)\), which gives a linear dependency relation.

Alternatively, let \(a, b \in \mathbb{R}\) and observe that
\[a(-2, 4, -6) + b(3, -6, 9) = (0, 0, 0) \implies (-2a, 4a, -6a) + (3b, -6b, 9b) = (0, 0, 0)\].

The last equality results in the system:
\[
\begin{align*}
-2a + 3b &= 0 \\
4a - 6b &= 0 \\
-6a + 9b &= 0
\end{align*}
\]

The RREF of the augmented matrix of this system is
\[
\begin{bmatrix}
1 & -3/2 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

which implies that \(a = \frac{3}{2}b\). Thus, the given set of vectors is linearly dependent.

7. \(\{(-1, 1, 2), (0, 2, -1), (3, 1, 2), (-1, -1, 1)\}\). These vectors are elements of \(\mathbb{R}^3\).

Since there are four vectors, it follows from Corollary 4.5.15 that the vectors are linearly dependent. Let \(v_1 = (-1, 1, 2), v_2 = (0, 2, -1), v_3 = (3, 1, 2), v_4 = (-1, -1, 1)\). Then
\[c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0\]
By CET (row 1), we obtain:

(b) By inspection, we see that since

\[
11. \text{(a)}
\]

Thus, it follows from Corollary 4.5.15 that the vectors are linearly independent.

9. \{(2, -1, 0, 1), (1, 0, -1, 2), (0, 3, 1, 2), (-1, 1, 2, 1)\}. These vectors are elements in \(\mathbb{R}^4\).

By CET (row 1), we obtain:

\[
\begin{vmatrix}
2 & 1 & 0 & -1 \\
-1 & 0 & 3 & 1 \\
0 & -1 & 1 & 2 \\
1 & 2 & 2 & 1
\end{vmatrix}
= 2
\begin{vmatrix}
0 & 3 & 1 \\
-1 & 1 & 2 \\
2 & 1
\end{vmatrix}
- \begin{vmatrix}
1 & 3 & 1 \\
0 & -1 & 1 \\
1 & 2 & 2
\end{vmatrix}
= 21 \neq 0.
\]

Thus, it follows from Corollary 4.5.15 that the vectors are linearly independent.

11. (a) Since \[
\begin{vmatrix}
2 & 1 & 0 & -1 \\
-1 & 0 & 3 & 1 \\
0 & -1 & 1 & 2 \\
1 & 2 & 2 & 1
\end{vmatrix}
= 0,
\]
the given set of vectors is linearly dependent.

(b) By inspection, we see that \(v_3 = -3v_2\). Hence \(v_2\) and \(v_3\) are colinear and therefore \(\text{span}\{v_2, v_3\}\) is the line through the origin that has the direction of \(v_2\). Further, since \(v_1\) is not proportional to either of these vectors, it does not lie along the same line, hence \(v_1\) is not in \(\text{span}\{v_2, v_3\}\).

13. \{(1, 0, 1, k), (-1, 0, k, 1), (2, 0, 1, 3)\}. These vectors are elements in \(\mathbb{R}^4\).

Let \(a, b, c \in \mathbb{R}\).

\[
a(1, 0, 1, k) + b(-1, 0, k, 1) + c(2, 0, 1, 3) = (0, 0, 0, 0)
\]

\[
\implies (a, 0, a, ka) + (-b, 0, kb, b) + (2c, 0, c, 3c) = (0, 0, 0, 0)
\]

\[
\implies (a - b + 2c, 0, a + kb + c, ka + b + 3c) = (0, 0, 0, 0).
\]

The last equality results in the system:

\[
\begin{align*}
a - b + 2c &= 0 \\ a + kb + c &= 0
\end{align*}
\]

Evaluating the determinant of the coefficient matrix, we obtain

\[
\begin{vmatrix}
1 & -1 & 2 \\
1 & k & 1 \\
k & k + 1 & 3 - 2k
\end{vmatrix}
= 2(k + 1)(2 - k).
\]

Consequently, the system has only the trivial solution, hence the given set of vectors are linearly independent if and only if \(k \neq 2, -1\).

15. Let \(a, b, c \in \mathbb{R}\).

\[
a \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
+ b \begin{bmatrix}
2 & -1 \\
0 & 1
\end{bmatrix}
+ c \begin{bmatrix}
3 & 6 \\
0 & 4
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
\implies \begin{bmatrix}
a + 2b + 3c & a - b + 6c \\
0 & a + b + 4c
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

The last equation results in the system:

\[
\begin{align*}
a + 2b + 3c &= 0 \\ a - b + 6c &= 0 \\ a + b + 4c &= 0
\end{align*}
\]

The RREF of the augmented matrix of this system is

\[
\begin{bmatrix}
1 & 0 & 5 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
which implies that the system has an infinite number of solutions. Consequently, the given matrices are linearly dependent.
17. Let \(a, b, c \in \mathbb{R}\). \(a \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + b \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} + c \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\)

\[\Rightarrow \begin{bmatrix} a - b + 2c & b + c \\ a + 2b + 5c & 2a + b + 7c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.\] The last equation results in the system:

\[
\begin{cases}
2a + 4b = 0 \\
3a + 6b = 0 \\
b + c = 0
\end{cases}
\]

The RREF of the augmented matrix of this system is \(\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\), which implies that the system has an infinite number of solutions. Consequently, the given matrices are linearly dependent.

19. Let \(a, b \in \mathbb{R}\). \(ap_1(x) + bp_2(x) = 0 \Rightarrow a(2 + 3x) + b(4 + 6x) = 0 \Rightarrow (2a + 4b) + (3a + 6b)x = 0.\)

Equating like coefficients, we obtain the system:

\[
\begin{cases}
2a + 4b = 0 \\
3a + 6b = 0 \\
b + c = 0
\end{cases}
\]

The RREF of the augmented matrix of this system is \(\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\), which implies that the system has an infinite number of solutions. Thus, the given vectors are linearly dependent.

21. Since \(\cos 2x = \cos^2 x - \sin^2 x, f_1(x) = f_3(x) - f_2(x)\) so it follows that \(f_1, f_2,\) and \(f_3\) are linearly dependent in \(C^\infty(-\infty, \infty).\)

23. Let \(v_1 = (1, 2, -1), v_2 = (3, 1, 5), v_3 = (0, 0, 0), v_4 = (-1, 2, 3).\) Since \(v_3 = 0,\) it is certainly true that \(\text{span}\{v_1, v_2, v_3\} = \text{span}\{v_1, v_2, v_3, v_4\}.\) Further, since \(\det[v_1, v_2, v_3] = -42 \neq 0,\) \(\{v_1, v_2, v_3\}\) is a linearly independent set.

25. Let \(v_1 = (1, 1, -1, 1), v_2 = (2, -1, 3, 1), v_3 = (1, 1, 2, 1), v_4 = (2, -1, 2, 1).\)

Since

\[
\begin{vmatrix}
1 & 2 & 1 & 2 \\
1 & -1 & 1 & -1 \\
-1 & 3 & 2 & 2 \\
1 & 1 & 1 & 1
\end{vmatrix}
= 0,
\]

the set \(\{v_1, v_2, v_3, v_4\}\) is linearly dependent. We now determine the linearly dependent relationship. The RREF of the augmented matrix corresponding to the system

\[
c_1(1, 1, -1, 1) + c_2(2, -1, 3, 1) + c_3(1, 1, 2, 1) + c_4(2, -1, 2, 1) = (0, 0, 0, 0)
\]

is

\[
\begin{bmatrix}
1 & 0 & 0 & \frac{1}{3} \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & -\frac{1}{3} \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

that a linearly dependent relationship between the given set of vectors is

\[-v_1 - 3v_2 + v_3 + 3v_4 = 0\]

so that

\[v_1 = -3v_2 + v_3 + 3v_4.\]

Consequently, \(\text{span}\{v_2, v_3, v_4\} = \text{span}\{v_1, v_2, v_3, v_4\},\) and \(\{v_2, v_3, v_4\}\) is a linearly independent set.

27. We first determine whether the given set of polynomials is linearly dependent. Let

\[p_1(x) = 2 - 5x,\ p_2(x) = 3 + 7x,\ p_3(x) = 4 - x.\]
Then
\[ c_1(2 - 5x) + c_2(3 + 7x) + c_3(4 - x) = 0 \]
requires
\[ 2c_1 + 3c_2 + 4c_3 = 0 \quad \text{and} \quad -5c_1 + 7c_2 - c_3 = 0. \]
This system has solution \((-31r, -18r, 29r)\), where \(r\) is a free variable. Consequently, the given set of polynomials is linearly dependent, and a linearly dependent relationship is
\[ -31p_1(x) - 18p_2(x) + 29p_3(x) = 0, \]
or equivalently,
\[ p_3(x) = \frac{1}{29}[31p_1(x) + 18p_2(x)]. \]

Hence, the linearly independent set of vectors \(\{2 - 5x, 3 + 7x, 4 - x\}\) spans the same subspace of \(P_1\) as that spanned by \(\{2 - 5x, 3 + 7x, 4 - x\}\).

29. \(W[f_1, f_2, f_3](x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2.\) Since \(W[f_1, f_2, f_3](x) \neq 0\) on \(I\), it follows that the functions are linearly independent on \(I\).

31. \(W[f_1, f_2, f_3](x) = \begin{vmatrix} 1 & 3x & x^2 - 1 \\ 0 & 3 & 2x \\ 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2x \\ 0 & 2 \end{vmatrix} = 6 \neq 0\) on \(I\). Consequently, \(\{f_1, f_2, f_3\}\) is a linearly independent set on \(I\) by Theorem 4.5.21.

33. On \([0, \infty), f_2 = 7f_1\), so that the functions are linearly dependent on this interval. Therefore \(W[f_1, f_2](x) = 0\) for \(x \in [0, \infty)\). However, on \((-\infty, 0)\), \(W[f_1, f_2](x) = \begin{vmatrix} 3x^3 & 7x^2 \\ 9x^2 & 14x \end{vmatrix} = -21x^4 \neq 0.\) Since the Wronskian is not zero for all \(x \in (-\infty, \infty)\), the functions are linearly independent on that interval.

35. We show that the Wronskian (the determinant can be computed by cofactor expansion along the first row) is identically zero:
\[
\begin{vmatrix}
e^x & e^{-x} & \cosh x \\
e^x & -e^{-x} & \sinh x \\
e^x & e^{-x} & \cosh x
\end{vmatrix} = -(\cosh x + \sinh x) - (\cosh x - \sinh x) + 2 \cosh x = 0.
\]
Thus, the Wronskian is identically zero on \((-\infty, \infty)\). Furthermore, \(\{f_1, f_2, f_3\}\) is a linearly dependent set because
\[-\frac{1}{2}f_1(x) - \frac{1}{2}f_2(x) + f_3(x) = -\frac{1}{2}e^x - \frac{1}{2}e^{-x} + \cosh x = -\frac{1}{2}e^x - \frac{1}{2}e^{-x} + \frac{e^x + e^{-x}}{2} = 0\] for all \(x \in I\).

37. (a) When \(x > 0, f_2'(x) = 1\) and when \(x < 0, f_2'(x) = -1;\) thus \(f_2'(0)\) does not exist, which implies that \(f_2 \notin C^1(-\infty, \infty)\).

(b) Let \(a, b \in \mathbb{R}\). On the interval \((-\infty, 0), ax + b(-x) = 0\), which has more than the trivial solution for \(a\) and \(b\). Thus, \(\{f_1, f_2\}\) is a linearly dependent set of vectors on \((-\infty, 0)\).

On the interval \([0, \infty), ax + bx = 0 \implies a + b = 0,\) which has more than the trivial solution for \(a\) and \(b). Therefore \(\{f_1, f_2\}\) is a linearly dependent set of vectors on \([0, \infty)\).

On the interval \((-\infty, \infty), a\) and \(b\) must satisfy: \(ax + b(-x) = 0\) and \(ax + bx = 0,\) that is,
\[
\begin{cases}
a - b = 0 \\
a + b = 0
\end{cases}
\] Since
this system has only \( a = b = 0 \) as its solution, \( \{ f_1, f_2 \} \) is a linearly independent set of vectors on \( (-\infty, \infty) \).

39. Let \( a, b, c \in \mathbb{R} \) and \( x \in (-\infty, \infty) \).

\[
a f_1(x) + b f_2(x) + c f_3(x) = 0 \implies \begin{cases} a(x - 1) + b(2x) + c(3) = 0 \text{ for } x \geq 1 \\
2a(x - 1) + b(2x) + c(3) = 0 \text{ for } x < 1 
\end{cases}
\]

Since the only solution to this system of equations is \( a = b = c = 0 \), it follows that the given functions are linearly independent on \( (-\infty, \infty) \).

The domain space may be divided into three types of intervals: (1) interval subsets of \( (-\infty, 1) \), (2) interval subset of \( [1, \infty) \), (3) intervals containing \( 1 \) where \( 1 \) is not an endpoint of the intervals.

For intervals of type (3):
Intervals such as type (3) are treated as above [with domain space of \( (-\infty, \infty) \)]: vectors are linearly independent.

For intervals of type (1):
\[
a(2(x - 1)) + b(2x) + c(3) = 0 \implies (2a + 2b)x + (-2a + 3c) = 0 \implies 2a + 2b = 0, \text{ and } -2a + 3c = 0.
\]

Since this system has three variables with only two equations, the solution to the system is not unique, hence intervals of type (1) result in linearly dependent vectors.

For intervals of type (2):
\[
a(x - 1) + b(2x) + c(3) = 0 \implies a + 2b = 0 \text{ and } -a + 3c = 0.
\]

As in the last case, this system has three variables with only two equations, so it must be the case that intervals of type (2) result in linearly dependent vectors.

41. (a) Let \( f_1(x) = e^{r_1 x}, f_2(x) = e^{r_2 x} \) and \( f_3(x) = e^{r_3 x} \). Then

\[
W[f_1, f_2, f_3](x) = e^{(r_1 + r_2 + r_3)x} (r_3 - r_1)(r_1 - r_2) \neq 0.
\]

If \( r_i \neq r_j \) for \( i \neq j \), then \( W[f_1, f_2, f_3](x) \) is never zero, and hence the functions are linearly independent on any interval. If, on the other hand, \( r_i = r_j \) with \( i \neq j \), then \( f_i - f_j = 0 \), so that the functions are linearly dependent. Thus, \( r_1, r_2, r_3 \) must all be different in order that \( f_1, f_2, f_3 \) are linearly independent.
We assume that \( v_1, v_2, \ldots, v_n \) are linearly independent. Let \( u_1 = a_1 v_1 + b_1 v_2, u_2 = a_2 v_1 + b_2 v_2, \) and \( u_3 = a_3 v_1 + b_3 v_2 \) where \( a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R} \). Let \( c_1, c_2, c_3 \in \mathbb{R} \). Then
\[
\begin{align*}
\Rightarrow c_1(a_1 v_1 + b_1 v_2) + c_2(a_2 v_1 + b_2 v_2) + c_3(a_3 v_1 + b_3 v_2) &= 0 \\
&\Rightarrow (c_1 a_1 + c_2 a_2 + c_3 a_3) v_1 + (c_1 b_1 + c_2 b_2 + c_3 b_3) v_2 = 0.
\end{align*}
\]
Now since \( v_1 \) and \( v_2 \) are linearly independent, \[
\begin{cases}
  c_1 a_1 + c_2 a_2 + c_3 a_3 = 0 \\
  c_1 b_1 + c_2 b_2 + c_3 b_3 = 0
\end{cases}
\]
There are an infinite number of solutions to this homogeneous system since there are three unknowns but only two equations. Hence, \( \{u_1, u_2, u_3\} \) is a linearly dependent set of vectors.

**45.** We assume that
\[
c_1(A v_1) + c_2(A v_2) + \cdots + c_n(A v_n) = 0.
\]
Our aim is to show that \( c_1 = c_2 = \cdots = c_n = 0 \). We manipulate the left side of the above equation as follows:
\[
\begin{align*}
  &c_1(A v_1) + c_2(A v_2) + \cdots + c_n(A v_n) = 0 \\
  &A(c_1 v_1) + A(c_2 v_2) + \cdots + A(c_n v_n) = 0 \\
  &A(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = 0.
\end{align*}
\]
Since \( A \) is invertible, we can left multiply the last equation by \( A^{-1} \) to obtain
\[
c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0.
\]
Since \( \{v_1, v_2, \ldots, v_n\} \) is linearly independent, we can now conclude directly that \( c_1 = c_2 = \cdots = c_n = 0 \), as required.
47. Assume that \( c_1 v_1 + c_2 v_2 + \cdots + c_k v_k + c_{k+1} v_{k+1} = 0 \). We must show that \( c_1 = c_2 = \cdots = c_{k+1} = 0 \). Let us suppose for the moment that \( c_{k+1} \neq 0 \). In that case, we can solve the above equation for \( v_{k+1} \):

$$v_{k+1} = -\frac{c_1}{c_{k+1}} v_1 - \frac{c_2}{c_{k+1}} v_2 - \cdots - \frac{c_k}{c_{k+1}} v_k.$$  

However, this contradicts the assumption that \( v_{k+1} \) does not belong to \( \text{span}\{v_1, v_2, \ldots, v_k\} \). Therefore, we conclude that \( c_{k+1} = 0 \). Our starting equation therefore reduces to \( c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0 \). Now the assumption that \( \{v_1, v_2, \ldots, v_k\} \) is linearly independent shows that \( c_1 = c_2 = \cdots = c_k = 0 \). Therefore, \( c_1 = c_2 = \cdots = c_k = c_{k+1} = 0 \), as required.

49. We first prove part 1 of Proposition 4.5.7. Suppose that we have a set \( \{u, v\} \) of two vectors in a vector space \( V \). If \( \{u, v\} \) is linearly dependent, then we have 

$$cu + dv = 0,$$

where \( c \) and \( d \) are not both zero. Without loss of generality, suppose that \( c \neq 0 \). Then we have 

$$u = -\frac{d}{c} v,$$

so that \( u \) and \( v \) are proportional. Conversely, if \( u \) and \( v \) are proportional, then \( v = cu \) for some constant \( c \). Thus, \( cu - v = 0 \), which shows that \( \{u, v\} \) is linearly dependent.

For part 2 of Proposition 4.5.7, suppose the zero vector \( 0 \) belongs to a set \( S \) of vectors in a vector space \( V \). Then \( 1 \cdot 0 \) is a linear dependency among the vectors in \( S \), and therefore \( S \) is linearly dependent.

51. Let \( S = \{p_1, p_2, \ldots, p_k\} \) and assume without loss of generality that the polynomials are listed in decreasing order by degree:

$$\deg(p_1) > \deg(p_2) > \cdots > \deg(p_k).$$

To show that \( S \) is linearly independent, assume that

$$c_1 p_1 + c_2 p_2 + \cdots + c_k p_k = 0.$$  

We wish to show that \( c_1 = c_2 = \cdots = c_k = 0 \). We require that each coefficient on the left side of the above equation is zero, since we have 0 on the right-hand side. Since \( p_1 \) has the highest degree, none of the terms \( c_2 p_2, c_3 p_3, \ldots, c_k p_k \) can cancel the leading coefficient of \( p_1 \). Therefore, we conclude that \( c_1 = 0 \). Thus, we now have 

$$c_2 p_2 + c_3 p_3 + \cdots + c_k p_k = 0,$$

and we can repeat this argument again now to show successively that \( c_2 = c_3 = \cdots = c_k = 0 \).

**Solutions to Section 4.6**

1. **FALSE.** It is not enough that \( S \) spans \( V \). It must also be the case that \( S \) is linearly independent.

3. **TRUE.** Any set of two non-proportional vectors in \( \mathbb{R}^2 \) will form a basis for \( \mathbb{R}^2 \).

5. **FALSE.** For example, if \( V = \mathbb{R}^2 \), then the set \( S = \{(1, 0), (2, 0), (3, 0)\} \), consisting of 3 > 2 vectors, fails to span \( V \), a 2-dimensional vector space.

7. **FALSE.** For instance, the two vectors \( 1 + x \) and \( 2 + 2x \) in \( P_3 \) are linearly dependent.

9. **FALSE.** Only linearly independent sets with fewer than \( n \) vectors can be extended to a basis for \( V \).
11. **FALSE.** The set of all $3 \times 3$ upper triangular matrices forms a 6-dimensional subspace of $M_3(\mathbb{R})$, not a 3-dimensional subspace. One basis is given by \{\(E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}\)\}.

**Problems:**

1. \(\dim[\mathbb{R}^2] = 2\). There are two vectors, so if they are to form a basis for \(\mathbb{R}^2\), they need to be linearly independent: 
   \[
   \begin{vmatrix}
   1 & -1 \\
   1 & 1
   \end{vmatrix} = 2 \neq 0.
   \]
   This implies that the vectors are linearly independent, hence they form a basis for \(\mathbb{R}^2\).

2. \(\dim[\mathbb{R}^3] = 3\). There are three vectors, so if they are to form a basis for \(\mathbb{R}^3\), they need to be linearly independent: 
   \[
   \begin{vmatrix}
   1 & 2 & 3 \\
   -1 & 5 & 11 \\
   1 & -2 & -5
   \end{vmatrix} = 0.
   \]
   This implies that the vectors are linearly dependent, hence they do not form a basis for \(\mathbb{R}^3\).

3. \(\dim[\mathbb{R}^4] = 4\). There are four vectors, so if they are to form a basis for \(\mathbb{R}^4\), they need to be linearly independent:
   \[
   \begin{vmatrix}
   1 & 2 & -1 & 2 \\
   1 & 1 & 1 & -1 \\
   0 & 3 & 1 & 1 \\
   2 & -1 & -2 & 2
   \end{vmatrix} = \begin{vmatrix}
   1 & 2 & -1 & 2 \\
   0 & -1 & 2 & -3 \\
   0 & 3 & 1 & 1 \\
   0 & -5 & 0 & -2
   \end{vmatrix} = \begin{vmatrix}
   -1 & 2 & -3 \\
   3 & 1 & 1 \\
   -5 & 0 & -2 \\
   -5 & 0 & -2
   \end{vmatrix} = -11.
   \]
   Since this determinant is nonzero, the given vectors are linearly independent. Consequently, they form a basis for \(\mathbb{R}^4\).

7. The general vector \(p(x) \in P_3\) can be represented as \(p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3\). Thus \(P_3 = \text{span}\{1, x, x^2, x^3\}\). Further, \(\{1, x, x^2, x^3\}\) is a linearly independent set since 
   \[
   W[1, x, x^2, x^3] = \begin{vmatrix}
   1 & x & x^2 & x^3 \\
   0 & 1 & 2x & 3x^2 \\
   0 & 0 & 2 & 6x \\
   0 & 0 & 0 & 6
   \end{vmatrix} = 12 \neq 0.
   \]
   Consequently, \(S = \{1, x, x^2, x^3\}\) is a basis for \(P_3\) and \(\dim[P_3] = 4\). Of course, \(S\) is not the only basis for \(P_3\).

9. \(Ax = 0 \implies \begin{vmatrix}
   1 & 3 \\
   -2 & -6 \\
   0 & 0
   \end{vmatrix} \begin{bmatrix}
   x_1 \\
   x_2
   \end{bmatrix} = \begin{bmatrix}
   0 \\
   0
   \end{bmatrix}\). The augmented matrix for this system is 
   \[
   \begin{vmatrix}
   1 & 3 & 0 \\
   -2 & -6 & 0 \\
   0 & 0 & 0
   \end{vmatrix}.
   \]
   Thus, \(x_1 + 3x_2 = 0\), or \(x_1 = -3x_2\). Let \(x_2 = r\) so that \(x_1 = 3r\) where \(r \in \mathbb{R}\). Consequently, \(S = \{x \in \mathbb{R}^2 : x = r(3, 1), r \in \mathbb{R}\} = \text{span}\{(3, 1)\}\). It follows that \(\{(3, 1)\}\) is a basis for \(S\) and \(\dim[S] = 1\).

11. \(Ax = 0 \implies \begin{vmatrix}
   1 & -1 & 4 \\
   2 & 3 & -2 \\
   1 & 2 & -2
   \end{vmatrix} \begin{bmatrix}
   x_1 \\
   x_2 \\
   x_3
   \end{bmatrix} = \begin{bmatrix}
   0 \\
   0 \\
   0
   \end{bmatrix}\). The RREF of the augmented matrix for this system is 
   \[
   \begin{vmatrix}
   1 & 0 & 2 \\
   0 & 1 & -2 \\
   0 & 0 & 0
   \end{vmatrix}.
   \]
   If we let \(x_3 = r\) then \((x_1, x_2, x_3) = (-2r, 2r, r) = r(-2, 2, 1)\), so that the solution set of the system is \(S = \{x \in \mathbb{R}^3 : x = r(-2, 2, 1), r \in \mathbb{R}\}\). Therefore we see that \(\{(-2, 2, 1)\}\) is a basis for \(S\) and \(\dim[S] = 1\).

13. If we let \(y = r\) and \(z = s\) where \(r, s \in \mathbb{R}\), then \(x = 3r - s\). It follows that any ordered triple in \(S\) can be written in the form:
   \((x, y, z) = (3r - s, r, s) = (3r, r, 0) + (-s, 0, s) = r(3, 1, 0) + s(-1, 0, 1), \) where \(r, s \in \mathbb{R}\). If we let \(v_1 = (3, 1, 0)\) and \(v_2 = (-1, 0, 1)\), then \(S = \{v \in \mathbb{R}^3 : v = r(3, 1, 0) + s(-1, 0, 1), r, s \in \mathbb{R}\} = \text{span}\{v_1, v_2\}\); moreover, \(v_1\)
and \( v_2 \) are linearly independent for if \( a, b \in \mathbb{R} \) and \( av_1 + bv_2 = 0 \), then

\[
    a(3, 1, 0) + b(-1, 0, 1) = (0, 0, 0),
\]

which implies that \((3a, a, 0) + (-b, 0, b) = (0, 0, 0)\), or \((3a - b, a, b) = (0, 0, 0)\). In other words, \( a = b = 0 \). Since \( \{v_1, v_2\} \) spans \( S \) and is linearly independent, it is a basis for \( S \). Also, \( \dim[S] = 2 \).

15. \( S = \{A \in M_2(\mathbb{R}) : A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, a, b, c \in \mathbb{R} \} \). Each vector in \( S \) can be written as

\[
    A = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\]

so that \( S = \text{span}\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \} \). Since the vectors in this set are clearly linearly independent, it follows that a basis for \( S \) is \( \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \} \), and therefore \( \dim[S] = 3 \).

17. We see directly that \( v_3 = 2v_1 \). Let \( v \) be an arbitrary vector in \( S \). Then

\[
    v = c_1v_1 + c_2v_2 + c_3v_3 = (c_1 + 2c_3)v_1 + c_2v_2 = d_1v_1 + d_2v_2,
\]

where \( d_1 = (c_1 + 2c_3) \) and \( d_2 = c_2 \). Hence \( S = \text{span}\{v_1, v_2\} \). Further, \( v_1 \) and \( v_2 \) are linearly independent for if \( a, b \in \mathbb{R} \) and \( av_1 + bv_2 = 0 \), then

\[
    a(1, 0, 1) + b(0, 1, 1) = (0, 0, 0) \implies (a, 0, a) + (0, b, b) = (0, 0, 0) \implies (a, b, a + b) = (0, 0, 0) \implies a = b = 0.
\]

Consequently, \( \{v_1, v_2\} \) is a basis for \( S \) and \( \dim[S] = 2 \).

19. The given set of matrices is linearly dependent because it contains the zero vector. Consequently, the matrices \( A_1 = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \), \( A_2 = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} \), \( A_3 = \begin{bmatrix} 5 & -6 \\ -5 & 1 \end{bmatrix} \) span the same subspace of \( M_2(\mathbb{R}) \) as that spanned by the original set. We now determine whether \( \{A_1, A_2, A_3\} \) is linearly independent. The vector equation:

\[
    c_1A_1 + c_2A_2 + c_3A_3 = 0
\]

leads to the linear system

\[
    c_1 - c_2 + 5c_3 = 0, 3c_1 + 4c_2 - 6c_3 = 0, -c_1 + c_2 - 5c_3 = 0, 2c_1 + c_2 + c_3 = 0.
\]

This system has solution \((-2r, 3r, r)\), where \( r \) is a free variable. Consequently, \( \{A_1, A_2, A_3\} \) is linearly dependent with linear dependency relation

\[
    -2A_1 + 3A_2 + A_3 = 0,
\]

or equivalently,

\[
    A_3 = 2A_1 - 3A_2.
\]

It follows that the set of matrices \( \{A_1, A_2\} \) spans the same subspace of \( M_2(\mathbb{R}) \) as that spanned by the original set of matrices. Further \( \{A_1, A_2\} \) is linearly independent by inspection, and therefore it is a basis for the subspace.

21. (a) We must show that every vector \((x, y) \in \mathbb{R}^2\) can be expressed as a linear combination of \( v_1 \) and \( v_2 \). Mathematically, we express this as

\[
    c_1(2, 1) + c_2(3, 1) = (x, y)
\]
which implies that
\[ 2c_1 + 3c_2 = x \quad \text{and} \quad c_1 + c_2 = y. \]

From this, we can solve for \( c_1 \) and \( c_2 \) in terms of \( x \) and \( y \):
\[ c_1 = 3y - x \quad \text{and} \quad c_2 = x - 2y. \]

Therefore, since we were able to solve for \( c_1 \) and \( c_2 \) in terms of \( x \) and \( y \), we see that the system of equations is consistent for all \( x \) and \( y \). Therefore, \( \{v_1, v_2\} \) spans \( \mathbb{R}^2 \).

(b) \[
\begin{vmatrix}
2 & 3 \\
1 & 1
\end{vmatrix} = -1 \neq 0,
\]
so the vectors are linearly independent.

(c) We can draw this conclusion from part (a) alone by using Theorem 4.6.12 or from part (b) alone by using Theorem 4.6.10.

23. \( \dim[P_2] = 3 \). There are 3 vectors, so \( \{p_1, p_2, p_3\} \) may be a basis for \( P_2 \) depending on \( \alpha \). To be a basis, the set of vectors must be linearly independent. Let \( a, b, c \in \mathbb{R} \). Then \( ap_1(x) + bp_2(x) + cp_3(x) = 0 \)
\[ \implies a(1 + \alpha x^2) + b(1 + x + x^2) + c(2 + x) = 0 \]
\[ \implies (a + b + 2c) + (b + c)x + (\alpha a + b)x^2 = 0. \]
Equating like coefficients in the last equality, we obtain the system:
\[ \begin{cases}
\alpha b + c = 0 \\
\alpha a + b = 0.
\end{cases} \]
Reduce the augmented matrix of this system.
\[
\begin{bmatrix}
1 & 1 & 2 & 0 \\
0 & 1 & 1 & 0 \\
\alpha & 1 & 0 & 0
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 1 & 2 & 0 \\
0 & 1 & 1 & 0 \\
0 & \alpha - 1 & 2\alpha & 0
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 1 & 2 & 0 \\
0 & 1 & 1 & 0 \\
0 & \alpha & \alpha + 1 & 0
\end{bmatrix}.
\]

For this system to have only the trivial solution, the last row of the matrix must not be a zero row. This means that \( \alpha \neq -1 \). Therefore, for the given set of vectors to be linearly independent (and thus a basis), \( \alpha \) can be any value except \(-1\).

25. \[ \text{W} [p_0, p_1, p_2] (x) = \begin{vmatrix}
1 & x \\
0 & 1 \\
0 & 3
\end{vmatrix} \]
\[ = 3 \neq 0, \]
so \( \{p_0, p_1, p_2\} \) is a linearly independent set. Since \( \dim[P_2] = 3 \), it follows that \( \{p_0, p_1, p_2\} \) is a basis for \( P_2 \).

27.

(a) \( Ax = 0 \implies \begin{bmatrix}
1 & 1 & -1 & 1 \\
2 & -3 & 5 & -6 \\
5 & 0 & 2 & -3
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}. \]

The augmented matrix for this linear system is
\[
\begin{bmatrix}
1 & 1 & -1 & 1 & 0 \\
2 & -3 & 5 & -6 & 0 \\
5 & 0 & 2 & -3 & 0
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 1 & -1 & 1 & 0 \\
0 & -5 & 7 & -8 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 1 & -1 & 1 & 0 \\
0 & -5 & 7 & -8 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 1 & -1 & 1 & 0 \\
0 & -5 & 7 & -8 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

We see that \( x_3 = r \) and \( x_4 = s \) are free variables, and therefore \( \text{nullspace}(A) \) is 2-dimensional. Now we can check directly that \( Av_1 = 0 \) and \( Av_2 = 0 \), and since \( v_1 \) and \( v_2 \) are linearly independent (they are non-proportional), they therefore form a basis for \( \text{nullspace}(A) \) by Corollary 4.6.13.
We see that $T$ is linearly independent vectors, which therefore form a basis for $S$. Now let $x,y,z,w$ be vectors:

$\begin{align*}
(x,y,z,w) &= c_1(-2,7,5,0) + c_2(3,-8,0,5),
\end{align*}$

where $c_1, c_2 \in \mathbb{R}$.

29.

(a) An arbitrary matrix in $S$ takes the form

$\begin{bmatrix}
  a & b & c \\
  d & e & a+b+c-d-e \\
  b+c-d & a+c-e & d+e-c
\end{bmatrix}$

$= a\begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix} + b\begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  1 & 0 & 0
\end{bmatrix} + c\begin{bmatrix}
  0 & 0 & 1 \\
  0 & 0 & 1 \\
  1 & 1 & 1
\end{bmatrix} + d\begin{bmatrix}
  0 & 0 & 0 \\
  1 & 0 & -1 \\
 -1 & 0 & 1
\end{bmatrix} + e\begin{bmatrix}
  0 & 0 & 0 \\
  0 & 1 & -1 \\
  0 & -1 & 1
\end{bmatrix}$.

Therefore, we have the following basis for $S$:

$\left\{ \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}, \begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  1 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
  0 & 0 & 1 \\
  0 & 0 & 1 \\
  1 & 1 & 1
\end{bmatrix}, \begin{bmatrix}
  0 & 0 & 0 \\
  1 & 0 & -1 \\
 -1 & 0 & 1
\end{bmatrix}, \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 1 & -1 \\
  0 & -1 & 1
\end{bmatrix} \right\}$.

From this, we see that $\dim[S] = 5$.

(b) We must include four additional matrices that are linearly independent and outside of $S$. The matrices $E_{11}, E_{12}, E_{11} + E_{13}$, and $E_{22}$ will suffice in this case.

31. We know that $\dim[M_n(\mathbb{R})] = n^2$. Let $S \in \text{Sym}_n(\mathbb{R})$ and let $[S_{ij}]$ be the matrix with ones in the $(i,j)$ and $(j,i)$ positions and zeroes elsewhere. Then the general $n \times n$ symmetric matrix can be expressed as:

$S = a_{11}S_{11} + a_{12}S_{12} + \cdots + a_{1n}S_{1n}$

$+ a_{22}S_{22} + a_{23}S_{23} + \cdots + a_{2n}S_{2n}$

$+ \cdots$

$+ a_{n-1,n-1}S_{n-1,n-1} + a_{n-1,n}S_{n-1,n} + a_{nn}S_{nn}.$

We see that $S$ has been resolved into a linear combination of $n + (n-1) + (n-2) + \cdots + 1 = n(n+1)/2$ linearly independent matrices, which therefore form a basis for $\text{Sym}_n(\mathbb{R})$; hence $\dim[\text{Sym}_n(\mathbb{R})] = n(n+1)/2$.

Now let $T \in \text{Skew}_n(\mathbb{R})$ and let $[T_{ij}]$ be the matrix with one in the $(i,j)$-position, negative one in the $(j,i)$-position, and zeroes elsewhere, including the main diagonal. Then the general $n \times n$ skew-symmetric matrix can be expressed as:

$T = a_{12}T_{12} + a_{13}T_{13} + a_{14}T_{14} + \cdots + a_{1n}T_{1n}$

$+ a_{23}T_{23} + a_{24}T_{24} + \cdots + a_{2n}T_{2n}$

$+ \cdots$

$+ a_{n-1,n}T_{n-1,n}$

We see that $T$ has been resolved into a linear combination of $n(n-1) + (n-2) + (n-3) + \cdots + 1 = (n-1)n/2$ linearly independent vectors, which therefore form a basis for $\text{Skew}_n(\mathbb{R})$; hence $\dim[\text{Skew}_n(\mathbb{R})] = (n-1)n/2$.

Consequently, using these results, we have

$\dim[\text{Sym}_n(\mathbb{R})] + \dim[\text{Skew}_n(\mathbb{R})] = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = n^2 = \dim[M_n(\mathbb{R})]$. 
33. Each vector in $S$ can be written as
\[
\begin{bmatrix}
a & b \\
b & a
\end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]
Consequently, a basis for $S$ is given by the linearly independent set \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}. To extend this basis to $M_2(\mathbb{R})$, we can choose, for example, the two vectors \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} and \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. Then the linearly independent set \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\} is a basis for $M_2(\mathbb{R})$.

35. Since $S$ is a basis for $P_{n-1}$, $S$ contains $n$ vectors. Therefore, $S \cup \{x^n\}$ is a set of $n + 1$ vectors, which is precisely the dimension of $P_n$. Moreover, $x^n$ does not lie in $P_{n-1} = \text{span}(S)$, and therefore, $S \cup \{x^n\}$ is linearly independent by Problem 47 in Section 4.5. By Corollary 4.6.13, we conclude that $S \cup \{x^n\}$ is a basis for $P_n$.

37. (a) Let $e_k$ denote the $k$th standard basis vector. Then a basis for $\mathbb{C}^n$ with scalars in $\mathbb{R}$ is given by
\[
\{e_1, e_2, \ldots, e_n, ie_1, ie_2, \ldots, ie_n\},
\]
and the dimension is $2n$.

(b) Using the notation in part (a), a basis for $\mathbb{C}^n$ with scalars in $\mathbb{C}$ is given by
\[
\{e_1, e_2, \ldots, e_n\},
\]
and the dimension is $n$.

**Solutions to Section 4.7**

True-False Review:

1. **TRUE.** This is the content of Theorem 4.7.1. The existence of such a linear combination comes from the fact that a basis for $V$ must span $V$, and the uniqueness of such a linear combination follows from the linear independence of the vectors comprising a basis.

3. **TRUE.** The number of columns in the change-of-basis matrix $P_{C \leftarrow B}$ is the number of vectors in $B$, while the number of rows of $P_{C \leftarrow B}$ is the number of vectors in $C$. Since all bases for the vector space $V$ contain the same number of vectors, this implies that $P_{C \leftarrow B}$ contains the same number of rows and columns.

5. **TRUE.** This follows from the linearity properties:
\[
[v - w]_B = [v + (-1)w]_B = [v]_B + [(-1)w]_B = [v]_B + (-1)[w]_B = [v]_B - [w]_B.
\]

7. **FALSE.** For instance, if we consider the bases $B = \{(1, 0), (0, 1)\}$ and $C = \{(0, 1), (1, 0)\}$ for $\mathbb{R}^2$, and if we let $v = (1, 0)$ and $w = (0, 1)$, then $v \neq w$, but $[v]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [w]_C$.

Problems:

1. Write
\[
(5, -10) = c_1(2, -2) + c_2(1, 4).
\]
Then
\[2c_1 + c_2 = 5 \quad \text{and} \quad -2c_1 + 4c_2 = -10.\]
Solving this system of equations gives \(c_1 = 3\) and \(c_2 = -1\). Thus,
\[
[v]_B = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.
\]

3. Write
\[(-9, 1, -8) = c_1(1, 0, 1) + c_2(1, 1, -1) + c_3(2, 0, 1).\]
Then
\[c_1 + c_2 + 2c_3 = -9 \quad \text{and} \quad c_2 = 1 \quad \text{and} \quad c_1 - c_2 + c_3 = -8.
\]

5. Write
\[(1, 7, 7) = c_1(3, -1, -1) + c_2(1, -6, 3) + c_3(0, 5, -1).\]
Then
\[3c_1 + c_2 = 1 \quad \text{and} \quad -c_1 - 6c_2 + 5c_3 = 7 \quad \text{and} \quad -c_1 + 3c_2 - c_3 = 7.
\]
Using Gaussian elimination to solve this system of equations gives \(c_1 = -1, c_2 = 4, \text{ and } c_3 = 6\). Thus,
\[
[v]_B = \begin{bmatrix} -1 \\ 4 \\ 6 \end{bmatrix}.
\]

7. Write
\[-4x^2 + 2x + 6 = c_1(x^2 + x) + c_2(2 + 2x) + c_3(1).\]
Equating the powers of \(x\) on each side, we have
\[c_1 = -4 \quad \text{and} \quad c_1 + 2c_2 = 2 \quad \text{and} \quad 2c_2 + c_3 = 6.
\]
Solving this system of equations, we find that \(c_1 = -4, c_2 = 3, \text{ and } c_3 = 0\). Hence,
\[
[p(x)]_B = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}.
\]

9. Write
\[4 - x + x^2 - 2x^3 = c_1(1) + c_2(1 + x) + c_3(1 + x + x^2) + c_4(1 + x + x^2 + x^3).
\]
Equating the powers of \(x\) on each side, we have
\[c_1 + c_2 + c_3 + c_4 = 4 \quad \text{and} \quad c_2 + c_3 + c_4 = -1 \quad \text{and} \quad c_3 + c_4 = 1 \quad \text{and} \quad c_4 = -2.
\]
Solving this system of equations, we find that \(c_1 = 5, c_2 = -2, c_3 = 3, \text{ and } c_4 = -2\). Hence,
\[
[p(x)]_B = \begin{bmatrix} 5 \\ -2 \\ 3 \\ -2 \end{bmatrix}.
\]
11. Write
\[
\begin{bmatrix}
-3 & -2 \\
-1 & 2
\end{bmatrix} = c_1 \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix} + c_2 \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix} + c_3 \begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix} + c_4 \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}.
\]
Equating the individual entries of the matrices on each side of this equation (upper left, upper right, lower left, and lower right, respectively) gives
\[
c_1 + c_2 + c_3 + c_4 = -3 \quad \text{and} \quad c_1 + c_2 + c_3 = -2 \quad \text{and} \quad c_1 + c_2 = -1 \quad \text{and} \quad c_1 = 2.
\]
Solving this system of equations, we find that \(c_1 = 2, c_2 = -3, c_3 = -1, \) and \(c_4 = -1.\) Thus,
\[
[A]_B = \begin{bmatrix}
2 \\
-3 \\
-1 \\
-1
\end{bmatrix}.
\]

13. Write
\[
\begin{bmatrix}
5 & 6 \\
7 & 8
\end{bmatrix} = c_1 \begin{bmatrix}
-1 & 1 \\
0 & 1
\end{bmatrix} + c_2 \begin{bmatrix}
1 & 3 \\
-1 & 0
\end{bmatrix} + c_3 \begin{bmatrix}
1 & 0 \\
1 & 2
\end{bmatrix} + c_4 \begin{bmatrix}
0 & -1 \\
2 & 3
\end{bmatrix}.
\]
Equating the individual entries of the matrices on each side of this equation (upper left, upper right, lower left, and lower right, respectively) gives
\[
-c_1 + c_2 + c_3 = 5 \quad \text{and} \quad c_1 + 3c_2 - c_4 = 6 \quad \text{and} \quad -c_2 + c_3 + 2c_4 = 7 \quad \text{and} \quad c_1 + 2c_3 + 3c_4 = 8.
\]
Solving this system of equations, we find that \(c_1 = -34/3, c_2 = 12, c_3 = -55/3, \) and \(c_4 = 56/3.\) Thus,
\[
[A]_B = \begin{bmatrix}
-34/3 \\
12 \\
-55/3 \\
56/3
\end{bmatrix}.
\]

15. Write
\[
a_0 + a_1 x + a_2 x^2 = c_1 (1 + x) + c_2 x(x - 1) + c_3 (1 + 2x^2).
\]
Equating powers of \(x\) on both sides of this equation, we have
\[
c_1 + c_3 = a_0 \quad \text{and} \quad c_1 - c_2 = a_1 \quad \text{and} \quad c_2 + 2c_3 = a_2.
\]
The augmented matrix corresponding to this system of equations is
\[
\begin{bmatrix}
1 & 0 & 1 & | a_0 \\
1 & -1 & 0 & | a_1 \\
0 & 1 & 2 & | a_2
\end{bmatrix}.
\]
We can reduce this to row-echelon form as
\[
\begin{bmatrix}
1 & 0 & 1 & | a_0 \\
0 & 1 & 2 & | -a_0 + a_1 + a_2 \\
0 & 0 & 1 & | a_2
\end{bmatrix}.
\]
Thus, solving by back-substitution, we have
\[
c_1 = 2a_0 - a_1 - a_2 \quad \text{and} \quad c_2 = 2a_0 - 2a_1 - a_2 \quad \text{and} \quad c_3 = -a_0 + a_1 + a_2.
\]
Hence, relative to the ordered basis \(B = \{p_1, p_2, p_3\},\) we have
\[
[p(x)]_B = \begin{bmatrix}
2a_0 - a_1 - a_2 \\
2a_0 - 2a_1 - a_2 \\
-a_0 + a_1 + a_2
\end{bmatrix}.
\]
17. Let $v_1 = (-5, -3)$ and $v_2 = (4, 28)$. Setting 
\[ (-5, -3) = c_1(6, 2) + c_2(1, -1) \]
and solving, we find $c_1 = -1$ and $c_2 = 1$. Thus, $[v_1]_C = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Next, setting 
\[ (4, 28) = c_1(6, 2) + c_2(1, -1) \]
and solving, we find $c_1 = 4$ and $c_2 = -20$. Thus, $[v_2]_C = \begin{bmatrix} 4 \\ -20 \end{bmatrix}$. Therefore, 
\[ P_{C \leftarrow B} = \begin{bmatrix} -1 & 4 \\ 1 & -20 \end{bmatrix}. \]

19. Let $v_1 = (-7, 4, 4)$, $v_2 = (4, 2, -1)$, and $v_3 = (-7, 5, 0)$. Setting 
\[ (-7, 4, 4) = c_1(1, 1, 0) + c_2(0, 1, 1) + c_3(3, -1, -1) \]
and solving, we find $c_1 = 0$, $c_2 = 5/3$ and $c_3 = -7/3$. Thus, $[v_1]_C = \begin{bmatrix} 0 \\ 5/3 \\ -7/3 \end{bmatrix}$. Next, setting 
\[ (4, 2, -1) = c_1(1, 1, 0) + c_2(0, 1, 1) + c_3(3, -1, -1) \]
and solving, we find $c_1 = 0$, $c_2 = 19/3$ and $c_3 = -7/3$. Setting 
\[ (4, 2, -1) = c_1(1, 1, 0) + c_2(0, 1, 1) + c_3(3, -1, -1) \]
and solving, we find $c_1 = 3$, $c_2 = -2/3$, and $c_3 = 1/3$. Thus, $[v_2]_C = \begin{bmatrix} 3 \\ -2/3 \\ 1/3 \end{bmatrix}$. Finally, setting 
\[ (-7, 5, 0) = c_1(1, 1, 0) + c_2(0, 1, 1) + c_3(3, -1, -1) \]
and solving, we find $c_1 = 5$, $c_2 = -4$, and $c_3 = -4$. Hence, $[v_3]_C = \begin{bmatrix} 5 \\ -4 \\ -4 \end{bmatrix}$. Therefore, 
\[ P_{C \leftarrow B} = \begin{bmatrix} 0 & 3 & 5 \\ -4 & -2 & 4 \\ -4 & 3 & -4 \end{bmatrix}. \]

21. Let $v_1 = -4 + x - 6x^2$, $v_2 = 6 + 2x^2$, and $v_3 = -6 - 2x + 4x^2$. Setting 
\[ -4 + x - 6x^2 = c_1(1 - x + 3x^2) + c_2(2) + c_3(3 + x^2) \]
and solving, we find $c_1 = -1$, $c_2 = 3$, and $c_3 = -3$. Thus, $[v_1]_C = \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix}$. Next, setting 
\[ 6 + 2x^2 = c_1(1 - x + 3x^2) + c_2(2) + c_3(3 + x^2) \]
and solving, we find $c_1 = 0$, $c_2 = 0$, and $c_3 = 2$. Thus, $\begin{bmatrix} v_2 \end{bmatrix}_C = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. Finally, setting

$$-6 - 2x + 4x^2 = c_1(1 - x + 3x^2) + c_2(2) + c_3(3 + x^2)$$

and solving, we find $c_1 = 2$, $c_2 = -1$, and $c_3 = -2$. Thus, $\begin{bmatrix} v_3 \end{bmatrix}_C = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$. Therefore,

$$P_{C \leftarrow B} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -1 \\ -3 & 2 & -2 \end{bmatrix}.$$

23. Let $v_1 = 2 + x^2$, $v_2 = -1 - 6x + 8x^2$, and $v_3 = -7 - 3x - 9x^2$. Setting

$$2 + x^2 = c_1(1 + x) + c_2(-x + x^2) + c_3(1 + 2x^2)$$

and solving, we find $c_1 = 3$, $c_2 = 3$, and $c_3 = -1$. Thus, $\begin{bmatrix} v_1 \end{bmatrix}_C = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$. Next, solving

$$-1 - 6x + 8x^2 = c_1(1 + x) + c_2(-x + x^2) + c_3(1 + 2x^2)$$

and solving, we find $c_1 = -4$, $c_2 = 2$, and $c_3 = 3$. Thus, $\begin{bmatrix} v_2 \end{bmatrix}_C = \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}$. Finally, solving

$$-7 - 3x - 9x^2 = c_1(1 + x) + c_2(-x + x^2) + c_3(1 + 2x^2)$$

and solving, we find $c_1 = -2$, $c_2 = 1$, and $c_3 = -5$. Thus, $\begin{bmatrix} v_3 \end{bmatrix}_C = \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$. Therefore,

$$P_{C \leftarrow B} = \begin{bmatrix} 3 & -4 & -2 \\ 3 & 2 & 1 \\ -1 & 3 & -5 \end{bmatrix}.$$

25. Let $v_1 = E_{12}$, $v_2 = E_{22}$, $v_3 = E_{21}$, and $v_4 = E_{11}$. We see by inspection that

$$\begin{bmatrix} v_1 \end{bmatrix}_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} v_2 \end{bmatrix}_C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} v_3 \end{bmatrix}_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} v_4 \end{bmatrix}_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$ 

Therefore,

$$P_{C \leftarrow B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$
27. We could simply compute the inverse of the matrix obtained in Problem 16. For instructive purposes, however, we proceed directly. Let \( w_1 = (6, 2) \) and \( w_2 = (1, -1) \). Setting
\[
(6, 2) = c_1(-5, -3) + c_2(4, 28)
\]
and solving, we obtain \( c_1 = -5/4 \) and \( c_2 = -1/16 \). Thus, \( [w_1]_B = \begin{bmatrix} -5/4 \\ -1/16 \end{bmatrix} \). Next, setting
\[
(1, -1) = c_1(-5, -3) + c_2(4, 28)
\]
and solving, we obtain \( c_1 = 25/4 \) and \( c_2 = -1/16 \). Thus, \( [w_2]_B = \begin{bmatrix} 25/4 \\ -1/16 \end{bmatrix} \). Therefore,
\[
P_{B\rightarrow C} = \begin{bmatrix} 1/7 & 2/7 \\ -2/7 & 3/7 \end{bmatrix}.
\]

29. We could simply compute the inverse of the matrix obtained in Problem 16. For instructive purposes, however, we proceed directly. Let \( w_1 = 1 - 2x \) and \( w_2 = 2 + x \). Setting
\[
1 - 2x = c_1(7 - 4x) + c_2(5x)
\]
and solving, we find \( c_1 = 1/7 \) and \( c_2 = -2/7 \). Thus, \( [w_1]_B = \begin{bmatrix} 1/7 \\ -2/7 \end{bmatrix} \). Setting
\[
2 + x = c_1(7 - 4x) + c_2(5x)
\]
and solving, we find \( c_1 = 2/7 \) and \( c_2 = 3/7 \). Thus, \( [w_2]_B = \begin{bmatrix} 2/7 \\ 3/7 \end{bmatrix} \). Therefore,
\[
P_{B\rightarrow C} = \begin{bmatrix} 1/7 & 2/7 \\ -2/7 & 3/7 \end{bmatrix}.
\]

31. We could simply compute the inverse of the matrix obtained in Problem 25. For instructive purposes, however, we proceed directly. Let \( w_1 = E_{22}, w_2 = E_{11}, w_3 = E_{21}, \) and \( w_4 = E_{12} \). We see by inspection that
\[
[w_1]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [w_2]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad [w_3]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad [w_4]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\]
Therefore
\[
P_{B\rightarrow C} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\]

33. We first compute \([v]_B\) and \([v]_C\) directly. Setting
\[
(-1, 2, 0) = c_1(-7, 4, 4) + c_2(4, 2, -1) + c_3(-7, 5, 0)
\]
and solving, we obtain \( c_1 = 3/43 \), \( c_2 = 12/43 \), and \( c_3 = 10/43 \). Thus, \( [v]_B = \begin{bmatrix} \frac{3}{43} \\ \frac{12}{43} \\ \frac{10}{43} \end{bmatrix} \). Setting

\((-1, 2, 0) = c_1(1, 1, 0) + c_2(0, 1, 1) + c_3(3, -1, -1)\)

and solving, we obtain \( c_1 = 2 \), \( c_2 = -1 \), and \( c_3 = -1 \). Thus, \( [v]_C = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \). Now, according to Problem 19, \( P_{C \leftarrow B} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -1 \\ -3 & 2 & -2 \end{bmatrix} \), so \( P_{C \leftarrow B}[v]_B = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -1 \\ -3 & 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = [v]_C \), which confirms Equation (4.7.6).

35. We first compute \([v]_B\) and \([v]_C\) directly. Setting

\[ 5 - x + 3x^2 = c_1(-4 + x - 6x^2) + c_2(6 + 2x^2) + c_3(-6 - 2x + 4x^2) \]

and solving, we obtain \( c_1 = 1 \), \( c_2 = 5/2 \), and \( c_3 = 1 \). Thus, \( [v]_B = \begin{bmatrix} 1 \\ \frac{5}{2} \\ 1 \end{bmatrix} \). Next, setting

\[ 5 - x + 3x^2 = c_1(1 - x + 3x^2) + c_2(2) + c_3(3 + x^2) \]

and solving, we obtain \( c_1 = 1 \), \( c_2 = 2 \), and \( c_3 = 0 \). Thus, \( [v]_C = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \). Now, according to Problem 21,

\( P_{C \leftarrow B} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -1 \\ -3 & 2 & -2 \end{bmatrix} \), so

\( P_{C \leftarrow B}[v]_B = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -1 \\ -3 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{5}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = [v]_C \),

which confirms Equation (4.7.6).

37. Write \( x = a_1v_1 + a_2v_2 + \cdots + a_nv_n \). We have

\[ cx = c(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = (ca_1)v_1 + (ca_2)v_2 + \cdots + (ca_n)v_n. \]

Hence,

\[ [cx]_B = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} = c \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = c[x]_B. \]
39. Let \( B = \{v_1, v_2, \ldots, v_n\} \) and let \( C = \{v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}\} \), where \( \sigma \) is a permutation of the set \( \{1, 2, \ldots, n\} \). We will show that \( P_{C \leftarrow B} \) contains exactly one 1 in each row and column, and zeroes elsewhere (the argument for \( P_{B \leftarrow C} \) is essentially identical, or can be deduced from the fact that \( P_{B \leftarrow C} \) is the inverse of \( P_{C \leftarrow B} \)).

Let \( i \) be in \( \{1, 2, \ldots, n\} \). The \( i \)th column of \( P_{C \leftarrow B} \) is \([v_i]_C\). Suppose that \( k_i \in \{1, 2, \ldots, n\} \) is such that \( \sigma(k_i) = i \). Then \([v_i]_C\) is a column vector with a 1 in the \( k_i \)th position and zeroes elsewhere. Since the values \( k_1, k_2, \ldots, k_n \) are distinct, we see that each column of \( P_{C \leftarrow B} \) contains a single 1 (with zeroes elsewhere) in a different position from any other column. Hence, when we consider all \( n \) columns as a whole, each position in the column vector must have a 1 occurring exactly once in one of the columns. Therefore, \( P_{C \leftarrow B} \) contains exactly one 1 in each row and column, and zeroes elsewhere.

### Solutions to Section 4.8

#### True-False Review:

1. **TRUE.** Note that \( \text{rowspace}(A) \) is a subspace of \( \mathbb{R}^n \) and \( \text{colspace}(A) \) is a subspace of \( \mathbb{R}^m \), so certainly if \( \text{rowspace}(A) = \text{colspace}(A) \), then \( \mathbb{R}^n \) and \( \mathbb{R}^m \) must be the same. That is, \( m = n \).

2. **FALSE.** The nonzero column vectors of the original matrix \( A \) that correspond to the nonzero column vectors of a row-echelon form of \( A \) form a basis for \( \text{colspace}(A) \).

3. **TRUE.** For an invertible \( n \times n \) matrix, \( \text{rank}(A) = n \). That means there are \( n \) nonzero rows in a row-echelon form of \( A \), and so \( \text{rowspace}(A) \) is \( n \)-dimensional. Therefore, we conclude that \( \text{rowspace}(A) = \mathbb{R}^n \).

#### Problems:

1. A row-echelon form of \( A \) is \[ \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \]. Consequently, a basis for \( \text{rowspace}(A) \) is \( \{(1, -2)\} \), whereas a basis for \( \text{colspace}(A) \) is \( \{(1, -3)\} \).

2. A row-echelon form of \( A \) is \[ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \]. Consequently, a basis for \( \text{rowspace}(A) \) is \( \{(1, 2, 3), (0, 1, 2)\} \), whereas a basis for \( \text{colspace}(A) \) is \( \{(1, 5, 9), (2, 6, 10)\} \).

3. A row-echelon form of \( A \) is \[ \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]. Consequently, a basis for \( \text{rowspace}(A) \) is \( \{(1, 2, -1, 3), (0, 0, 0, 1)\} \), whereas a basis for \( \text{colspace}(A) \) is \( \{(1, 3, 1, 5), (3, 5, -1, 7)\} \).

7. We determine a basis for the rowspace of the matrix \[ \begin{bmatrix} 1 & -1 & 2 \\ 5 & -4 & 1 \\ 7 & -5 & -4 \end{bmatrix} \]. A row-echelon form of this matrix is \[ \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{bmatrix} \]. Consequently, a basis for the subspace spanned by the given vectors is \( \{(1, -1, 2), (0, 1, -9)\} \).

9. We determine a basis for the rowspace of the matrix \[ \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 3 & -4 \\ 1 & 2 & -6 & 10 \end{bmatrix} \]. A row-echelon form of this matrix is \[ \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]. Consequently, a basis for the subspace spanned by the given vectors is \( \{(1, 1, -1, 2)\} \).
\{(1,1,−1,2), (0,1,−5,8)\}.

11. A row-echelon form of \( A \) is 
\[
\begin{bmatrix}
1 & -3 \\
0 & 0
\end{bmatrix}.
\]
Consequently, a basis for \( \text{rowspace}(A) \) is \( \{(1, -3)\} \), whereas a basis for \( \text{colspace}(A) \) is \( \{(-3, 1)\} \). Both of these subspaces are lines in the \( xy \)-plane.

13. If \( A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \), \( \text{colspace}(A) \) is spanned by \( (1, 2) \), but if we permute the two rows of \( A \), we obtain a new matrix whose column space is spanned by \( (2, 1) \). On the other hand, if we multiply the first row by 2, then we obtain a new matrix whose column space is spanned by \( (1, 0) \). Therefore, in all cases, \( \text{colspace}(A) \) is altered by the row operations performed.

### Solutions to Section 4.9

**True-False Review:**

1. **FALSE.** For example, consider the \( 7 \times 3 \) zero matrix, \( 0_{7 \times 3} \). We have \( \text{rank}(0_{7 \times 3}) = 0 \), and therefore by the Rank-Nullity Theorem, \( \text{nullity}(0_{7 \times 3}) = 3 \). But \( |m - n| = |7 - 3| = 4 \). Many other examples can be given. In particular, provided that \( m > 2n \), the \( m \times n \) zero matrix will show the falsity of the statement.

3. **TRUE.** By the Rank-Nullity Theorem, \( \text{rank}(A) = 7 \), and therefore, \( \text{rowspace}(A) \) is a 7-dimensional subspace of \( \mathbb{R}^7 \). Hence, \( \text{rowspace}(A) = \mathbb{R}^7 \).

5. **TRUE.** An invertible matrix \( A \) must have \( \text{nullspace}(A) = \{0\} \), but if \( \text{colspace}(A) \) is also \( \{0\} \), then \( A \) would be the zero matrix, which is certainly not invertible.

7. **FALSE.** For instance, if we take \( A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \), then \( \text{nullity}(A) \cdot \text{nullity}(B) = 1 \cdot 1 = 1 \), but \( AB = 0_2 \), and \( \text{nullity}(AB) = 2 \).

9. **TRUE.** If \( y \) belongs to \( \text{nullspace}(A) \), then \( A y = 0 \). Hence, if \( A x_p = b \), then
\[
A(y + x_p) = Ay + A x_p = 0 + b = b,
\]
which demonstrates that \( y + x_p \) is also a solution to the linear system \( A x = b \).

**Problems:**

1. The matrix is already in row-echelon form. A vector \( (x, y, z, w) \) in \( \text{nullspace}(A) \) must satisfy \( x - 6z - w = 0 \). We see that \( y, z, \) and \( w \) are free variables, and
\[
\text{nullspace}(A) = \left\{ \begin{bmatrix} 6z + w \\ y \\ z \\ w \end{bmatrix} : y, z, w \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]
Therefore, \( \text{nullity}(A) = 3 \). Moreover, since this row-echelon form contains one nonzero row, \( \text{rank}(A) = 1 \). Since the number of columns of \( A \) is 4 = 1 + 3, the Rank-Nullity Theorem is verified.

3. We bring \( A \) to row-echelon form:
\[
\text{REF}(A) = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}.
\]
Since there are no unpivoted columns, there are no free variables in the associated homogeneous linear system, and so
\[
\text{nullspace}(A) = \{\mathbf{0}\}.
\]
Therefore, \(\text{nullity}(A) = 0\). Since \(\text{REF}(A)\) contains three nonzero rows, \(\text{rank}(A) = 3\). Since the number of columns of \(A\) is \(3 = 3 + 0\), the Rank-Nullity Theorem is verified.

5. Since all rows (or columns) of this matrix are proportional to the first one, \(\text{rank}(A) = 1\). Since \(A\) has two columns, we conclude from the Rank-Nullity Theorem that
\[
\text{nullity}(A) = 2 - \text{rank}(A) = 2 - 1 = 1.
\]

7. Since the second and third columns are not proportional and the first column is all zeros, we have \(\text{rank}(A) = 2\). Since \(A\) has three columns, we conclude from the Rank-Nullity Theorem that
\[
\text{nullity}(A) = 3 - \text{rank}(A) = 3 - 2 = 1.
\]

9. The augmented matrix for this linear system is
\[
\begin{bmatrix}
1 & 3 & -1 & 4 \\
2 & 7 & 9 & 11 \\
1 & 5 & 21 & 10 \\
\end{bmatrix}
\]
We quickly reduce this augmented matrix to row-echelon form:
\[
\begin{bmatrix}
1 & 3 & -1 & 4 \\
0 & 1 & 11 & 3 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
A solution \((x, y, z)\) to the system will have a free variable corresponding to the third column: \(z = t\). Then \(y + 11t = 3\), so \(y = 3 - 11t\). Finally, \(x + 3y - z = 4\), so \(x = 4 + t - 3(3 - 11t) = -5 + 34t\). Thus, the solution set is
\[
\left\{ \begin{bmatrix}
-5 + 34t \\
3 - 11t \\
t
\end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix}
34 \\
-11 \\
1
\end{bmatrix} t + \begin{bmatrix}
-5 \\
3 \\
0
\end{bmatrix} : t \in \mathbb{R} \right\}.
\]
Observe that \(\mathbf{x}_p = \begin{bmatrix}
-5 \\
3 \\
0
\end{bmatrix}\) is a particular solution to \(A\mathbf{x} = \mathbf{b}\), and that \(\begin{bmatrix}
34 \\
-11 \\
1
\end{bmatrix}\) forms a basis for \(\text{nullspace}(A)\). Therefore, the set of solution vectors obtained does indeed take the form (4.9.3).

11. The augmented matrix for this linear system is
\[
\begin{bmatrix}
1 & 1 & -2 & -3 \\
3 & -1 & -7 & 2 \\
1 & 1 & 1 & 0 \\
2 & 2 & -4 & -6 \\
\end{bmatrix}
\]
We quickly reduce this augmented matrix to row-echelon form:
\[
\begin{bmatrix}
1 & 1 & -2 & -3 \\
0 & 1 & \frac{7}{3} & \frac{-11}{3} \\
0 & 0 & 1 & 1
\end{bmatrix}
\]
There are no free variables in the solution set (since none of the first three columns is unpivoted), and we find the solution set by back-substitution:
\[
\left\{ \begin{bmatrix}
2 \\
-3 \\
1
\end{bmatrix} \right\}.
\]
It is easy to see that this is indeed a particular solution:
\[
\mathbf{x}_p = \begin{bmatrix}
2 \\
-3 \\
1
\end{bmatrix}.
\]
Since the row-echelon form of \(A\) has three nonzero rows, \(\text{rank}(A) = 3\). Thus, \(\text{nullity}(A) = 0\).
Hence, nullspace(A) = {0}. Thus, the only term in the expression (4.9.3) that appears in the solution is x_p, and this is precisely the unique solution we obtained in the calculations above.

13. By the Rank-Nullity Theorem, rank(A) = 7 – nullity(A) = 7 – 4 = 3, and hence, colspace(A) is 3-dimensional. But since A has three rows, colspace(A) is a subspace of \( \mathbb{R}^3 \). Therefore, since the only 3-dimensional subspace of \( \mathbb{R}^3 \) is \( \mathbb{R}^3 \) itself, we conclude that colspace(A) = \( \mathbb{R}^3 \). Now rowspace(A) is also 3-dimensional, but it is a subspace of \( \mathbb{R}^5 \). Therefore, it is not accurate to say that rowspace(A) = \( \mathbb{R}^3 \).

15. If rowspace(A) = nullspace(A), then we know that rank(A) = nullity(A). Therefore, rank(A) + nullity(A) must be even. But rank(A) + nullity(A) is the number of columns of A. Therefore, A contains an even number of columns.

17. We know that rank(A) + nullity(A) = 8. But since A only has three rows, rank(A) ≤ 3. Therefore, nullity(A) ≥ 5. However, since nullspace(A) is a subspace of \( \mathbb{R}^8 \), nullity(A) ≤ 8. Therefore, 5 ≤ nullspace(A) ≤ 8. There are many examples of a 3 × 8 matrix A with nullity(A) = 5; one example is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The only 3 × 8 matrix with nullity(A) = 8 is \( 0_{3 \times 8} \), the 3 × 8 zero matrix.

---

**Solutions to Section 4.10**

**True-False Review:**

1. **TRUE.** This follows from the equivalence of (a) and (m) in the Invertible Matrix Theorem.

3. **FALSE.** If the matrix has \( n \) linearly independent columns, then by the equivalence of (a) and (j) in the Invertible Matrix Theorem, such a matrix would be invertible. But if that were so, then by part (m) of the Invertible Matrix Theorem, such a matrix would have to have \( n \) linearly independent rows.

5. **TRUE.** If rowspace(A) \( \neq \mathbb{R}^n \), then by the equivalence of (a) and (n) in the Invertible Matrix Theorem, A is not invertible. Therefore, A is not row-equivalent to the identity matrix. Since B is row-equivalent to A, then B is not row-equivalent to the identity matrix, and therefore, B is not invertible. Hence, by part (k) of the Invertible Matrix Theorem, we conclude that colspace(B) \( \neq \mathbb{R}^n \).

7. **FALSE.** The matrix \([A|B]\) has \( 2n \) columns, but only \( n \) rows, and therefore \( \text{rank}([A|B]) \leq n \). Hence, by the Rank-Nullity Theorem, nullity([A|B]) \( \geq n > 0 \).

9. **FALSE.** For instance, the matrix

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

is of the form given, but satisfies any (and all) of the statements of the Invertible Matrix Theorem.

---

**Solutions to Section 4.11**

1. **FALSE.** The converse of this statement is true, but for the given statement, many counterexamples exist. For instance, the vectors \( \mathbf{v} = (1, 1) \) and \( \mathbf{w} = (1, 0) \) in \( \mathbb{R}^2 \) are linearly independent, but they are not orthogonal.

3. **TRUE.** We have

\[
\langle c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2, \mathbf{w} \rangle = \langle c_1 \mathbf{v}_1, \mathbf{w} \rangle + \langle c_2 \mathbf{v}_2, \mathbf{w} \rangle = c_1 \langle \mathbf{v}_1, \mathbf{w} \rangle + c_2 \langle \mathbf{v}_2, \mathbf{w} \rangle = c_1 \cdot 0 + c_2 \cdot 0 = 0.
\]
5. **FALSE.** For example, if $V$ is the inner product space of integrable functions on $(-\infty, \infty)$, then the formula
\[
\langle f, g \rangle = \int_{a}^{b} f(t)g(t)dt
\]
is a valid any product for any choice of real numbers $a < b$. See also Problem 9 in this section, in which a “non-standard” inner product on $\mathbb{R}^2$ is given.

7. **FALSE.** This definition of $\langle p, q \rangle$ will not satisfy the requirements of an inner product. For instance, if we take $p = x$, then $\langle p, p \rangle = 0$, but $p \neq 0$.

**Problems:**

1. $\langle v, w \rangle = 8$, $||v|| = 3\sqrt{3}$, $||w|| = \sqrt{7}$. Hence,
\[
\cos \theta = \frac{\langle v, w \rangle}{||v||||w||} = \frac{8}{3\sqrt{21}} \implies \theta \approx 0.95 \text{ radians}.
\]

3. $\langle v, w \rangle = (2 + i)(-1 - i) + (3 - 2i)(1 + 3i) + (4 + i)(3 + i) = 19 + 11i$. $||v|| = \sqrt{\langle v, v \rangle} = \sqrt{35}$, $||w|| = \sqrt{\langle w, w \rangle} = \sqrt{22}$.

5. We need only demonstrate one example showing that some property of an inner product is violated by the given formula. Set $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then according to the given formula, we have $\langle A, A \rangle = -2$, violating the requirement that $\langle u, u \rangle \geq 0$ for all vectors $u$.

7. $\langle A, B \rangle = 13$, $||A|| = \sqrt{33}$, $||B|| = \sqrt{7}$.

9. Property 1:
\[
\langle u, u \rangle = 2u_1u_1 + u_1u_2 + u_2u_1 + 2u_2u_2 = 2u_1^2 + 2u_1u_2 + 2u_2^2 = (u_1 + u_2)^2 + u_1^2 + u_2^2 \geq 0.
\]
\[
\langle u, u \rangle = 0 \iff (u_1 + u_2)^2 + u_1^2 + u_2^2 = 0 \iff u_1 = 0 = u_2 \iff u = 0.
\]
Property 2:
\[
\langle u, v \rangle = 2u_1v_1 + u_1v_2 + u_2v_1 + 2u_2v_2 = 2u_2v_2 + u_2v_1 + u_1v_2 + 2u_1v_1 = \langle v, u \rangle.
\]
Property 3:
\[
k(u, v) = k(2u_1v_1 + u_1v_2 + u_2v_1 + 2u_2v_2) = 2ku_1v_1 + ku_1v_2 + ku_2v_1 + 2ku_2v_2
\]
\[
= 2(ku_1)v_1 + (ku_1)v_2 + (ku_2)v_1 + 2(ku_2)v_2 = \langle ku, v \rangle
\]
\[
= 2ku_1v_1 + ku_1v_2 + ku_2v_1 + 2ku_2v_2
\]
\[
= 2u_1(kv_1) + u_1(kv_2) + u_2(kv_1) + 2u_2(kv_2) = \langle u, kv \rangle.
\]

Property 4:
\[
\langle (u + v), w \rangle = \langle (u_1 + v_1, u_2 + v_2), (w_1, w_2) \rangle
\]
\[
= 2(u_1 + v_1)w_1 + (u_1 + v_1)w_2 + (u_2 + v_2)w_1 + 2(u_2 + v_2)w_2
\]
\[
= 2u_1w_1 + 2v_1w_1 + u_1w_2 + v_1w_2 + u_2w_1 + v_2w_1 + 2u_2w_2 + 2v_2w_2
\]
\[
= 2u_1w_1 + u_1w_2 + u_2w_1 + 2u_2w_2 + 2v_1w_1 + v_1w_2 + v_2w_1 + 2v_2w_2
\]
\[
= \langle u, w \rangle + \langle v, w \rangle.
\]
Therefore $\langle u, v \rangle = 2u_1v_1 + u_1v_2 + u_2v_1 + 2u_2v_2$ defines an inner product on $\mathbb{R}^2$.

11. (a) Using the defined inner product:
\[
\langle v, w \rangle = 2 \cdot 2 \cdot 3 + 2 \cdot 6 + (-1)3 + 2(-1)6 = 9 \neq 0.
\]

(b) Using the standard inner product:
\[
\langle v, w \rangle = 2 \cdot 3 + (-1)6 = 0.
\]
13. (a) Show symmetry: \( \langle v, w \rangle = \langle w, v \rangle \).
\[
\langle v, w \rangle = \langle (v_1, v_2), (w_1, w_2) \rangle = v_1 w_1 - v_2 w_2 = w_1 v_1 - w_2 v_2 = \langle (w_1, w_2), (v_1, v_2) \rangle = \langle w, v \rangle.
\]

(b) Show \( k \langle v, w \rangle = \langle kv, w \rangle \).
Note that \( kv = k(v_1, v_2) = (kv_1, kv_2) \) and \( kw = k(w_1, w_2) = (kw_1, kw_2) \).
\[
\langle kv, w \rangle = \langle (kv_1, kv_2), (w_1, w_2) \rangle = (kv_1) w_1 - (kv_2) w_2 = k(v_1 w_1 - v_2 w_2) = k\langle (v_1, v_2), (w_1, w_2) \rangle = k\langle v, w \rangle.
\]

Also,
\[
\langle v, kw \rangle = \langle (v_1, v_2), (kw_1, kw_2) \rangle = v_1 (kw_1) - v_2 (kw_2) = k(v_1 w_1 - v_2 w_2) = k\langle (v_1, v_2), (w_1, w_2) \rangle = k\langle v, w \rangle.
\]

(c) Show \( \langle (u + v), w \rangle = \langle u, w \rangle + \langle v, w \rangle \).
Let \( w = (w_1, w_2) \) and note that \( u + v = (u_1 + v_1, u_2 + v_2) \).
\[
\langle (u + v), w \rangle = \langle (u_1 + v_1, u_2 + v_2), (w_1, w_2) \rangle = (u_1 + v_1) w_1 - (u_2 + v_2) w_2
\]
\[
= u_1 w_1 + v_1 w_1 - u_2 w_2 - v_2 w_2 = u_1 w_1 - u_2 w_2 + v_1 w_1 - v_2 w_2
\]
\[
= \langle (u_1, v_1), (w_1, w_2) \rangle + \langle (v_1, v_2), (w_1, w_2) \rangle = \langle u, w \rangle + \langle v, w \rangle.
\]
Property 1 fails since, for example, \( \langle u, u \rangle < 0 \) whenever \( |u_2| > |u_1| \).

15. \( \langle v, v \rangle < 0 \implies \langle (v_1, v_2), (v_1, v_2) \rangle < 0 \implies v_1^2 - v_2^2 < 0 \implies v_1^2 < v_2^2 \). In this space, timelike vectors are given by \( \{(v_1, v_2) \in \mathbb{R}^2 : v_1^2 < v_2^2 \} \).

17.

![Figure 0.0.34: Figure for Exercise 17](image)

19. We have
\[
\langle v, 0 \rangle = \langle v, 0 + 0 \rangle = \langle 0 + 0, v \rangle = \langle 0, v \rangle + \langle 0, v \rangle = \langle v, 0 \rangle + \langle v, 0 \rangle = 2\langle v, 0 \rangle,
\]
which implies that \( \langle v, 0 \rangle = 0 \).

21. For all \( v, w \in V \) and \( v_1, w_i \in \mathbb{C} \).
\[ |v + w|^2 = \langle v + w, v + w \rangle = \langle v, v + w \rangle + \langle w, v + w \rangle \] by Property 4
\[ = \langle v, v + w \rangle + \langle v + w, w \rangle \] by Property 2
\[ = \langle v, v \rangle + \langle w, v \rangle + \langle v, w \rangle + \langle w, w \rangle \] by Property 4
\[ = \langle v, v \rangle + \langle w, v \rangle + \langle v, w \rangle + \langle w, w \rangle \]
\[ = \|v\|^2 + \|w\|^2 + 2\Re\{\langle v, w \rangle\} \]
\[ = \|v\|^2 + 2\Re\{\langle v, w \rangle\} + \|v\|^2. \]

**Solutions to Section 4.12**

1. **TRUE.** An orthonormal basis is simply an orthogonal basis consisting of unit vectors.

3. **TRUE.** We can verify easily that
\[ \int_0^\pi \cos t \sin t \, dt = \frac{\sin^2 \frac{\pi}{2}}{2} = 0, \]
which means that \( \{\cos x, \sin x\} \) is an orthogonal set. Moreover, since they are non-proportional functions, they are linearly independent. Therefore, they comprise an orthogonal basis for the 2-dimensional inner product space \( \text{span}\{\cos x, \sin x\} \).

5. **TRUE.** This is the content of Theorem 4.12.7.

7. **TRUE.** We have
\[ P(w_1 + w_2, v) = \frac{w_1 + w_2}{\|v\|^2} \begin{pmatrix} v \\ v \end{pmatrix} = \frac{w_1}{\|v\|^2} \begin{pmatrix} v \\ v \end{pmatrix} + \frac{w_2}{\|v\|^2} \begin{pmatrix} v \\ v \end{pmatrix} = P(w_1, v) + P(w_2, v). \]

**Problems:**

1. \( \langle (2, -1, 1), (1, 1, -1) \rangle = 2 + (-1) + (-1) = 0; \)
\( \langle (2, -1, 1), (0, 1, 1) \rangle = 0 + (-1) + 1 = 0; \)
\( \langle (1, 1, -1), (0, 1, 1) \rangle = 0 + 1 + (-1) = 0. \)
Since each vector in the set is orthogonal to every other vector in the set, the vectors form an orthogonal set. To generate an orthonormal set, we divide each vector by its norm:
\[ \|(2, -1, 1)\| = \sqrt{4 + 1 + 1} = \sqrt{6}, \]
\[ \|(1, 1, -1)\| = \sqrt{1 + 1 + 1} = \sqrt{3}, \]
and
\[ \|(0, 1, 1)\| = \sqrt{0 + 1 + 1} = \sqrt{2}. \]
Thus, the corresponding orthonormal set is:
\[ \left\{ \frac{\sqrt{6}}{6}(2, -1, 1), \frac{\sqrt{3}}{3}(1, 1, -1), \frac{\sqrt{2}}{2}(0, 1, 1) \right\}. \]

3. \( \langle (1, 2, -1, 0), (0, 1, 0, 2) \rangle = 1 + 0 + (-1) + 0 = 0; \)
\( \langle (1, 2, -1, 0), (-1, 1, 1, 0) \rangle = -1 + 2 + (-1) + 0 = 0; \)
\( \langle (1, 2, -1, 0), (1, -1, -1, 0) \rangle = 1 + (-2) + 1 + 0 = 0; \)
\( \langle (1, 0, 1, 2), (-1, 1, 1, 0) \rangle = -1 + 0 + 1 + 0 = 0; \)
\( \langle (1, 0, 1, 2), (1, -1, -1, 0) \rangle = 1 + 0 + (-1) + 0 = 0; \)
\[ \langle -1,1,1,0 \rangle, (1,-1,-1,0) \rangle = -1 + (-1) + (-1) = -3. \]

Hence, this is not an orthogonal set of vectors.

5. We require that \( \langle v_1, v_2 \rangle = \langle v_1, w \rangle = \langle v_2, w \rangle = 0. \) Let \( w = (a,b,c) \) where \( a, b, c \in \mathbb{R} \).
\[ \langle v_1, v_2 \rangle = \langle (1,2,3), (1,1,-1) \rangle = 0. \]
\[ \langle v_1, w \rangle = \langle (1,2,3), (a,b,c) \rangle = a + 2b + 3c = 0. \]
\[ \langle v_2, w \rangle = \langle (1,1,-1), (a,b,c) \rangle = a + b - c = 0. \]

Letting the free variable \( c = t \in \mathbb{R} \), the system has the solution \( a = 5t, b = -4t \), and \( c = t \). Consequently, \( \{(1,2,3), (1,1,-1), (5t,-4t,t)\} \) will form an orthogonal set whenever \( t \neq 0 \). To determine the corresponding orthonormal set, we must divide each vector by its norm:
\[ \|v_1\| = \sqrt{1 + 4 + 9} = \sqrt{14}, \|v_2\| = \sqrt{1 + 1 + 1} = \sqrt{3}, \|w\| = \sqrt{25t^2 + 16t^2 + t^2} = \sqrt{42t^2} = |t|\sqrt{42} = t\sqrt{42} \]
if \( t \geq 0 \). Setting \( t = 1 \), an orthonormal set is:
\[ \left\{ \frac{\sqrt{14}}{14}(1,2,3), \frac{\sqrt{3}}{3}(1,1,-1), \frac{\sqrt{42}}{42}(5,-4,1) \right\}. \]

7. \( \langle (1-i,1+i,i), (0,i,1-i) \rangle = (1-i) \cdot 0 + (1+i)(-i) + i(1+i) = 0. \)
\( \langle (1-i,1+i,i), (-3+3i,2+2i,2i) \rangle = (1-i)(-3-3i) + (1+i)(2-2i) + i(-2i) = 0. \)
\( \langle (0,i,1-i), (-3+3i,2+2i,2i) \rangle = 0 + i(2-2i) + (1-i)(-2i) = (2i+2) + (-2i-2) = 0. \)

Hence, the vectors are orthogonal. To obtain a corresponding orthonormal set, we divide each vector by its norm.
\[ \|v_1\| = \sqrt{(1-i)^2(1+i)^2 + (1+i)^2(-i) + i(1+i)^2} = \sqrt{1+1+1+1} = \sqrt{4}. \]
\[ \|v_2\| = \sqrt{0 + i(-i) + (1-i)^2(1+i)} = \sqrt{1+1+1} = \sqrt{3}. \]
\[ \|v_3\| = \sqrt{(-3+3i)(-3-3i) + (2+2i)(2-2i) + 2i(-2i)} = \sqrt{9+9+4+4} = \sqrt{30}. \]

Consequently, an orthonormal set is:
\[ \left\{ \frac{\sqrt{5}}{5}(1-i,1+i,i), \frac{\sqrt{3}}{3}(0,i,1-i), \frac{\sqrt{30}}{30}(-3+3i,2+2i,2i) \right\}. \]

9. \( \langle f_1, f_2 \rangle = \langle 1, \sin \pi x \rangle = \int_{-1}^{1} \sin \pi x dx = \left[ -\cos \pi x \right]_{-1}^{1} = 0. \)
\( \langle f_1, f_3 \rangle = \langle 1, \cos \pi x \rangle = \int_{-1}^{1} \cos \pi x dx = \left[ \sin \pi x \right]_{-1}^{1} = 0. \)
\( \langle f_2, f_3 \rangle = \langle \sin \pi x, \cos \pi x \rangle = \int_{-1}^{1} \sin \pi x \cos \pi x dx = \frac{1}{2} \int_{-1}^{1} \sin 2\pi x dx = \left[ \frac{-1}{4\pi} \cos 2\pi x \right]_{-1}^{1} = 0. \)

Thus, the vectors are orthogonal.
\[ \|f_1\| = \sqrt{\int_{-1}^{1} 1 dx} = \sqrt{2}. \]
\[ \|f_2\| = \sqrt{\int_{-1}^{1} \sin^2 \pi x dx} = \sqrt{\int_{-1}^{1} \frac{1 - \cos 2\pi x}{2} dx} = \sqrt{\left[ \frac{x}{2} \right]_{-1}^{1} - \left[ \frac{1}{4\pi} \sin 2\pi x \right]_{-1}^{1}} = 1. \]
\[ \|f_3\| = \sqrt{\int_{-1}^{1} \cos^2 \pi x dx} = \sqrt{\int_{-1}^{1} \frac{1 + \cos 2\pi x}{2} dx} = \sqrt{\left[ \frac{x}{2} \right]_{-1}^{1} + \left[ \frac{1}{4\pi} \sin 2\pi x \right]_{-1}^{1}} = 1. \]

Consequently, \( \left\{ \frac{\sqrt{5}}{2}, \sin \pi x, \cos \pi x \right\} \) is an orthonormal set of functions on \([-1,1] \).
11. \( \langle f_1, f_2 \rangle = \int_{-1}^{1} \sin \pi x \sin 2\pi x dx = \frac{1}{2} \int_{-1}^{1} (\cos 3\pi x - \cos \pi x) dx = 0. \)

\( \langle f_1, f_3 \rangle = \int_{-1}^{1} \sin \pi x \sin 3\pi x dx = \frac{1}{2} \int_{-1}^{1} (\cos 4\pi x - \cos 2\pi x) dx = 0. \)

\( \langle f_2, f_3 \rangle = \int_{-1}^{1} \sin 2\pi x \sin 3\pi x dx = \frac{1}{2} \int_{-1}^{1} (\cos 5\pi x - \cos \pi x) dx = 0. \)

Therefore, \( \{f_1, f_2, f_3\} \) is an orthogonal set.

\[ ||f_1|| = \sqrt{\int_{-1}^{1} \sin^2 \pi x dx} = \sqrt{\int_{-1}^{1} \frac{1}{2} (1 - \cos 2\pi x) dx} = \sqrt{\frac{1}{2} \left[ x - \frac{\sin 2\pi x}{2\pi} \right]_{-1}^{1}} = 1. \]

\[ ||f_2|| = \sqrt{\int_{-1}^{1} \sin^2 2\pi x dx} = \sqrt{\int_{-1}^{1} \frac{1}{2} (1 - \cos 4\pi x) dx} = \sqrt{\frac{1}{2} \left[ x - \frac{\sin 4\pi x}{4\pi} \right]_{-1}^{1}} = 1. \]

\[ ||f_3|| = \sqrt{\int_{-1}^{1} \sin^2 3\pi x dx} = \sqrt{\int_{-1}^{1} \frac{1}{2} (1 - \cos 6\pi x) dx} = \sqrt{\frac{1}{2} \left[ x - \frac{\sin 6\pi x}{6\pi} \right]_{-1}^{1}} = 1. \]

Thus, it follows that \( \{f_1, f_2, f_3\} \) is an orthonormal set of vectors on \([-1, 1]\).

13. It is easily verified that \( \langle A_1, A_2 \rangle = 0, \langle A_1, A_3 \rangle = 0, \langle A_2, A_3 \rangle = 0 \). Thus we require \( a, b, c, d \) such that

\[ \langle A_1, A_2 \rangle = 0 \implies a + b - c + 2d = 0, \]
\[ \langle A_2, A_3 \rangle = 0 \implies -a + b + 2c + d = 0, \]
\[ \langle A_3, A_4 \rangle = 0 \implies a - 3b + 2d = 0. \]

Solving this system for \( a, b, c, \) and \( d \), we obtain: \( a = \frac{3}{2} c, \quad b = -\frac{1}{2} c, \quad d = 0 \). Thus,

\[ A_4 = \begin{bmatrix} \frac{3}{2} c & -\frac{1}{2} c \\ c & 0 \end{bmatrix} = 2c \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} = k \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}, \]

where \( k \) is any nonzero real number.

15. Let \( v_1 = (2, 1, -2) \) and \( v_2 = (1, 3, -1) \).

\( u_1 = v_1 = (2, 1, -2), \quad ||u_1|| = \sqrt{4 + 1 + 4} = 3. \)

\( \langle v_2, u_1 \rangle = \langle (1, 3, -1), (2, 1, -2) \rangle = 2 \cdot 1 + 1 \cdot 3 + (-2) \cdot (-1) = 7. \)

\( u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{||u_1||^2} u_1 = (1, 3, -1) - \frac{7}{9} (2, 1, -2) = \frac{5}{9} (-1, 4, 1). \)

\[ ||u_2|| = \frac{5}{9} \sqrt{1 + 16 + 1} = \frac{5\sqrt{2}}{3}. \]

Hence, an orthonormal basis is:

\[ \left\{ \frac{1}{3} (2, 1, -2), \frac{\sqrt{2}}{6} (-1, 4, 1) \right\}. \]

17. Let \( v_1 = (1, 0, -1, 0), \quad v_2 = (1, 1, -1, 0) \) and \( v_3 = (-1, 1, 0, 1) \).

\( u_1 = v_1 = (1, 0, -1, 0), \quad ||u_1|| = \sqrt{1 + 0 + 1 + 0} = \sqrt{2}. \)

\( \langle v_2, u_1 \rangle = \langle (1, 1, -1, 0), (1, 0, -1, 0) \rangle = 1 \cdot 1 + 1 \cdot 0 + (-1)(-1) + 0 \cdot 0 = 2. \)

\( u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{||u_1||^2} u_1 = (1, 1, -1, 0) - (1, 0, -1, 0) = (0, 1, 0, 0). \)

\[ ||u_2|| = \sqrt{0 + 1 + 0 + 0} = 1. \]

\( \langle v_3, u_1 \rangle = \langle (-1, 1, 0, 1), (1, 0, -1, 0) \rangle = (-1)1 + 1 \cdot 0 + 0(-1) + 1 \cdot 0 = -1. \)

\( \langle v_3, u_2 \rangle = \langle (-1, 1, 0, 1), (0, 1, 0, 0) \rangle = (-1)0 + 1 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 = 1. \)

\( u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{||u_1||^2} u_1 - \frac{\langle v_3, u_2 \rangle}{||u_2||^2} u_2 = (-1, 1, 0, 1) + \frac{1}{2} (1, 0, -1, 0) - (0, 1, 0, 0) = \frac{1}{2} (-1, 0, -1, 2); \)
\( \|u_3\| = \frac{1}{2} \sqrt{1 + 0 + 1 + 4} = \frac{\sqrt{6}}{2}. \) Hence, an orthonormal basis is:

\[
\left\{ \frac{\sqrt{2}}{2} (1, 0, -1, 0), (0, 1, 0, 0), \frac{\sqrt{6}}{6} (-1, 0, -1, 2) \right\}.
\]

19. Let \( v_1 = (1, 1, -1, 0), v_2 = (-1, 0, 1, 1) \) and \( v_3 = (2, -1, 2, 1). \)
\( u_1 = v_1 = (1, 1, -1, 0), \|u_1\| = \sqrt{1 + 1 + 1 + 0} = \sqrt{3}. \)
\( \langle v_2, u_1 \rangle = \langle (-1, 0, 1, 1), (1, 1, -1, 0) \rangle = -1 \cdot 1 + 0 \cdot 1 + 1(-1) + 1 \cdot 0 = -2. \)
\( u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1 = (-1, 0, 1, 1) - \frac{-2}{3} (1, 1, -1, 0) = \frac{1}{3}(-1, 2, 1, 3). \)
\( \|u_2\| = \frac{1}{3} \sqrt{1 + 4 + 1 + 9} = \frac{\sqrt{15}}{3}. \)
\( \langle v_3, u_1 \rangle = \langle (2, -1, 2, 1), (1, 1, -1, 0) \rangle = 2 \cdot 1 - 1 \cdot 1 + 2(-1) + 1 \cdot 0 = -1. \)
\( \langle v_3, u_2 \rangle = \langle (2, -1, 2, 1), (-1/3, 2/3, 1/3, 1) \rangle = 2(-1/3) + (-1)(2/3) + 2(1/3) + 1 \cdot 1 = \frac{1}{3}. \)
\( u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} u_2 = (2, -1, 2, 1) + \frac{1}{3}(1, 1, -1, 0) - \frac{1}{15}(-1, 2, 1, 3) = \frac{4}{5}(3, -1, 2, 1); \)
\( \|u_3\| = \frac{4\sqrt{15}}{5}. \) Hence, an orthonormal basis is:

\[
\left\{ \frac{\sqrt{3}}{3} (1, 1, -1, 0), \frac{\sqrt{15}}{15} (-1, 2, 1, 3), \frac{\sqrt{15}}{15} (3, -1, 2, 1) \right\}.
\]

21. Let \( v_1 = (1 - i, 0, i) \) and \( v_2 = (1, 1 + i, 0). \)
\( u_1 = v_1 = (1 - i, 0, i), \|u_1\| = \sqrt{(1 - i)(1 + i) + 0 + i(-i)} = \sqrt{3}. \)
\( \langle v_2, u_1 \rangle = \langle (1, 1 + i, 0), (1 - i, 0, i) \rangle = 1(1 + i) + (1 + i)0 + 0(-i) = 1 + i. \)
\( u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1 = (1, 1 + i, 0) - \frac{1 + i}{3} (1 - i, 0, i) = \frac{1}{3}(1, 3 + 3i, 1 - i). \)
\( \|u_2\| = \frac{1}{3} \sqrt{1 + (3 + 3i)(3 - 3i) + (1 - i)(1 + i)} = \frac{\sqrt{21}}{3}. \) Hence, an orthonormal basis is:

\[
\left\{ \frac{\sqrt{3}}{3} (1 - i, 0, i), \frac{\sqrt{21}}{21}(1, 3 + 3i, 1 - i) \right\}.
\]

23. Let \( f_1 = 1, f_2 = x \) and \( f_3 = x^2. \)
\( g_1 = f_1 = 1; \)
\( \|g_1\|^2 = \int_0^1 dx = 1; \langle f_2, g_1 \rangle = \int_0^1 x dx = \left[ \frac{x^2}{2} \right]_0^1 = \frac{1}{2}. \)
\( g_2 = f_2 - \frac{\langle f_2, g_1 \rangle}{\|g_1\|^2} g_1 = x - \frac{1}{2} = \frac{1}{2}(2x - 1). \)
\( \|g_2\|^2 = \int_0^1 \left( x - \frac{1}{2} \right)^2 dx = \int_0^1 \left( x^2 - x + \frac{1}{4} \right) dx = \left[ \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4} \right]_0^1 = \frac{1}{12}. \)
\( \langle f_3, g_1 \rangle = \int_0^1 x^2 dx = \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3}. \)
\( \langle f_3, g_2 \rangle = \int_0^1 x^2 \left( x - \frac{1}{2} \right) dx = \left[ \frac{x^4}{4} - \frac{x^3}{6} \right]_0^1 = \frac{1}{12}. \)
\[ g_3 = f_3 - \frac{\langle f_3, g_1 \rangle}{\|g_1\|^2} g_1 - \frac{\langle f_3, g_2 \rangle}{\|g_2\|^2} g_2 = x^2 - \frac{1}{3} - \left( x - \frac{1}{2} \right) = \frac{1}{6} (6x^2 - 6x + 1). \] Thus, an orthogonal basis is given by:
\[
\left\{ 1, \frac{1}{2} (2x - 1), \frac{1}{6} (6x^2 - 6x + 1) \right\}.
\]

25. Let \( f_1 = 1, f_2 = \sin x \) and \( f_3 = \cos x \) for all \( x \) in \([-\frac{\pi}{2}, \frac{\pi}{2}]\).
\[
\langle f_1, f_2 \rangle = \int_{-\pi/2}^{\pi/2} \sin x \, dx = \frac{\pi}{2} - \frac{\pi}{2} = 0. \] Therefore, \( f_1 \) and \( f_2 \) are orthogonal.
\[
\langle f_2, f_3 \rangle = \int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} 2 \sin x \, dx = \left[ -\cos 2x \right]_{-\pi/2}^{\pi/2} = 0. \] Therefore, \( f_2 \) and \( f_3 \) are orthogonal.
\[
\langle f_1, f_3 \rangle = \int_{-\pi/2}^{\pi/2} \cos x \, dx = \left[ \sin x \right]_{-\pi/2}^{\pi/2} = 2.
\]
Let \( g_1 = f_1 \) so that \( \|g_1\|^2 = \int_{-\pi/2}^{\pi/2} dx = \pi. \)
\[
g_2 = f_2 = \sin x, \quad \|g_2\|^2 = \int_{-\pi/2}^{\pi/2} \sin^2 x \, dx = \int_{-\pi/2}^{\pi/2} \frac{1 - \cos 2x}{2} \, dx = \frac{\pi}{2}.
\]
\[
g_3 = f_3 - \frac{\langle f_3, g_1 \rangle}{\|g_1\|^2} g_1 - \frac{\langle f_3, g_2 \rangle}{\|g_2\|^2} g_2 = \cos x - \frac{2}{\pi} \cdot 1 + 0 \cdot \sin x = \frac{1}{\pi} (\pi \cos x - 2). \] Thus, an orthogonal basis for the subspace of \( C^0[-\pi/2, \pi/2] \) spanned by \( \{1, \sin x, \cos x\} \) is:
\[
\left\{ 1, \sin x, \frac{1}{\pi} (\pi \cos x - 2) \right\}.
\]

27. Given \( A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \) and \( A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \). Using the Gram-Schmidt procedure:
\[
B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \langle A_2, B_1 \rangle = 5, \text{ and } \|B_1\|^2 = 5.
\]
\[
B_2 = A_2 - \frac{\langle A_2, B_1 \rangle}{\|B_1\|^2} B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]
Also, \( \langle A_3, B_1 \rangle = 5, \langle A_3, B_2 \rangle = 0, \text{ and } \|B_2\|^2 = 5, \) so that
\[
B_3 = A_3 - \frac{\langle A_3, B_1 \rangle}{\|B_1\|^2} B_1 - \frac{\langle A_3, B_2 \rangle}{\|B_2\|^2} B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\] Thus, an orthogonal basis for the subspace of \( M_2(\mathbb{R}) \) spanned by \( A_1, A_2, \) and \( A_3 \) is:
\[
\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},
\]
which is the subspace of all symmetric matrices in \( M_2(\mathbb{R}) \).

29. Given \( p_1(x) = 1 + x^2, p_2(x) = 2 - x + x^3, \) and \( p_3(x) = -x + 2x^2 \). Using the Gram-Schmidt procedure:
\[
q_1 = 1 + x^2, \quad \langle p_2, q_1 \rangle = 2 \cdot 1 + (-1)0 + 0 \cdot 1 + 1 \cdot 2 = 2, \quad \text{and } \|q_1\|^2 = 1^2 + 1^2 = 2. \] So,
\[
q_2 = p_2 - \frac{\langle p_2, q_1 \rangle}{\|q_1\|^2} q_1 = 2 - x + x^3 - (1 + x^2) = 1 - x - x^2 + x^3. \] Also,
\[
\langle p_3, q_1 \rangle = 0 \cdot 1 + (-1)0 + 2 \cdot 1 + 0^2 = 2,
\]
\[
\langle p_3, q_2 \rangle = 0 \cdot 1 + (-1)^2 + 2(-1) + 0 \cdot 1 = -1, \text{ and }\]
\[ \|q_2\|^2 = 1^2 + (-1)^2 + (-1)^2 + 1^2 = 4 \] so that
\[ q_3 = p_3 - \frac{\langle p_3, q_1 \rangle}{\|q_1\|^2} q_1 - \frac{\langle p_3, q_2 \rangle}{\|q_2\|^2} q_2 = -x + 2x^2 - (1 + x^2) + \frac{1}{4}(1 - x - x^2 + x^3) = \frac{1}{4}(-3 - 5x + 3x^2 + x^3). \] Thus, an orthogonal basis for the subspace spanned by \( p_1, p_2, \) and \( p_3 \) is \( \{1 + x^2, 1 - x - x^2 + x^3, -3 - 5x + 3x^2 + x^3\}. \)

31. It was shown in Remark 3 following Definition 4.12.1 that each vector \( u_i \) is a unit vector. Moreover, for \( i \neq j, \)
\[ \langle u_i, u_j \rangle = \frac{1}{\|v_i\|} \frac{1}{\|v_j\|} \langle v_i, v_j \rangle = 0, \] since \( \langle v_i, v_j \rangle = 0. \) Therefore \( \{u_1, u_2, \ldots, u_k\} \) is an orthonormal set of vectors.

33. We must show that \( W^\perp \) is closed under addition and closed under scalar multiplication.

Closure under addition: Let \( v_1 \) and \( v_2 \) belong to \( W^\perp. \) This means that \( \langle v_1, w \rangle = \langle v_2, w \rangle = 0 \) for all \( w \in W. \) Therefore,
\[ \langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle = 0 + 0 = 0 \] for all \( w \in W. \) Therefore, \( v_1 + v_2 \in W^\perp. \)

Closure under scalar multiplication: Let \( v \) belong to \( W^\perp \) and let \( c \) be a scalar. This means that \( \langle v, w \rangle = 0 \) for all \( w \in W. \) Therefore,
\[ \langle cv, w \rangle = c \langle v, w \rangle = c \cdot 0 = 0, \] which shows that \( cv \in W^\perp. \)

35. In this case, \( W^\perp \) consists of all \( (x, y, z, w) \in \mathbb{R}^4 \) such that \( \langle (x, y, z, w), (0, 1, -1, 3) \rangle = \langle (x, y, z, w), (1, 0, 0, 3) \rangle = 0. \) This requires that \( y - z + 3w = 0 \) and \( x + 3w = 0. \) We can associated an augmented matrix with this system of linear equations:
\[ \begin{bmatrix} 0 & 1 & -1 & 3 & 0 \\ 1 & 0 & 0 & 3 & 0 \end{bmatrix}. \] Note that \( z \) and \( w \) are free variables: \( z = s \) and \( w = t. \) Then \( y = s - 3t \) and \( x = -3t. \) Thus,
\[ W^\perp = \{(3t, s - 3t, s, t) : s, t \in \mathbb{R} \} = \{t(-3, -3, 0, 1) + s(0, 1, 1, 0) : s, t \in \mathbb{R} \} = \text{span}\{(3, -3, 0, 1), (0, 1, 1, 0)\}. \]

37. Suppose \( v \) belongs to both \( W \) and \( W^\perp. \) Then \( \langle v, v \rangle = 0 \) by definition of \( W^\perp, \) which implies that \( v = 0 \) by the first axiom of an inner product. Therefore \( W \) and \( W^\perp \) can contain no common elements aside from the zero vector.

39. (a) Using technology we find that
\[ \int_{-\pi}^{\pi} \sin nx \, dx = 0, \int_{-\pi}^{\pi} \cos nx \, dx = 0, \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0. \]
Further, for \( m \neq n, \)
\[ \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0. \]
Consequently the given set of vectors is orthogonal on \([\pi, \pi].\)

(b) Multiplying \( (4.12.7) \) by \( \cos mx \) and integrating over \([\pi, \pi]\) yields
\[ \int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos mx \, dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \cos mx \, dx. \]
Assuming that interchange of the integral and infinite summation is permissible, this can be written

\[ \int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos mx \, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx) \cos mx \, dx. \]

which reduces to

\[ \int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos mx \, dx + a_m \int_{-\pi}^{\pi} \cos^2 mx \, dx \]

where we have used the results from part (a). When \( m = 0 \), this gives

\[ \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \, dx = \pi a_0 \implies a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx, \]

whereas for \( m \neq 0 \),

\[ \int_{-\pi}^{\pi} f(x) \cos mx \, dx = a_m \int_{-\pi}^{\pi} \cos mx \, dx = \pi a_m \implies a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx. \]

(c) Multiplying (4.12.7) by \( \sin(mx) \), integrating over \([-\pi, \pi]\), and interchanging the integration and summation yields

\[ \int_{-\pi}^{\pi} f(x) \sin mx \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx) \sin mx \, dx. \]

Using the results from (a), this reduces to

\[ \int_{-\pi}^{\pi} f(x) \sin mx \, dx = b_n \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi b_n \implies b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx. \]

(d) The Fourier coefficients for \( f \) are

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, dx = 0, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \, dx = 0, \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx = -\frac{2}{n} \cos n\pi = \frac{2}{n} (-1)^{n+1}. \]

The Fourier series for \( f \) is

\[ \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx. \]

(e) The approximations using the first term, the first three terms, the first five terms, and the first ten terms in the Fourier series for \( f \) are shown in the accompanying figures.

These figures suggest that the Fourier series is converging to the function \( f(x) \) at all points in the interval \((-\pi, \pi)\).

**Solutions to Section 4.13**
Problems:
1. Write $v = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5$. Then we have
\[
(r + s)v = (r + s)(a_1, a_2, a_3, a_4, a_5) \\
= ((r + s)a_1, (r + s)a_2, (r + s)a_3, (r + s)a_4, (r + s)a_5) \\
= (ra_1 + sa_1, ra_2 + sa_2, ra_3 + sa_3, ra_4 + sa_4, ra_5 + sa_5) \\
= r(a_1, a_2, a_3, a_4, a_5) + s(a_1, a_2, a_3, a_4, a_5) \\
= rv + sv.
\]

3. **NO.** This set of polynomials is not closed under scalar multiplication. For example, the polynomial $p(x) = 2x$ belongs to the set, but $\frac{1}{3}p(x) = \frac{2}{3}x$ does not belong to the set (since $\frac{2}{3}$ is not an even integer).

5. **NO.** We can see immediately that the zero vector $(0, 0, 0)$ is not a solution to this linear system (the first equation is not satisfied by the zero vector), and therefore, we know at once that this set cannot be a vector space.

7. **NO.** This set is not closed under addition. For example, the vectors \[
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}
\]
both belong to the set (their entries are all nonzero), but
\[
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix},
\]
which does not belong to the set (some entries are zero, and some are nonzero). So closure under addition fails, and therefore, this set does not form a vector space.

9. **YES.** The set of functions $f : [0, 1] \rightarrow [0, 1]$ such that $f(0) = f\left(\frac{1}{3}\right) = f\left(\frac{2}{3}\right) = f(1) = 0$ is a subspace of the vector space of all functions $[0, 1] \rightarrow [0, 1]$. We confirm this by checking that this set is closed under addition and scalar multiplication:
Closure under Addition: Let \( g \) and \( h \) be functions such that
\[
g(0) = g\left(\frac{1}{4}\right) = g\left(\frac{1}{2}\right) = g\left(\frac{3}{4}\right) = g(1) = 0
\]
and
\[
h(0) = h\left(\frac{1}{4}\right) = h\left(\frac{1}{2}\right) = h\left(\frac{3}{4}\right) = h(1) = 0.
\]
Now
\[
(g + h)(0) = g(0) + h(0) = 0 + 0 = 0,
\]
\[
(g + h)\left(\frac{1}{4}\right) = g\left(\frac{1}{4}\right) + h\left(\frac{1}{4}\right) = 0 + 0 = 0,
\]
\[
(g + h)\left(\frac{1}{2}\right) = g\left(\frac{1}{2}\right) + h\left(\frac{1}{2}\right) = 0 + 0 = 0,
\]
\[
(g + h)\left(\frac{3}{4}\right) = g\left(\frac{3}{4}\right) + h\left(\frac{3}{4}\right) = 0 + 0 = 0,
\]
\[
(g + h)(1) = g(1) + h(1) = 0 + 0 = 0.
\]
Thus, \( g + h \) belongs to the set, and so the set is closed under addition.

Closure under Scalar Multiplication: Let \( g \) be a function with
\[
g(0) = g\left(\frac{1}{4}\right) = g\left(\frac{1}{2}\right) = g\left(\frac{3}{4}\right) = g(1) = 0,
\]
and let \( k \) be a scalar. Then
\[
(kg)(0) = k \cdot g(0) = k \cdot 0 = 0,
\]
\[
(kg)\left(\frac{1}{4}\right) = k \cdot g\left(\frac{1}{4}\right) = k \cdot 0 = 0,
\]
\[
(kg)\left(\frac{1}{2}\right) = k \cdot g\left(\frac{1}{2}\right) = k \cdot 0 = 0,
\]
\[
(kg)\left(\frac{3}{4}\right) = k \cdot g\left(\frac{3}{4}\right) = k \cdot 0 = 0,
\]
\[
(kg)(1) = k \cdot g(1) = k \cdot 0 = 0.
\]
Thus, \( kg \) belongs to the set, and so the set is closed under scalar multiplication.

11. NO. This set is not closed under addition. For example, if we let
\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
\]
then \( A^2 = A \) is symmetric, and \( B^2 = 0 \) is symmetric, but
\[
(A + B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}
\]
is not symmetric, so \( A + B \) is not in the set. Thus, the set in question is not closed under addition.
13. **NO.** This set is not closed under addition, nor under scalar multiplication. For instance, the point $(5, -3, 4)$ is a distance 5 from $(0, -3, 4)$, so the point $(5, -3, 4)$ lies in the set. But the point $(10, -6, 8) = 2(5, -3, 4)$ is a distance
\[
\sqrt{(10 - 0)^2 + (-6 + 3)^2 + (8 - 4)^2} = \sqrt{100 + 9 + 16} = \sqrt{125} \neq 5
\]
from $(0, -3, 4)$, so $(10, -6, 8)$ is not in the set. So the set is not closed under scalar multiplication, and hence does not form a subspace.

15. We must show that $W$ is closed under addition and closed under scalar multiplication:

*Closure under Addition:* Let $(a, 2^a)$ and $(b, 2^b)$ be elements of $W$. Now consider the sum of these elements:
\[
(a, 2^a) + (b, 2^b) = (a + b, 2^a 2^b) = (a + b, 2^{a+b}) \in W,
\]
which shows that $W$ is closed under addition.

*Closure under Scalar Multiplication:* Let $(a, 2^a)$ be an element of $W$, and let $k$ be a scalar. Then
\[
k(a, 2^a) = (ka, (2^a)^k) = (ka, 2^{ka}) \in W,
\]
which shows that $W$ is closed under scalar multiplication.

Thus, $W$ is a subspace of $V$.

17. We show that $S = \{(1, 4), (2, 1)\}$ is linearly independent and spans $V$.

* $S$ is linearly independent: Assume that
\[
c_1(1, 4) + c_2(2, 1) = (0, 1).
\]
This can be written
\[
(c_1, 4^{c_1}) + (2c_2, 1^{c_2}) = (0, 1)
\]
or
\[
(c_1 + 2c_2, 4^{c_1}) = (0, 1).
\]
In order for $4^{c_1} = 1$, we must have $c_1 = 0$. And then in order for $c_1 + 2c_2 = 0$, we must have $c_2 = 0$. Therefore, $S$ is linearly independent.

* $S$ spans $V$: Consider an arbitrary vector $(a_1, a_2) \in V$, where $a_2 > 0$. We must find constants $c_1$ and $c_2$ such that
\[
c_1(1, 4) + c_2(2, 1) = (a_1, a_2).
\]
Thus,
\[
(c_1, 4^{c_1}) + (2c_2, 1^{c_2}) = (a_1, a_2)
\]
or
\[
(c_1 + 2c_2, 4^{c_1}) = (a_1, a_2).
\]
Hence,
\[
c_1 + 2c_2 = a_1 \quad \text{and} \quad 4^{c_1} = a_2.
\]
From the second equation, we conclude that
\[
c_1 = \log_4(a_2).
\]
Thus, from the first equation,

\[ c_2 = \frac{1}{2} (a_1 - \log_4(a_2)). \]

Hence, since we were able to find constants \( c_1 \) and \( c_2 \) in order that

\[ c_1(1,4) + c_2(2,1) = (a_1, a_2), \]

we conclude that \{(1, 4), (2, 1)\} spans \( V \).

19. **NO.** This set is not closed under scalar multiplication. For example, \((1, 1)\) belongs to \( W \), but \(2 \cdot (1, 1) = (2, 2)\) does not belong to \( W \).

21. **NO.** The zero vector (zero matrix) is not an orthogonal matrix. Any subspace must contain a zero vector.

23. **YES.** We show that \( W \) is closed under addition and closed under scalar multiplication.

*Closure under Addition:* Assume that \( f \) and \( g \) belong to \( W \). Thus,

\[
\int_a^b f(x)dx = 0 \quad \text{and} \quad \int_a^b g(x)dx = 0.
\]

Hence,

\[
\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx = 0 + 0 = 0,
\]

so \( f + g \in W \).

*Closure under Scalar Multiplication:* Assume that \( f \) belongs to \( W \) and \( k \) is a scalar. Thus,

\[
\int_a^b f(x)dx = 0.
\]

Therefore,

\[
\int_a^b (kf)(x)dx = \int_a^b kf(x)dx = k \int_a^b f(x)dx = k \cdot 0 = 0,
\]

so \( kf \in W \).

25. (a) **NO,** (b) **YES.** Since 3 vectors are required to span \( \mathbb{R}^3 \), \( S \) cannot span \( V \). However, since the vectors are not proportional, they are linearly independent.

27. (a) **NO,** (b) **YES.** Since we have only 3 vectors in a 4-dimensional vector space, they cannot possibly span \( \mathbb{R}^4 \). To check linear independence, we place the vectors into the columns of a matrix:

\[
\begin{bmatrix}
6 & 1 & 1 \\
-3 & 1 & -8 \\
2 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}.
\]

The first three rows for the same invertible matrix as in the previous problem, so the reduced row-echelon form is

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix},
\]

so there are no free variables, and hence, the column vectors form a linearly independent set.
29. (a) YES, (b) YES. Consider the linear equation
\[ c_1(2x - x^3) + c_2(1 + x + x^2) + 3c_3 + c_4x = 0. \]
Then
\[ (c_2 + 3c_3) + (2c_1 + c_2 + c_4)x + c_2x^2 - c_1x^3 = 0. \]
From the latter equation, we see that \( c_1 = c_2 = 0 \) (looking at the \( x^2 \) and \( x^3 \) coefficients) and thus, \( c_3 = c_4 = 0 \) (looking at the constant term and \( x \) coefficient). Thus, \( c_1 = c_2 = c_3 = c_4 = 0 \), and hence, \( S \) is linearly independent. Since we have four linearly independent vectors in the 4-dimensional vector space \( P_3 \), we conclude that these vectors also span \( P_3 \).

31. (a) NO, (b) YES. The vector space \( M_{2 \times 3}(\mathbb{R}) \) is 6-dimensional, and since only four vectors belong to \( S \), \( S \) cannot possibly span \( M_{2 \times 3}(\mathbb{R}) \). On the other hand, if we form the linear system with augmented matrix
\[
\begin{bmatrix}
-1 & 3 & -1 & -11 & 0 \\
0 & 2 & -2 & -6 & 0 \\
0 & 1 & -3 & -5 & 0 \\
0 & 1 & 3 & 1 & 0 \\
1 & 2 & 2 & -2 & 0 \\
1 & 3 & 1 & -5 & 0 \\
\end{bmatrix},
\]
row reduction shows that each column of a row-echelon form contains a pivot, and therefore, the vectors are linearly independent.

33. Assume that \( \{v_1, v_2, v_3\} \) is linearly independent, and that \( v_4 \) does not lie in \( \text{span}\{v_1, v_2, v_3\} \). We will show that \( \{v_1, v_2, v_3, v_4\} \) is linearly independent. To do this, assume that
\[ c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0. \]
We must prove that \( c_1 = c_2 = c_3 = c_4 = 0 \).
If \( c_4 \neq 0 \), then we rearrange the above equation to show that
\[ v_4 = -\frac{c_1}{c_4}v_1 - \frac{c_2}{c_4}v_2 - \frac{c_3}{c_4}v_3, \]
which implies that \( v_4 \in \text{span}\{v_1, v_2, v_3\} \), contrary to our assumption. Therefore, we know that \( c_4 = 0 \). Hence the equation above reduces to
\[ c_1v_1 + c_2v_2 + c_3v_3 = 0, \]
and the linear independence of \( \{v_1, v_2, v_3\} \) now implies that \( c_1 = c_2 = c_3 = 0 \). Therefore, \( c_1 = c_2 = c_3 = c_4 = 0 \), as required.

35.
(a): Our proof here actually shows that the set of \( n \times n \) skew-symmetric matrices forms a subspace of \( M_n(\mathbb{R}) \) for all positive integers \( n \). We show that \( W \) is closed under addition and scalar multiplication:

Closure under Addition: Suppose that \( A \) and \( B \) are in \( W \). This means that \( A^T = -A \) and \( B^T = -B \). Then
\[ (A + B)^T = A^T + B^T = (-A) + (-B) = -(A + B), \]
so \( A + B \) is skew-symmetric. Therefore, \( A + B \) belongs to \( W \), and \( W \) is closed under addition.

Closure under Scalar Multiplication: Suppose that \( A \) is in \( W \) and \( k \) is a scalar. We know that \( A^T = -A \). Then
\[ (kA)^T = k(A^T) = k(-A) = -(kA), \]
so $kA$ is skew-symmetric. Therefore, $kA$ belongs to $W$, and $W$ is closed under scalar multiplication.

Therefore, $W$ is a subspace.

(b): An arbitrary $3 \times 3$ skew-symmetric matrix takes the form

$$
\begin{pmatrix}
0 & a & b \\
-a & 0 & c \\
-b & -c & 0
\end{pmatrix} = a
\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + b
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix} + c
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}.
$$

This results in $0_3$ if and only if $a = b = c = 0$, so the three matrices appearing on the right-hand side are linearly independent. Moreover, the equation above also demonstrates that $W$ is spanned by the three matrices appearing on the right-hand side. Therefore, these matrices are a basis for $W$:

$$
\text{Basis} = \left\{ \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}, E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23} \right\}.
$$

Hence, $\dim[W] = 3$.

(c): Since $\dim[M_3(\mathbb{R})] = 9$, we must add an additional six (linearly independent) $3 \times 3$ matrices to form a basis for $M_3(\mathbb{R})$. Using the notation prior to Example 4.6.3, we can use the matrices $E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23}$ to extend the basis in part (b) to a basis for $M_3(\mathbb{R})$:

$$
\text{Basis for } M_3(\mathbb{R}) = \left\{ \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}, E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23} \right\}.
$$

37.

(a): We must verify the axioms (A1)-(A10) for a vector space:

Axiom (A1): Assume that $(v_1, w_1)$ and $(v_2, w_2)$ belong to $V \oplus W$. Since $v_1 +_V v_2 \in V$ and $w_1 +_W w_2 \in W$, then the sum

$$(v_1, w_1) + (v_2, w_2) = (v_1 +_V v_2, w_1 +_W w_2)$$

lies in $V \oplus W$.

Axiom (A2): Assume that $(v, w)$ belongs to $V \oplus W$, and let $k$ be a scalar. Since $k \cdot_V v \in V$ and $k \cdot_W w \in W$, the scalar multiplication

$$k \cdot (v, w) = (k \cdot_V v, k \cdot_W w)$$

lies in $V \oplus W$.

For the remainder of the axioms, we will omit the $\cdot_V$ and $\cdot_W$ notations. They are to be understood.

Axiom (A3): Assume that $(v_1, w_1), (v_2, w_2) \in V \oplus W$. Then

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1),$$

as required.
Axiom (A4): Assume that \((v_1, w_1), (v_2, w_2), (v_3, w_3) \in V \oplus W\). Then
\[
((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3)
\]
\[
= ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3)
\]
\[
= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3))
\]
\[
= (v_1, w_1) + ((v_2, w_2) + (v_3, w_3)),
\]
as required.

Axiom (A5): We claim that the zero vector in \(V \oplus W\) is \((0_V, 0_W)\), where \(0_V\) is the zero vector in the vector space \(V\) and \(0_W\) is the zero vector in the vector space \(W\). To check this, let \((v, w) \in V \oplus W\). Then
\[
(0_V, 0_W) + (v, w) = (0_V + v, 0_W + w) = (v, w),
\]
which confirms that \((0_V, 0_W)\) is the zero vector for \(V \oplus W\).

Axiom (A6): We claim that the additive inverse of the vector \((v, w) \in V \oplus W\) is the vector \((-v, -w)\), where \(-v\) is the additive inverse of \(v\) in the vector space \(V\) and \(-w\) is the additive inverse of \(w\) in the vector space \(W\). We check this:
\[
(v, w) + (-v, -w) = (v + (-v), w + (-w)) = (0_V, 0_W),
\]
as required.

Axiom (A7): For every vector \((v, w) \in V \oplus W\), we have
\[
1 \cdot (v, w) = (1 \cdot v, 1 \cdot w) = (v, w),
\]
where in the last step we have used the fact that Axiom (A7) holds in each of the vector spaces \(V\) and \(W\).

Axiom (A8): Let \((v, w)\) be a vector in \(V \oplus W\), and let \(r\) and \(s\) be scalars. Using the fact that Axiom (A8) holds in \(V\) and \(W\), we have
\[
(rs)(v, w) = ((rs)v, (rs)w)
\]
\[
= (r(sv), r(sw))
\]
\[
= r(sv, sw)
\]
\[
= r(s(v, w)).
\]

Axiom (A9): Let \((v_1, w_1)\) and \((v_2, w_2)\) be vectors in \(V \oplus W\), and let \(r\) be a scalar. Then
\[
r((v_1, w_1) + (v_2, w_2)) = r(v_1 + v_2, w_1 + w_2)
\]
\[
= (rv_1 + rv_2, rw_1 + rw_2)
\]
\[
= (rv_1, rw_1) + (rv_2, rw_2)
\]
\[
= r(v_1, w_1) + r(v_2, w_2)),
\]
as required.

Axiom (A10): Let \((v, w) \in V \oplus W\), and let \(r\) and \(s\) be scalars. Then
\[
(r + s)(v, w) = ((r + s)v, (r + s)w)
\]
\[
= (rv + sv, rw + sw)
\]
\[
= (rv, rw) + (sv, sw)
\]
\[
= r(v, w) + s(v, w),
\]
as required.
as required.

(b): We show that \( \{(v, 0) : v \in V\} \) is a subspace of \( V \oplus W \), by checking closure under addition and closure under scalar multiplication:

**Closure under Addition:** Suppose \((v_1, 0)\) and \((v_2, 0)\) belong to \(\{(v, 0) : v \in V\}\), where \(v_1, v_2 \in V\). Then

\[
(v_1, 0) + (v_2, 0) = (v_1 + v_2, 0) \in \{(v, 0) : v \in V\},
\]

which shows that the set is closed under addition.

**Closure under Scalar Multiplication:** Suppose \((v, 0) \in \{(v, 0) : v \in V\}\) and \(k\) is a scalar. Then \(k(v, 0) = (kv, 0)\) is again in the set. Thus, \(\{(v, 0) : v \in V\}\) is closed under scalar multiplication.

Therefore, \(\{(v, 0) : v \in V\}\) is a subspace of \(V \oplus W\).

(c): Let \(\{v_1, v_2, \ldots, v_n\}\) be a basis for \(V\), and let \(\{w_1, w_2, \ldots, w_m\}\) be a basis for \(W\). We claim that

\[
S = \{(v_i, 0) : 1 \leq i \leq n\} \cup \{(0, w_j) : 1 \leq j \leq m\}
\]

is a basis for \(V \oplus W\). To show this, we will verify that \(S\) is a linearly independent set that spans \(V \oplus W\):

**Check that \(S\) is linearly independent:** Assume that

\[
c_1(v_1, 0) + c_2(v_2, 0) + \cdots + c_n(v_n, 0) + d_1(0, w_1) + d_2(0, w_2) + \cdots + d_m(0, w_m) = (0, 0).
\]

We must show that

\[
c_1 = c_2 = \cdots = c_n = d_1 = d_2 = \cdots = d_m = 0.
\]

Adding the vectors on the left-hand side, we have

\[
(c_1v_1 + c_2v_2 + \cdots + c_nv_n, d_1w_1 + d_2w_2 + \cdots + d_mw_m) = (0, 0),
\]

so that

\[
c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0 \quad \text{and} \quad d_1w_1 + d_2w_2 + \cdots + d_mw_m = 0.
\]

Since \(\{v_1, v_2, \ldots, v_n\}\) is linearly independent,

\[
c_1 = c_2 = \cdots = c_n = 0,
\]

and since \(\{w_1, w_2, \ldots, w_m\}\) is linearly independent,

\[
d_1 = d_2 = \cdots = d_m = 0.
\]

Thus, \(S\) is linearly independent.

**Check that \(S\) spans \(V \oplus W\):** Let \((v, w) \in V \oplus W\). We must express \((v, w)\) as a linear combination of the vectors in \(S\). Since \(\{v_1, v_2, \ldots, v_n\}\) spans \(V\), there exist scalars \(c_1, c_2, \ldots, c_n\) such that

\[
v = c_1v_1 + c_2v_2 + \cdots + c_nv_n,
\]

and since \(\{w_1, w_2, \ldots, w_m\}\) spans \(W\), there exist scalars \(d_1, d_2, \ldots, d_m\) such that

\[
w = d_1w_1 + d_2w_2 + \cdots + d_mw_m.
\]

Then

\[
(v, w) = c_1(v_1, 0) + c_2(v_2, 0) + \cdots + c_n(v_n, 0) + d_1(0, w_1) + d_2(0, w_2) + \cdots + d_m(0, w_m).
\]
Therefore, \((v, w)\) is a linear combination of vectors in \(S\), so \(S\) spans \(V \oplus W\).
Therefore, \(S\) is a basis for \(V \oplus W\). Since \(S\) contains \(n + m\) vectors, \(\dim[V \oplus W] = n + m\).

39. Let \(A\) be an \(m \times n\) matrix. By the Rank-Nullity Theorem,
\[
\dim[\text{colspace}(A)] + \dim[\text{nullspace}(A)] = n.
\]
Since, by assumption, \(\text{colspace}(A) = \text{nullspace}(A) = r\), \(n = 2r\) must be even.

41. A row-echelon form of \(A\) is given by \[
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}.
\]
Thus, a basis for the rowspace of \(A\) is given by \(\{(1, 2)\}\), a basis for the columnspace of \(A\) is given by \(\left\{ \begin{bmatrix} -3 \\ -6 \end{bmatrix} \right\}\), and a basis for the nullspace of \(A\) is given by \(\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}\). All three subspaces are one-dimensional.

43. A row-echelon form of \(A\) is given by \[
\begin{bmatrix}
1 & -2.5 & -5 \\
0 & 1 & 1.3 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}.
\]
Thus, a basis for the rowspace of \(A\) is given by \(\{(1, -2.5, -5), (0, 1, 1.3), (0, 0, 1)\}\), and a basis for the columnspace of \(A\) is given by \(\left\{ \begin{bmatrix} -4 \\ 0 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 5 \\ 10 \end{bmatrix} \right\}\).

Moreover, we have that the nullspace of \(A\) is 0-dimensional, and so the basis is empty.

45. We will obtain bases for the rowspace, columnspace, and nullspace and orthonormalize them. A row-echelon form of \(A\) is given by \[
\begin{bmatrix}
1 & 2 & 6 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
We see that a basis for the rowspace of \(A\) is given by \(\{(1, 2, 6), (0, 1, 2)\}\).

We apply Gram-Schmidt to this set, and thus we need to replace \((0, 1, 2)\) by
\[
(0, 1, 2) - \frac{14}{41}(1, 2, 6) = \left( -\frac{14}{41}, \frac{13}{41}, -\frac{2}{41} \right).
\]
So an orthogonal basis for the rowspace of \(A\) is given by \(\left\{ (1, 2, 6), \left( -\frac{14}{41}, \frac{13}{41}, -\frac{2}{41} \right) \right\}\).

To replace this with an orthonormal basis, we must normalize each vector. The first one has norm \(\sqrt{41}\) and the second one has norm \(\sqrt{41}\). Hence, an orthonormal basis for the rowspace of \(A\) is
\[
\left\{ \frac{\sqrt{41}}{41}(1, 2, 6), \frac{\sqrt{41}}{3} \left( -\frac{14}{41}, \frac{13}{41}, -\frac{2}{41} \right) \right\}.
\]
Returning to the row-echelon form of $A$ obtained above, we see that a basis for the columnspace of $A$ is
\[
\begin{bmatrix}
1 \\
2 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
2 \\
1 \\
1 \\
0
\end{bmatrix}
\]
We apply Gram-Schmidt to this set, and thus we need to replace $\begin{bmatrix}
2 \\
1 \\
0 \\
1
\end{bmatrix}$ by
\[
\begin{bmatrix}
2 \\
1 \\
0 \\
1
\end{bmatrix} - \frac{4}{6} \begin{bmatrix}
1 \\
2 \\
0 \\
1
\end{bmatrix} = \frac{4}{3} \begin{bmatrix}
1 \\
-1 \\
3 \\
-2
\end{bmatrix}.
\]
So an orthogonal basis for the columnspace of $A$ is given by
\[
\begin{bmatrix}
1 \\
2 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
3 \\
\frac{4}{3} \\
\frac{1}{3}
\end{bmatrix}
\]
The norms of these vectors are, respectively, $\sqrt{6}$ and $\sqrt{30}/3$. Hence, we normalize the above orthogonal basis to obtain the orthonormal basis for the columnspace:
\[
\begin{bmatrix}
\frac{\sqrt{6}}{6} \\
1 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
\frac{\sqrt{30}}{30} \\
\frac{4}{3} \\
\frac{-1}{3} \\
\frac{-2}{3}
\end{bmatrix}
\]
Returning once more to the row-echelon form of $A$ obtained above, we see that, in order to find the nullspace of $A$, we must solve the equations $x + 2y + 6z = 0$ and $y + 2z = 0$. Setting $z = t$ as a free variable, we find that $y = -2t$ and $x = -2t$. Thus, a basis for the nullspace of $A$ is $\{(-2, -2, 1)\}$, which can be normalized to
\[
\left\{ \left( -\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \right\}.
\]
47. Let $x_1 = (5, -1, 2)$ and let $x_2 = (7, 1, 1)$. Using the Gram-Schmidt process, we have
\[
v_1 = x_1 = (5, -1, 2)
\]
and
\[
v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\|v_1\|^2} v_1 = (7, 1, 1) - \frac{36}{30} (5, -1, 2) = (7, 1, 1) - (6, -\frac{6}{5}, \frac{12}{5}) = (1, \frac{11}{5}, -\frac{7}{5}).
\]
Hence, an orthogonal basis is given by
\[
\{(5, -1, 2), (1, \frac{11}{5}, -\frac{7}{5})\}.
57. Any of the conditions (a)-(p) appearing in the Invertible Matrix Theorem would be appropriate at this point in the text.

**Solutions to Section 5.1**
True-False Review:

1. **FALSE.** The conditions $T(u + v) = T(u) + T(v)$ and $T(c \cdot v) = cT(v)$ must hold for all vectors $u, v$ in $V$ and for all scalars $c$, not just "for some".

3. **FALSE.** This will only necessarily hold for a linear transformation, not for more general mappings.

5. **TRUE.** Since
   \[0 = T(0) = T(v + (-v)) = T(v) + T(-v),\]
we conclude that $T(-v) = -T(v)$.

Problems:

1. Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and $c \in \mathbb{R}$.
   
   
   \[
   T((x_1, x_2) + (y_1, y_2)) = T(x_1 + y_1, x_2 + y_2) = (x_1 + y_1 + 2x_2 + 2y_2, 2x_1 + 2y_1 - x_2 - y_2) = (x_1 + 2x_2, 2x_1 - x_2) + (y_1 + 2y_2, 2y_1 - y_2) = T(x_1, x_2) + T(y_1, y_2).
   \]
   
   
   Consequently, $T$ is a linear transformation.

3. Let $y_1, y_2 \in C^2(I)$ and $c \in \mathbb{R}$. Then,
   
   
   \[
   T(y_1 + y_2) = (y_1 + y_2)'' - 16(y_1 + y_2) = (y_1'' - 16y_1) + (y_2'' - 16y_2) = T(y_1) + T(y_2).
   \]
   
   
   Consequently, $T$ is a linear transformation.

5. Let $f, g \in V$ and $c \in \mathbb{R}$. Then
   \[
   T(f + g) = \int_a^b (f + g)(x)dx = \int_a^b f(x) + f(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx = T(f) + T(g).
   \]
   
   
   Therefore, $T$ is a linear transformation.

7. Let $A, B \in M_n(\mathbb{R})$ and $c \in \mathbb{R}$. Then,
   
   
   \[
   S(A + B) = (A + B) + (A + B)^T = A + A^T + B + B^T = S(A) + S(B).
   \]
   
   
   \[
   S(cA) = (cA) + (cA)^T = c(A + A^T) = cS(A).
   \]
   
   Consequently, $S$ is a linear transformation.

9. Let $x = (x_1, x_2), y = (y_1, y_2)$ be in $\mathbb{R}^2$. Then
   
   
   \[
   T(x + y) = T(x_1 + x_2 + y_1 + y_2) = (x_1 + x_2 + y_1 + y_2, 2),
   \]
   
   whereas
   
   \[
   T(x) + T(y) = (x_1 + x_2, 2) + (y_1 + y_2, 2) = (x_1 + x_2 + y_1 + y_2, 4).
   \]
   
   We see that $T(x + y) \neq T(x) + T(y)$, hence $T$ is not a linear transformation.

11. If $T(x_1, x_2) = (3x_1 - 2x_2, x_1 + 5x_2)$, then $A = [T(e_1), T(e_2)] = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$. 

13. If \( T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, x_3 - x_1) \), then \( A = [T(e_1), T(e_2), T(e_3)] = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \).

15. If \( T(x_1, x_2, x_3) = (x_3 - x_1, -x_1, 3x_1 + 2x_3, 0) \), then \( A = [T(e_1), T(e_2), T(e_3)] = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ 3 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \).

17. \( T(x) = Ax = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ 3x_1 + x_2 - 2x_3 \end{bmatrix} \). which we write as

\[ T(x_1, x_2, x_3) = (2x_1 - x_2 + 5x_3, 3x_1 + x_2 - 2x_3). \]

19. \( T(x) = Ax = \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} x \\ -2x \\ 0 \\ x \end{bmatrix} \), which we write as

\[ T(x) = (-3x, -2x, 0, x). \]

21. Let \( u \) be a fixed vector in \( V \), \( v_1, v_2 \in V \), and \( c \in \mathbb{R} \). Then

\[ T(v_1 + v_2) = \langle u, v_1 + v_2 \rangle = \langle u, v_1 \rangle + \langle u, v_2 \rangle = T(v_1) + T(v_2). \]

\[ T(cv_1) = \langle u, cv_1 \rangle = c \langle u, v_1 \rangle = c T(v_1). \] Thus, \( T \) is a linear transformation.

23. (a) If \( D = [v_1, v_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \) then \( \det(D) = -2 \neq 0 \), so by Corollary 4.5.15 the vectors \( v_1 = (1, 1) \) and \( v_2 = (1, -1) \) are linearly independent. Since \( \dim[\mathbb{R}^2] = 2 \), it follows from Theorem 4.6.10 that \( \{v_1, v_2\} \) is a basis for \( \mathbb{R}^2 \).

(b) Let \( x = (x_1, x_2) \) be an arbitrary vector in \( \mathbb{R}^2 \). Since \( \{v_1, v_2\} \) forms a basis for \( \mathbb{R}^2 \), there exist \( c_1 \) and \( c_2 \) such that

\[ (x_1, x_2) = c_1(1, 1) + c_2(1, -1), \]

that is, such that

\[ c_1 + c_2 = x_1, \quad c_1 - c_2 = x_2. \]

Solving this system yields \( c_1 = \frac{1}{2}(x_1 + x_2), \quad c_2 = \frac{1}{2}(x_1 - x_2) \). Thus,

\[ (x_1, x_2) = \frac{1}{2}(x_1 + x_2)v_1 + \frac{1}{2}(x_1 - x_2)v_2, \]

so that

\[ T[(x_1, x_2)] = T \left[ \frac{1}{2}(x_1 + x_2)v_1 + \frac{1}{2}(x_1 - x_2)v_2 \right] \]

\[ = \frac{1}{2}(x_1 + x_2)T(v_1) + \frac{1}{2}(x_1 - x_2)T(v_2) \]

\[ = \frac{1}{2}(x_1 + x_2)(2, 3) + \frac{1}{2}(x_1 - x_2)(-1, 1) \]

\[ = \left( \frac{x_1}{2} + \frac{3x_2}{2}, 2x_1 + x_2 \right). \]

In particular, when \( (4, -2) \) is substituted for \( (x_1, x_2) \), it follows that \( T(4, -2) = (-1, 6) \).
25. The matrix of $T$ is the $2 \times 4$ matrix $[T(e_1), T(e_2), T(e_3), T(e_4)]$. Therefore, we must determine $T(1,0,0,0)$, $T(0,1,0,0)$, $T(0,0,1,0)$, and $T(0,0,0,1)$, which we can determine from the given information by using the linear transformation properties. We are given that

$$T(1,0,0,0) = (3,-2).$$

Next,

$$T(0,1,0,0) = T(1,1,0,0) - T(1,0,0,0) = (5,1) - (3,-2) = (2,3),$$

and

$$T(0,0,1,0) = T(1,1,1,0) - T(1,1,0,0) = (-1,0) - (5,1) = (-6,-1),$$

Therefore, we have the matrix of $T$:

$$
\begin{bmatrix}
3 & 2 & -6 & 3 \\
-2 & 3 & -1 & 2
\end{bmatrix}.
$$

27. The matrix of $T$ is the $4 \times 3$ matrix $[T(e_1), T(e_2), T(e_3)]$. Therefore, we must determine $T(1,0,0)$, $T(0,1,0)$, and $T(0,0,1)$, which we can determine from the given information by using the linear transformation properties. A quick calculation shows that $(1,0,0) = \frac{1}{3}(0,-1,4) - \frac{1}{4}(0,3,3) + \frac{1}{4}(4,4,-1)$,

$$T(1,0,0) = \frac{1}{4}T(0,-1,4) - \frac{1}{4}T(0,3,3) + \frac{1}{4}T(4,4,-1) = \frac{1}{4}(2,5,-2,1) - \frac{1}{4}(-1,0,0,5) + \frac{1}{4}(-3,1,1,3) = (0,3,0,\frac{1}{4}),$$

Similarly, $(0,1,0) = -\frac{1}{5}(0,-1,4) + \frac{4}{15}(0,3,3)$, so

$$T(0,1,0) = -\frac{1}{5}(2,5,-2,1) + \frac{4}{15}(-1,0,0,5) = (-\frac{2}{3},1,-\frac{2}{5},\frac{17}{15}).$$

Finally, $(0,0,1) = \frac{1}{5}(0,-1,4) + \frac{1}{15}(0,3,3)$, so

$$T(0,0,1) = \frac{1}{5}T(0,-1,4) + \frac{1}{15}T(0,3,3) = \frac{1}{5}(2,5,-2,1) + \frac{1}{15}(-1,0,0,5) = (\frac{1}{3},1,-\frac{2}{5},\frac{8}{15}).$$

Therefore, we have the matrix of $T$:

$$
\begin{bmatrix}
0 & -2/3 & 1/3 \\
3/2 & -1 & 1 \\
0 & 2/5 & -2/5 \\
-1/4 & 17/15 & 8/15
\end{bmatrix}.
$$

29. Using the linearity of $T$, we have $T(2v_1 + 3v_2) = v_1 + v_2$ and $T(v_1 + v_2) = 3v_1 - v_2$. That is, $2T(v_1) + 3T(v_2) = v_1 + v_2$ and $T(v_1) + T(v_2) = 3v_1 - v_2$. Solving this system for the unknowns $T(v_1)$ and $T(v_2)$, we obtain $T(v_2) = 3v_2 - 5v_1$ and $T(v_1) = 8v_1 - 4v_2$.

31. Let $v \in V$. Since $\{v_1, v_2\}$ is a basis for $V$, there exists $a, b \in \mathbb{R}$ such that $v = av_1 + bv_2$. Hence

$$T(v) = T(av_1 + bv_2) = aT(v_1) + bT(v_2) = a(3v_1 - v_2) + b(v_1 + 2v_2) = 3av_1 - av_2 + bv_1 + 2bv_2 = (3a + b)v_1 + (2b - a)v_2.$$

33. Let $v$ be any vector in $V$. Since $\{v_1, v_2, \ldots, v_k\}$ is a basis for $V$, we can write $v = c_1v_1 + c_2v_2 + \cdots + c_kv_k$ for suitable scalars $c_1, c_2, \ldots, c_k$. Then

$$T(v) = T(c_1v_1 + c_2v_2 + \cdots + c_kv_k) = c_1T(v_1) + c_2T(v_2) + \cdots + c_kT(v_k) = c_10 + c_20 + \cdots + c_k0 = 0.$$
as required.

35. \((T_1 + T_2)(x) = T_1(x) + T_2(x) = Ax + Bx = (A + B)x = \begin{bmatrix} 5 & 6 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 + 6x_2 \\ 2x_1 - 2x_2 \end{bmatrix}.\)

Hence, \((T_1 + T_2)(x_1, x_2) = (5x_1 + 6x_2, 2x_1 - 2x_2).\)

\((cT_1)(x) = cT_1(x) = c(Ax) = (cA)x = \begin{bmatrix} 3c & c \\ -c & 2c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3cx_1 + cx_2 \\ -cx_1 + 2cx_2 \end{bmatrix}.\)

Hence, \((cT_1)(x_1, x_2) = (3cx_1 + cx_2, -cx_1 + 2cx_2).\)

37. Problem 34 establishes that if \(T_1\) and \(T_2\) are in \(L(V,W)\) and \(c\) is any scalar, then \(T_1 + T_2\) and \(cT_1\) are in \(L(V,W)\). Consequently, Axioms (A1) and (A2) are satisfied.

A3: Let \(v\) be any vector in \(L(V,W)\). Then

\[(T_1 + T_2)(v) = T_1(v) + T_2(v) = T_2(v) + T_1(v) = (T_2 + T_1)(v).\]

Hence \(T_1 + T_2 = T_2 + T_1\), therefore the addition operation is commutative.

A4: Let \(T_3 \in L(V,W)\). Then

\[[(T_1 + T_2) + T_3](v) = (T_1 + T_2)(v) + T_3(v) = [T_1(v) + T_2(v)] + T_3(v) = T_1(v) + [T_2(v) + T_3(v)] = T_1(v) + (T_2 + T_3)(v) = [T_1 + (T_2 + T_3)](v).\]

Hence \((T_1 + T_2) + T_3 = T_1 + (T_2 + T_3)\), therefore the addition operation is associative.

A5: The zero vector in \(L(V,W)\) is the zero transformation, \(O: V \rightarrow W\), defined by

\[O(v) = 0, \text{ for all } v \text{ in } V,\]

where \(0\) denotes the zero vector in \(V\). To show that \(O\) is indeed the zero vector in \(L(V,W)\), let \(T\) be any transformation in \(L(V,W)\). Then

\[(T + O)(v) = T(v) + O(v) = T(v) + 0 = T(v) \text{ for all } v \in V,\]

so that \(T + O = T\).

A6: The additive inverse of the transformation \(T \in L(V,W)\) is the linear transformation \(-T\) defined by

\([T + (-T)](v) = T(v) + (-T)(v) = T(v) + (-1)T(v) = T(v) - T(v) = 0,\]

for all \(v \in V\), so that \(T + (-T) = O\).

A7-A10 are all straightforward verifications.

**Solutions to Section 5.2**

**True-False Review:**

1. **FALSE.** For example, \(T(x_1, x_2) = (0, 0)\) is a linear transformation that maps every line to the origin.

3. **FALSE.** A shear parallel to the \(x\)-axis composed with a shear parallel to the \(y\)-axis is given by matrix

\[
\begin{bmatrix}
1 & k \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
k & 1
\end{bmatrix} =
\begin{bmatrix}
1 + kl & k \\
l & 1
\end{bmatrix},
\]

which is not a shear.

5. **FALSE.** For example,

\[
R_{xy} \cdot R_x = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix} =
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix},
\]
and this matrix is not in the form of a stretch.

**Problems:**

1. $T(1, 1) = (1, -1), \ T(2, 1) = (1, -2), \ T(2, 2) = (2, -2), \ T(1, 2) = (2, -1)$. 

3. $T(1, 1) = (2, 0), \ T(2, 1) = (3, -1), \ T(2, 2) = (4, 0), \ T(1, 2) = (3, 1)$. 

5. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \implies T(x) = Ax$ corresponds to a shear parallel to the $y$-axis.

7. $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \implies T(x) = Ax$ corresponds to a shear parallel to the $y$-axis.

9. $\begin{bmatrix} 1 & -3 \\ -2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1. $A_{12}(2)$  
2. $M_{2}(1/2)$  
3. $A_{21}(3)$.
which corresponds to a shear parallel to the $x$-axis, followed by a stretch in the $y$-direction, followed by a shear parallel to the $y$-axis.

11. \[
\begin{bmatrix}
1 & 0 \\
0 & -2 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 2 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}
\]. So $T(x) = Ax$ corresponds to a reflection in the $x$-axis followed by a stretch in the $y$-direction.

13. 
\[
R(\theta) = \begin{bmatrix}
\cos \theta & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & \sec \theta \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\cos \theta & -\tan \theta \\
0 & \sin \theta \\
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
\end{bmatrix}
\]
which coincides with the matrix of the transformation of $\mathbb{R}^2$ corresponding to a rotation through an angle $\theta$ in the counter-clockwise direction.

Solutions to Section 5.3

True-False Review:

1. FALSE. The statement should read 
\[\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = \dim[V],\]
not $\dim[W]$ on the right-hand side.

3. FALSE. The solution set to the homogeneous linear system $Ax = 0$ is $\text{Ker}(T)$, not $\text{Rng}(T)$.

5. TRUE. From the given information, we see that $\text{Ker}(T)$ is at least 2-dimensional, and therefore, since $M_{23}$ is 6-dimensional, the Rank-Nullity Theorem requires that $\text{Rng}(T)$ have dimension at most $6 - 2 = 4$.

Problems:

1. \[T(7, 5, -1) = \begin{bmatrix}
1 & -1 & 2 \\
1 & -2 & -3 \\
\end{bmatrix}
\begin{bmatrix}
7 \\
5 \\
-1 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\implies (7, 5, -1) \in \text{Ker}(T).\]

\[T(-21, -15, 2) = \begin{bmatrix}
1 & -1 & 2 \\
1 & -2 & -3 \\
\end{bmatrix}
\begin{bmatrix}
-21 \\
-15 \\
2 \\
\end{bmatrix}
= \begin{bmatrix}
-2 \\
3 \\
\end{bmatrix}
\implies (-21, -15, 2) \notin \text{Ker}(T).\]

\[T(35, 25, -5) = \begin{bmatrix}
1 & -1 & 2 \\
1 & -2 & -3 \\
\end{bmatrix}
\begin{bmatrix}
35 \\
25 \\
-5 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\implies (35, 25, -5) \in \text{Ker}(T).\]

3. $\text{Ker}(T) = \{ x \in \mathbb{R}^3 : T(x) = 0 \} = \{ x \in \mathbb{R}^3 : Ax = 0 \}$. The augmented matrix of the system $Ax = 0$
is: 
\[
\begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & 2 & 0 \\
2 & -1 & 1 & 0 \\
\end{bmatrix}
\]
with reduced row-echelon form 
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
Thus $x_1 = x_2 = x_3 = 0$, so $\text{Ker}(T) = \{ 0 \}$. Geometrically, this describes a point (the origin) in $\mathbb{R}^3$.

\[\dim[\text{Ker}(T)] = 0.\]
For the given transformation, Rng\(T\) = colspace\(A\). From the preceding reduced row-echelon form of \(A\), we see that colspace\(A\) is generated by the first three column vectors of \(A\). Consequently, Rng\(T\) = \(\mathbb{R}^3\), dim[Rng\(T\)] = dim[\(\mathbb{R}^3\)] = 3, and Theorem 5.3.8 is satisfied since 
\[
dim[Ker(T)] + \dim[Rng(T)] = 0 + 3 = \dim[\mathbb{R}^3].
\]

5. Ker\(T\) = \(\{x \in \mathbb{R}^3 : T(x) = 0\}\) = \(\{x \in \mathbb{R}^3 : Ax = 0\}\). The augmented matrix of the system \(Ax = 0\) is:
\[
\begin{bmatrix}
1 & -1 & 2 & 0 \\
-3 & 3 & -6 & 0
\end{bmatrix}
\]
with reduced row-echelon form of
\[
\begin{bmatrix}
1 & -1 & 2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
Thus
\[
\text{Ker}(T) = \{x \in \mathbb{R}^3 : x = r(1, 1, 0) + s(-2, 0, 1), r, s \in \mathbb{R}\}.
\]
Geometrically, this describes the plane through the origin in \(\mathbb{R}^3\), which is spanned by the linearly independent set \((1, 1, 0), (-2, 0, 1)\).
\[
dim[\text{Ker}(T)] = 2.
\]

For the given transformation, Rng\(T\) = colspace\(A\). From the preceding reduced row-echelon form of \(A\), we see that a basis for colspace\(A\) is given by the first column vector of \(A\). Consequently,
\[
\text{Rng}(T) = \{y \in \mathbb{R}^2 : y = t(1, -3), t \in \mathbb{R}\}.
\]
Geometrically, this is the line through the origin in \(\mathbb{R}^2\) spanned by \((1, -3)\).
\[
dim[\text{Rng}(T)] = 1 \text{ and Theorem 5.3.8 is satisfied since } \dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = 2 + 1 = 3 = \dim[\mathbb{R}^3].
\]

7. The matrix of \(T\) in Problem 24 of Section 5.1 is \(A = \begin{bmatrix}
-5/3 & -2/3 \\
1/3 & 1/2 \\
5/3 & -1/3 \\
-1 & 1
\end{bmatrix}\). Thus,
\[
\text{Ker}(T) = \text{nullspace}(A) = \{0\}
\]
and
\[
\text{Rng}(T) = \text{colspace}(A) = \text{span}\left\{\begin{bmatrix}
-5/3 \\
1/3 \\
5/3 \\
-1
\end{bmatrix}, \begin{bmatrix}
-2/3 \\
1/2 \\
-1/3 \\
1
\end{bmatrix}\right\}.
\]

9. The matrix of \(T\) in Problem 26 of Section 5.1 is \(A = \begin{bmatrix}
32 & 3 & 0 \\
-5 & 2 & -3 \\
4 & -7 & 6
\end{bmatrix}\). Thus,
\[
\text{Ker}(T) = \text{nullspace}(A) = \{0\}
\]
and
\[
\text{Rng}(T) = \text{colspace}(A) = \mathbb{R}^3.
\]
11. (a) Ker\(T\) = \(\{v \in \mathbb{R}^3 : \langle u, v \rangle = 0\}\). For \(v\) to be in the kernel of \(T\), \(u\) and \(v\) must be orthogonal. Since \(u\) is any fixed vector in \(\mathbb{R}^3\), then \(v\) must lie in the plane orthogonal to \(u\). Hence \(\dim[\text{Ker}(T)] = 2\).

(b) Rng\(T\) = \(\{y \in \mathbb{R} : y = \langle u, v \rangle, v \in \mathbb{R}^3\}\), and \(\dim[\text{Rng}(T)] = 1\).

13. Ker\(T\) = \(\{A \in M_n(\mathbb{R}) : AB - BA = 0\}\) = \(\{A \in M_2(\mathbb{R}) : AB = BA\}\). This is the set of matrices that commute with \(B\).

15. Ker\(T\) = \(\{p \in P_2 : T(p) = 0\}\) = \(\{ax^2 + bx + c \in P_2 : (a + b) + (b - c)x = 0\}\), for all \(x\). Thus, \(a, b,\) and \(c\) must satisfy: \(a + b = 0\) and \(b - c = 0\) \(\implies a = -b\) and \(b = c\). Letting \(c = r \in \mathbb{R}\), we have
\[ax^2 + bx + c = r(-x^2 + x + 1).\] Thus, \(\text{Ker}(T) = \{r(-x^2 + x + 1) : r \in \mathbb{R}\}\) and \(\dim[\text{Ker}(T)] = 1.\)

\[\text{Rng}(T) = \{T(ax^2 + bx + c) : a, b, c \in \mathbb{R}\} = \{(a + b) + (b - c)x : a, b, c \in \mathbb{R}\} = \{c_1 + c_2x : c_1, c_2 \in \mathbb{R}\}.\]

Consequently, a basis for \(\text{Rng}(T)\) is \(\{1, x\}\), so that \(\text{Rng}(T) = P_1\), and \(\dim[\text{Rng}(T)] = 2.\)

17. \[
T(v) = 0 \iff T(av_1 + bv_2 + cv_3) = 0
\]
\[
\iff a(2w_1 - w_2) + b(w_1 - w_2) + c(w_1 + 2w_2) = 0
\]
\[
\iff (2a + b + c)w_1 + (-a - b + 2c)w_2 = 0
\]
\[
\iff 2a + b + c = 0 \text{ and } a - b + 2c = 0.
\]

Reducing the augmented matrix of the system yields:

\[
\begin{bmatrix}
2 & 1 & 1 & 0 \\
1 & -1 & 2 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -1 & 2 & 0 \\
2 & 1 & 1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0
\end{bmatrix}.
\]

Setting \(c = r \implies b = r, \ a = -r.\) Thus, \(\text{Ker}(T) = \{v \in V : v = r(-v_1 + v_2 + v_3), r \in \mathbb{R}\}\) and \(\dim[\text{Ker}(T)] = 1.\)

\[\text{Rng}(T) = \{T(v) : v \in V\} = \{(2a + b + c)w_1 + (-a - b + 2c)w_2 : a, b, c \in V\} = \text{span}\{w_1, w_2\} = W.\]

Consequently, \(\dim[\text{Rng}(T)] = 2.\)

**Solutions to Section 5.4**

**True-False Review:**

1. **FALSE.** Many one-to-one linear transformations \(T : P_3 \to M_{32}\) can be constructed. One possible example would be to define

\[T(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{bmatrix}
a_0 & a_1 \\
a_2 & a_3 \\
0 & 0
\end{bmatrix}.
\]

It is easy to check that with \(T\) so defined, \(T\) is a one-to-one linear transformation.

3. **TRUE.** Both \(\text{Ker}(T_1)\) and \(\text{Ker}(T_2T_1)\) are subspaces of \(V_1\), and since if \(T_1(v_1) = 0\), then \((T_2T_1)v_1 = T_2(T_1(v_1)) = T_2(0) = 0\), we see that every vector in \(\text{Ker}(T_1)\) belongs to \(\text{Ker}(T_2T_1)\). Therefore, \(\text{Ker}(T_1)\) is a subspace of \(\text{Ker}(T_2T_1)\).

5. **TRUE.** Since \(M_2(\mathbb{R})\) is 4-dimensional, \(\text{Rng}(T)\) can be at most 4-dimensional. However, \(P_4\) is 5-dimensional. Therefore, any such linear transformation \(T\) cannot be onto.

7. **FALSE.** This linear transformation is onto one-to-one. The reason is essentially because the derivative of any constant is zero. Therefore, \(\ker(T)\) consists of all constant functions, and therefore, \(\ker(T) \neq \{0\}\).

9. **TRUE.** Recall that \(\dim[\mathbb{R}^n] = n\) and \(\dim[\mathbb{R}^m] = m\). In order for such an isomorphism to exist, \(\mathbb{R}^n\) and \(\mathbb{R}^m\) must have the same dimension; that is, \(m = n\).

11. **FALSE.** In order for this to be true, it would also have to be assumed that \(T_1\) is onto. For example, suppose \(V_1 = V_2 = V_3 = \mathbb{R}^2\). If we define \(T_2(x, y) = (x, y)\) for all \((x, y)\) in \(\mathbb{R}^2\), then \(T_2\) is onto. However, if we define \(T_1(x, y) = (0, 0)\) for all \((x, y)\) in \(\mathbb{R}^2\), then \((T_2T_1)(x, y) = T_2(T_1(x, y)) = T_2(0, 0) = (0, 0)\) for all \((x, y)\) in \(\mathbb{R}^2\). Therefore \(T_2T_1\) is not onto, since \(\text{Rng}(T_2T_1) = \{(0, 0)\}\), even though \(T_2\) itself is onto.
Problems:

1. \( T_1 T_2(x) = T_1(T_2(x)) = T_1(Bx) = (AB)x \), so \( T_1 T_2 = AB \). Similarly, \( T_2 T_1 = BA \).

\[
AB = \begin{bmatrix}
-1 & 2 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 5 \\
-2 & 0
\end{bmatrix}
= \begin{bmatrix}
-5 & -5 \\
1 & 15
\end{bmatrix}.
\]

\( BA = \begin{bmatrix}
1 & 5 \\
-2 & 0
\end{bmatrix}
\begin{bmatrix}
-1 & 2 \\
3 & 1
\end{bmatrix}
= \begin{bmatrix}
14 & 7 \\
2 & -4
\end{bmatrix} \).

2. \( T_1 T_2(x) = (AB)x = \begin{bmatrix}
-5 & -5 \\
1 & 15
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
-5x_1 - 5x_2 \\
x_1 + 15x_2
\end{bmatrix} = \begin{bmatrix}
-5(x_1 + x_2), x_1 + 15x_2
\end{bmatrix} \) and

\( T_2 T_1(x) = (BA)x = \begin{bmatrix}
14 & 7 \\
2 & -4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
14x_1 + 7x_2 \\
2x_1 - 4x_2
\end{bmatrix} = \begin{bmatrix}
7(2x_1 + x_2), 2(x_1 - 2x_2)
\end{bmatrix} \). Clearly, \( T_1 T_2 \neq T_2 T_1 \).

3. \( \text{Ker}(T_1) = \{ x \in \mathbb{R}^2 : Ax = 0 \} \). \( Ax = 0 \implies \begin{bmatrix}
1 & -1 \\
2 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}. \) This matrix equation results in the system: \( x_1 - x_2 = 0 \) and \( 2x_1 - 2x_2 = 0 \), or equivalently, \( x_1 = x_2 \). Thus,

\[ \text{Ker}(T_1) = \{ x \in \mathbb{R}^2 : x = r(1, 1) \text{ where } r \in \mathbb{R} \}. \]

Geometrically, this is the line through the origin in \( \mathbb{R}^2 \) spanned by (1, 1).

\( \text{Ker}(T_2) = \{ x \in \mathbb{R}^2 : Bx = 0 \} \). \( Bx = 0 \implies \begin{bmatrix}
2 & 1 \\
3 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}. \) This matrix equation results in the system: \( 2x_1 + x_2 = 0 \) and \( 3x_1 - x_2 = 0 \), or equivalently, \( x_1 = x_2 \) and \( x_1 = 0 \). Thus, \( \text{Ker}(T_2) = \{ 0 \}. \) Geometrically, this is a point (the origin).

\( \text{Ker}(T_1 T_2) = \{ x \in \mathbb{R}^2 : (AB)x = 0 \} \). \( (AB)x = 0 \implies \begin{bmatrix}
1 & -1 \\
2 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}. \) This matrix equation results in the system: \( -x_1 + 2x_2 = 0 \) and \( -2x_1 + 4x_2 = 0 \), or equivalently, \( x_1 = 2x_2 \).

Thus, \( \text{Ker}(T_1 T_2) = \{ x \in \mathbb{R}^2 : x = s(2, 1) \text{ where } s \in \mathbb{R} \}. \) Geometrically, this is the line through the origin in \( \mathbb{R}^2 \) spanned by the vector (2, 1).

\( \text{Ker}(T_2 T_1) = \{ x \in \mathbb{R}^2 : (BA)x = 0 \} \). \( (BA)x = 0 \implies \begin{bmatrix}
2 & 1 \\
3 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}. \) This matrix equation results in the system: \( 4x_1 - 4x_2 = 0 \) and \( x_1 - x_2 = 0 \), or equivalently, \( x_1 = x_2 \).

Thus, \( \text{Ker}(T_2 T_1) = \{ x \in \mathbb{R}^2 : x = t(1, 1) \text{ where } t \in \mathbb{R} \}. \) Geometrically, this is the line through the origin in \( \mathbb{R}^2 \) spanned by the vector (1, 1).

5. (a) \( (T_1(f))(x) = \frac{d}{dx} [\sin(x - a)] = \cos(x - a). \)

\( (T_2(f))(x) = \int_a^x \sin(t - a)dt = [-\cos(t - a)]_a^x = 1 - \cos(x - a). \)

\( (T_1 T_2)(f)(x) = \frac{d}{dx} [1 - \cos(x - a)] = \sin(x - a) = f(x). \)

\( (T_2 T_1)(f)(x) = \int_a^x \cos(t - a)dt = [\sin(t - a)]_a^x = \sin(x - a) = f(x). \) Consequently,

\( (T_1 T_2)(f) = (T_2 T_1)(f) = f. \)
(b) \( T_1 T_2(f) = T_1(T_2(f)) = T_1 \left( \int_a^x f(x) \, dx \right) = \frac{d}{dx} \left( \int_a^x f(x) \, dx \right) = f(x). \)

\( T_2 T_1(g) = T_2(T_1(g)) = T_2 \left( \frac{dg(x)}{dx} \right) = \int_a^x \frac{dg(x)}{dx} \, dx = g(x) - g(a). \)

7. Let \( \mathbf{v} \in V \) so there exists \( a, b \in \mathbb{R} \) such that \( \mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 \). Then \( T_2 T_1(\mathbf{v}) = T_2[aT_1(\mathbf{v}_1) + bT_1(\mathbf{v}_2)] = T_2[a(3\mathbf{v}_1 + \mathbf{v}_2)] = 3a(-5\mathbf{v}_2) + a(-\mathbf{v}_1 + 6\mathbf{v}_2) = -a\mathbf{v}_1 - 9a\mathbf{v}_2. \)

9. \( \ker(T) = \{ \mathbf{x} \in \mathbb{R}^2 : T(\mathbf{x}) = \mathbf{0} \} = \{ \mathbf{x} \in \mathbb{R}^2 : A \mathbf{x} = \mathbf{0} \}. \) The augmented matrix of the system \( A \mathbf{x} = \mathbf{0} \) is:

\[
\begin{bmatrix}
1 & 2 & 0 \\
-2 & -4 & 0
\end{bmatrix},
\]

with reduced row-echelon form of \( \begin{bmatrix} 1 & 2 & \bigmid & 0 \\ 0 & 0 & \bigmid & 0 \end{bmatrix} \). It follows that

\( \ker(T) = \{ \mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = t(-2, 1), t \in \mathbb{R} \}. \)

By Theorem 5.4.7, since \( \ker(T) \neq \{ \mathbf{0} \} \), \( T \) is not one-to-one. This also implies that \( T^{-1} \) does not exist. For the given transformation, \( \text{Rng}(T) = \{ \mathbf{y} \in \mathbb{R}^2 : A \mathbf{x} = \mathbf{y} \ \text{is consistent} \}. \) The augmented matrix of the system \( A \mathbf{x} = \mathbf{y} \) is

\[
\begin{bmatrix}
1 & 2 & y_1 \\
-2 & -4 & y_2
\end{bmatrix}
\]

with reduced row-echelon form of \( \begin{bmatrix} 1 & 2 & \bigmid & y_1 \\ 0 & 0 & \bigmid & 2y_1 + y_2 \end{bmatrix} \). The last row of this matrix implies that \( 2y_1 + y_2 = 0 \) is required for consistency. Therefore, it follows that

\( \text{Rng}(T) = \{ (y_1, y_2) \in \mathbb{R}^2 : 2y_1 + y_2 = 0 \} = \{ \mathbf{y} \in \mathbb{R}^2 : \mathbf{y} = s(1, -2), s \in \mathbb{R} \}. \)

\( T \) is not onto because \( \dim[\text{Rng}(T)] = 1 \neq 2 = \dim[\mathbb{R}^2]. \)

11. Reducing \( A \) to row-echelon form, we obtain

\[
\text{REF}(A) = \begin{bmatrix}
1 & 3 & 5 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}.
\]

We quickly find that \( \ker(T) = \text{nullspace}(A) = \text{span}\{ (1, -2, 1) \}. \)

Moreover,

\( \text{Rng}(T) = \text{colspace}(A) = \text{span}\left\{ \begin{bmatrix} 0 & \bigmid & 1 \\ 3 & \bigmid & 4 \\ 5 & \bigmid & 4 \\ 2 & \bigmid & 1 \end{bmatrix} \right\}. \)

Based on these calculations, we see that \( T \) is neither one-to-one nor onto.

13. Show \( T \) is a linear transformation:
Let \( \mathbf{x}, \mathbf{y} \in V \) and \( c, \lambda \in \mathbb{R} \) where \( \lambda \neq 0 \).
\( T(\mathbf{x} + \mathbf{y}) = \lambda(\mathbf{x} + \mathbf{y}) = \lambda \mathbf{x} + \lambda \mathbf{y} = T(\mathbf{x}) + T(\mathbf{y}), \) and
\( T(c \mathbf{x}) = \lambda(c \mathbf{x}) = c(\lambda \mathbf{x}) = cT(\mathbf{x}) \). Thus, \( T \) is a linear transformation.

Show \( T \) is one-to-one:
\( \mathbf{x} \in \ker(T) \iff T(\mathbf{x}) = \mathbf{0} \iff \lambda \mathbf{x} = \mathbf{0} \iff \mathbf{x} = \mathbf{0}, \) since \( \lambda \neq 0 \). Thus, \( \ker(T) = \{ \mathbf{0} \}. \) By Theorem 5.4.7, \( T \)

is one-to-one.

Show \( T \) is onto:
\( \dim[\ker(T)] + \dim[\text{Rng}(T)] = \dim[V] \iff \dim[\{ \mathbf{0} \}] + \dim[\text{Rng}(T)] = \dim[V] \)
\( \iff 0 + \dim[\text{Rng}(T)] = \dim[V] \iff \dim[\text{Rng}(T)] = \dim[V] \iff \text{Rng}(T) = V. \) Thus, \( T \) is onto by Definition 5.3.3.
Find $T^{-1}$:

$T^{-1}(x) = \frac{1}{\lambda}x$ since $T(T^{-1}(x)) = T \left( \frac{1}{\lambda}x \right) = \lambda \left( \frac{1}{\lambda}x \right) = x$.

15. $T$ is not one-to-one:

Ker$(T) = \{ p \in P_2 : T(p) = 0 \} = \{ ax^2 + bx + c : c + (a - b)x = 0 \}$. Thus, $a, b,$ and $c$ must satisfy the system $c = 0$ and $a - b = 0$. Consequently, Ker$(T) = \{ r(x^2 + x) : r \in \mathbb{R} \} \neq \{ 0 \}$. Thus, by Theorem 5.4.7, $T$ is not one-to-one. $T^{-1}$ does not exist because $T$ is not one-to-one.

$T$ is onto:

Since Ker$(T) = \{ r(x^2 + x) : r \in \mathbb{R} \}$, we see that dim[Ker$(T)$] = 1.

Thus, dim[Ker$(T)$] + dim[Rng$(T)$] \(=\) dim[$P_2$] \(\Rightarrow\) 1 + dim[Rng$(T)$] \(=\) 3 \(\Rightarrow\) dim[Rng$(T)$] \(=\) 2. Since Rng$(T)$ is a subspace of $P_1$,

$$2 = \text{dim[Rng}(T)\text{)]} \leq \text{dim}[P_1] = 2,$$

and so equality holds: Rng$(T) = P_1$. Thus, $T$ is onto by Theorem 5.4.7.

17. Let $v \in V$. Then there exists $a, b \in \mathbb{R}$ such that $v = av_1 + bv_2$.

$$(T_1T_2)v = T_1[aT_2(v_1) + bT_2(v_2)] = T_1 \left[ \frac{a}{2}(v_1 + v_2) + \frac{b}{2}(v_1 - v_2) \right]$$

$$= \frac{a + b}{2} T_1(v_1) + \frac{a - b}{2} T_1(v_2) = \frac{a + b}{2} (v_1 + v_2) + \frac{a - b}{2} (v_1 - v_2)$$

$$= \left( \frac{a + b}{2} + \frac{a - b}{2} \right) v_1 + \left( \frac{a + b}{2} - \frac{a - b}{2} \right) v_2 = av_1 + bv_2 = v.$$

$$(T_2T_1)v = T_2[aT_1(v_1) + bT_1(v_2)] = T_2 \left[ a(v_1 + v_2) + b(v_1 - v_2) \right]$$

$$= T_2[(a + b)v_1 + (a - b)v_2] = (a + b)T_2(v_1) + (a - b)T_2(v_2)$$

$$= \frac{a + b}{2} (v_1 + v_2) + \frac{a - b}{2} (v_1 - v_2) = av_1 + bv_2 = v.$$

Since $(T_1T_2)v = v$ and $(T_2T_1)v = v$ for all $v \in V$, it follows that $T_2$ is the inverse of $T_1$, thus $T_2 = T_1^{-1}$.

19. Let $S$ denote the subspace of $M_2(\mathbb{R})$ consisting of all upper triangular matrices. An arbitrary vector in $S$ can be written as

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, we define an isomorphism $T : \mathbb{R}^3 \rightarrow S$ by

$$T(a, b, c) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}.$$

21. Let $S$ denote the subspace of $M_2(\mathbb{R})$ consisting of all symmetric matrices. An arbitrary vector in $S$ can be written as

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, we can define an isomorphism $T : \mathbb{R}^3 \rightarrow S$ by

$$T(a, b, c) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

23. A typical vector in $V$ takes the form $p = a_0 + a_2x^2 + a_3x^4 + a_6x^6 + a_8x^8$. Therefore, we can define $T : V \rightarrow \mathbb{R}^5$ via

$$T(p) = (a_0, a_2, a_4, a_6, a_8).$$
It is routine to verify that $T$ is an invertible linear transformation. Therefore, we have $n = 5$.

25. We have

$$T_2^{-1}(x) = A^{-1}x = \begin{bmatrix} -1/3 & -1/6 \\ 1/3 & 2/3 \end{bmatrix} x.$$  

27. We have

$$T_4^{-1}(x) = A^{-1}x = \begin{bmatrix} 8 & -29 & 3 \\ -5 & 19 & -2 \\ 2 & -8 & 1 \end{bmatrix} x.$$  

29. The matrix of $T_3T_2$ is

$$\begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 13 & 18 \\ 4 & 8 & 6 \\ 23 & 43 & 11 \end{bmatrix}.\]  

The matrix of $T_3T_4$ is

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 25 & 23 \\ 5 & 14 & 15 \\ 12 & 19 & 1 \end{bmatrix}.$$  

31. We first prove that if $T$ is one-to-one, then $T$ is onto. The assumption that $T$ is one-to-one implies that $\dim[\ker(T)] = 0$. Hence, by Theorem 5.3.8,

$$\dim[W] = \dim[V] = \dim[Rng(T)],$$

which implies that $\text{Rng}(T) = W$. That is, $T$ is onto.

Next we show that if $T$ is onto, then $T$ is one-to-one. We have $\text{Rng}(T) = W$, so that $\dim[\text{Rng}(T)] = \dim[W] = \dim[V]$. Hence, by Theorem 5.3.8, we have $\dim[\ker(T)] = 0$. Hence, $\ker(T) = \{0\}$, which implies that $T$ is one-to-one.

33. Let $T : V \rightarrow V$ be a one-to-one linear transformation. Since $T$ is one-to-one, it follows from Theorem 5.4.7 that $\ker(T) = \{0\}$, so $\dim[\ker(T)] = 0$. By Theorem 5.3.8 and substitution,

$$\dim[\ker(T)] + \dim[\text{Rng}(T)] = \dim[V] \Rightarrow 0 + \dim[\text{Rng}(T)] = \dim[V] \Rightarrow \dim[\text{Rng}(T)] = \dim[V],$$

and since $\text{Rng}(T)$ is a subspace of $V$, it follows that $\text{Rng}(T) = V$, thus $T$ is onto. $T^{-1}$ exists because $T$ is both one-to-one and onto.

35. To prove that $T$ is onto, let $w$ be an arbitrary vector in $W$. We must find a vector $v$ in $V$ such that $T(v) = w$. Since $\{w_1, w_2, \ldots, w_m\}$ spans $W$, we can write

$$w = c_1w_1 + c_2w_2 + \cdots + c_mw_m$$

for some scalars $c_1, c_2, \ldots, c_m$. Therefore

$$T(c_1v_1 + c_2v_2 + \cdots + c_mv_m) = c_1T(v_1) + c_2T(v_2) + \cdots + c_mT(v_m) = c_1w_1 + c_2w_2 + \cdots + c_mw_m = w,$$

which shows that $v = c_1v_1 + c_2v_2 + \cdots + c_mv_m$ maps under $T$ to $w$, as desired.

37. This problem is not correct.

39.

(a) To show that $T_2T_1 : V_1 \rightarrow V_1$ is one-to-one, we show that $\ker(T_2T_1) = \{0\}$. Suppose that $v_1 \in \ker(T_2T_1)$. This means that $(T_2T_1)(v_1) = 0$. Hence, $T_2(T_1(v_1)) = 0$. However, since $T_2$ is one-to-one, we conclude that $T_1(v_1) = 0$. Next, since $T_1$ is one-to-one, we conclude that $v_1 = 0$, which shows that the only vector in $\ker(T_2T_1)$ is $0$, as expected.
(b) To show that $T_2T_1 : V_1 \to V_3$ is onto, we begin with an arbitrary vector $v_3$ in $V_3$. Since $T_2 : V_2 \to V_3$ is onto, there exists $v_2$ in $V_2$ such that $T_2(v_2) = v_3$. Moreover, since $T_1 : V_1 \to V_2$ is onto, there exists $v_1$ in $V_1$ such that $T_1(v_1) = v_2$. Therefore,

$$(T_2T_1)(v_1) = T_2(T_1(v_1)) = T_2(v_2) = v_3,$$

and therefore, we have found a vector, namely $v_1$, in $V_1$ that is mapped under $T_2T_1$ to $v_3$. Hence, $T_2T_1$ is onto.

(c) This follows immediately from parts (a) and (b).

**Solutions to Section 5.5**

**True-False Review:**

1. FALSE. The matrix representation is an $m \times n$ matrix, not an $n \times m$ matrix.

3. FALSE. The correct equation is given in (5.5.2).

5. TRUE. Many examples are possible. A fairly simple one is the following. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T_1(x, y) = (x, y)$, and let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T_2(x, y) = (y, x)$. Clearly, $T_1$ and $T_2$ are different linear transformations. Now if $B_1 = \{(1, 0), (0, 1)\} = C_1$, then $[T_1]_{B_1}^{C_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $B_2 = \{(1, 0), (0, 1)\}$ and $C_2 = \{(0, 1), (1, 0)\}$, then $[T_2]_{B_2}^{C_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, although $T_1 \neq T_2$, we found suitable bases $B_1, C_1, B_2, C_2$ such that $[T_1]_{B_1}^{C_1} = [T_2]_{B_2}^{C_2}$.

**Problems:**

1.

(a): We must determine $T(1)$, $T(x)$, and $T(x^2)$, and find the components of the resulting vectors in $\mathbb{R}^2$ relative to the basis $C$. We have

$$T(1) = (1, 2), \quad T(x) = (0, 1), \quad T(x^2) = (-3, -2).$$

Therefore, relative to the standard basis $C$ on $\mathbb{R}^2$, we have

$$[T(1)]_C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [T(x)]_C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad [T(x^2)]_C = \begin{bmatrix} -3 \\ -2 \end{bmatrix}.$$

Therefore,

$$[T]_B^C = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 1 & -2 \end{bmatrix}.$$

(b): We must determine $T(1)$, $T(1+x)$, and $T(1+x+x^2)$, and find the components of the resulting vectors in $\mathbb{R}^2$ relative to the basis $C$. We have

$$T(1) = (1, 2), \quad T(1+x) = (1, 3), \quad T(1+x+x^2) = (-2, 1).$$

Setting

$$T(1) = (1, 2) = c_1(1, -1) + c_2(2, 1)$$
and solving, we find $c_1 = -1$ and $c_2 = 1$. Thus, $[T(1)]_C = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Next, setting

$$T(1 + x) = (1, 3) = c_1(1, -1) + c_2(2, 1)$$

and solving, we find $c_1 = -5/3$ and $c_2 = 4/3$. Thus, $[T(1 + x)]_C = \begin{bmatrix} -5/3 \\ 4/3 \end{bmatrix}$. Finally, setting

$$T(1 + x + x^2) = (-2, 1) = c_1(1, -1) + c_2(2, 1)$$

and solving, we find $c_1 = -4/3$ and $c_2 = -1/3$. Thus, $[T(1 + x + x^2)]_C = \begin{bmatrix} -4/3 \\ -1/3 \end{bmatrix}$. Putting the results for $[T(1)]_C$, $[T(1 + x)]_C$, and $[T(1 + x + x^2)]_C$ into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} -1 & -5/3 & -4/3 \\ 1 & 2/3 & -2 \end{bmatrix}.$$  

3. (a): We must determine $T(1, 0, 0)$, $T(0, 1, 0)$, and $T(0, 0, 1)$, and find the components of the resulting vectors relative to the basis $C$. We have

$$T(1, 0, 0) = \cos x, \quad T(0, 1, 0) = 3 \sin x, \quad T(0, 0, 1) = -2 \cos x + \sin x.$$  

Therefore, relative to the basis $C$, we have

$$[T(1, 0, 0)]_C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [T(0, 1, 0)]_C = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad [T(0, 0, 1)]_C = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$  

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 1 \end{bmatrix}.$$  

(b): We must determine $T(2, -1, -1)$, $T(1, 3, 5)$, and $T(0, 4, -1)$, and find the components of the resulting vectors relative to the basis $C$. We have

$$T(2, -1, -1) = 4 \cos x - 4 \sin x, \quad T(1, 3, 5) = -9 \cos x + 14 \sin x, \quad T(0, 4, -1) = 2 \cos x + 11 \sin x.$$  

Setting

$$4 \cos x - 4 \sin x = c_1(\cos x - \sin x) + c_2(\cos x + \sin x)$$

and solving, we find $c_1 = 4$ and $c_2 = 0$. Therefore $[T(2, -1, -1)]_C = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. Next, setting

$$-9 \cos x + 14 \sin x = c_1(\cos x - \sin x) + c_2(\cos x + \sin x)$$

and solving, we find $c_1 = -23/2$ and $c_2 = 5/2$. Therefore, $[T(1, 3, 5)]_C = \begin{bmatrix} -23/2 \\ 5/2 \end{bmatrix}$. Finally, setting

$$2 \cos x + 11 \sin x = c_1(\cos x - \sin x) + c_2(\cos x + \sin x)$$
and solving, we find \( c_1 = -9/2 \) and \( c_2 = 13/2 \). Therefore \( [T(0, 4, -1)]_C = \begin{bmatrix} -9/2 \\ 13/2 \end{bmatrix} \). Putting these results into the columns of \([T]_B^C\), we obtain

\[
[T]_B^C = \begin{bmatrix}
4 & -\frac{23}{2} & -\frac{9}{2} \\
0 & \frac{43}{2} & \frac{13}{2}
\end{bmatrix}.
\]

5.

(a): We must determine \( T(1), T(x), T(x^2), \) and \( T(x^3) \), and find the components of the resulting vectors relative to the standard basis \( C \) on \( P_2 \). We have

\[
T(1) = 0, \quad T(x) = 1, \quad T(x^2) = 2x, \quad T(x^3) = 3x^2.
\]

Therefore, if \( C \) is the standard basis on \( P_2 \), then we have

\[
[T(1)]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x^2)]_C = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad [T(x^3)]_C = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.
\]

Putting these results into the columns of \([T]_B^C\), we obtain

\[
[T]_B^C = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}.
\]

(b): We must determine \( T(x^3), T(x^3 + 1), T(x^3 + x), \) and \( T(x^3 + x^2) \), and find the components of the resulting vectors relative to the given basis \( C \) on \( P_2 \). We have

\[
T(x^3) = 3x^2, \quad T(x^3 + 1) = 3x^2, \quad T(x^3 + x) = 3x^2 + 1, \quad T(x^3 + x^2) = 3x^2 + 2x.
\]

Setting

\[
3x^2 = c_1(1) + c_2(1 + x) + c_3(1 + x + x^2)
\]

and solving, we find \( c_1 = 0, c_2 = -3, \) and \( c_3 = 3 \). Therefore \( [T(x^3)]_C = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \). Likewise, \( [T(x^3 + 1)]_C = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \).

Next, setting

\[
3x^2 + 1 = c_1(1) + c_2(1 + x) + c_3(1 + x + x^2)
\]

and solving, we find \( c_1 = 1, c_2 = -3, \) and \( c_3 = 3 \). Therefore, \( [T(x^3 + x)]_C = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \). Finally, setting

\[
3x^2 + 2x = c_1(1) + c_2(1 + x) + c_3(1 + x + x^2)
\]

and solving, we find \( c_1 = -2, c_2 = -1, \) and \( c_3 = 3 \). Therefore, \( [T(x^3 + 2x)]_C = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} \). Putting these results into the columns of \([T]_B^C\), we obtain

\[
[T]_B^C = \begin{bmatrix}
0 & 0 & 1 & -2 \\
-3 & -3 & -3 & -1 \\
3 & 3 & 3 & 3
\end{bmatrix}.
\]
7. (a): We must determine \( T(E_{11}) \), \( T(E_{12}) \), \( T(E_{21}) \), and \( T(E_{22}) \), and find the components of the resulting vectors relative to the standard basis \( C \) on \( M_2(\mathbb{R}) \). We have

\[
T(E_{11}) = E_{11}, \quad T(E_{12}) = 2E_{12} - E_{21}, \quad T(E_{21}) = 2E_{21} - E_{12}, \quad T(E_{22}) = E_{22}.
\]

Therefore, we have

\[
[T(E_{11})]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(E_{12})]_C = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad [T(E_{21})]_C = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad [T(E_{22})]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

Putting these results into the columns of \([T]_B^C\), we obtain

\[
[T]_B^C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

(b): Let us denote the four matrices in the ordered basis for \( B \) as \( A_1 \), \( A_2 \), \( A_3 \), and \( A_4 \), respectively. We must determine \( T(A_1) \), \( T(A_2) \), \( T(A_3) \), and \( T(A_4) \), and find the components of the resulting vectors relative to the standard basis \( C \) on \( M_2(\mathbb{R}) \). We have

\[
T(A_1) = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}, \quad T(A_2) = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}, \quad T(A_3) = \begin{bmatrix} 0 & -8 \\ 7 & -2 \end{bmatrix}, \quad T(A_4) = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}.
\]

Therefore, we have

\[
[T(A_1)]_C = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}, \quad [T(A_2)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(A_3)]_C = \begin{bmatrix} 0 \\ -8 \\ -2 \end{bmatrix}, \quad [T(A_4)]_C = \begin{bmatrix} 0 \\ 7 \\ -2 \end{bmatrix}.
\]

Putting these results into the columns of \([T]_B^C\), we obtain

\[
[T]_B^C = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 0 & -8 & 7 \\ -2 & 3 & 7 & -2 \\ -3 & 2 & -2 & 0 \end{bmatrix}.
\]

9. (a): Let us first compute \([T]_B^C\). We must determine \( T(1) \) and \( T(x) \), and find the components of the resulting vectors relative to the standard basis \( C = \{ E_{11}, E_{12}, E_{21}, E_{22} \} \) on \( M_2(\mathbb{R}) \). We have

\[
T(1) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad T(x) = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}.
\]

Therefore

\[
[T(1)]_C = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad [T(x)]_C = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}.
\]
Hence,

\[ [T]_B^C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & -2 \\ -1 & 1 \end{bmatrix}. \]

Now,

\[ [p(x)]_B = [-2 + 3x]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}. \]

Therefore, we have

\[ [T(p(x))]_C = [T]_B^C[p(x)]_B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -6 \\ 5 \end{bmatrix}. \]

Thus,

\[ T(p(x)) = \begin{bmatrix} -5 & 0 \\ -6 & 5 \end{bmatrix}. \]

(b): We have

\[ T(p(x)) = T(-2 + 3x) = \begin{bmatrix} -5 & 0 \\ -6 & 5 \end{bmatrix}. \]

11.

(a): Let us first compute \([T]_B^C\). We must determine \(T(E_{11}), T(E_{12}), T(E_{21}),\) and \(T(E_{22})\), and find the components of the resulting vectors relative to the standard basis \(C = \{E_{11}, E_{12}, E_{21}, E_{22}\}\). We have

\[ T(E_{11}) = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}, \quad T(E_{12}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T(E_{21}) = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}, \quad T(E_{22}) = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}. \]

Therefore,

\[ [T(E_{11})]_C = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \quad [T(E_{12})]_C = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad [T(E_{21})]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad [T(E_{22})]_C = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}. \]

Putting these results into the columns of \([T]_B^C\), we obtain

\[ [T]_B^C = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 \end{bmatrix}. \]

Now,

\[ [A]_B = \begin{bmatrix} -7 \\ 2 \\ 1 \\ -3 \end{bmatrix}. \]
and therefore,

\[
[T(A)]_C = [T]_B^C [A]_B = \begin{bmatrix}
2 & -1 & 0 & 1 \\
-1 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 \\
-1 & -1 & 3 & 0
\end{bmatrix} \begin{bmatrix}
-7 \\
2 \\
1 \\
-3
\end{bmatrix} = \begin{bmatrix}
-19 \\
-2 \\
0 \\
8
\end{bmatrix}.
\]

Hence,

\[
T(A) = \begin{bmatrix}
-19 & -2 \\
0 & 8
\end{bmatrix}.
\]

(b): We have

\[
T(A) = \begin{bmatrix}
-19 & -2 \\
0 & 8
\end{bmatrix}.
\]

13.

(a): Let us first compute \([T]_B^C\). We must determine \(T(E_{ij})\) for \(1 \leq i, j \leq 3\), and find components of the resulting vectors relative to the standard basis \(C = \{1\}\). We have

\[
T(E_{11}) = T(E_{22}) = T(E_{33}) = 1 \quad \text{and} \quad T(E_{12}) = T(E_{13}) = T(E_{23}) = T(E_{21}) = T(E_{31}) = T(E_{32}) = 0.
\]

Therefore

\[
[T(E_{11})]_C = [T(E_{22})]_C = [T(E_{33})]_C = [1] \quad \text{and all other component vectors are [0].}
\]

Putting these results into the columns of \([T]_B^C\), we obtain

\[
[T]_B^C = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

Now,

\[
[A]_B = \begin{bmatrix}
2 \\
-6 \\
0 \\
1 \\
4 \\
-4 \\
0 \\
0 \\
-3
\end{bmatrix}.
\]

and therefore,

\[
[T(A)]_C = [T]_B^C [A]_B = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix} = [3],
\]

\[
\begin{bmatrix}
2 \\
-6 \\
0 \\
1 \\
4 \\
-4 \\
0 \\
0 \\
-3
\end{bmatrix}
\]
Hence, $T(A) = 3$.

(b): We have $T(A) = 3$.

15.

(a): Let us first compute $[T]_B^C$. We must determine $T(1), T(x), T(x^2),$ and $T(x^3)$, and find the components of the resulting vectors relative to the standard basis $C = \{1\}$. We have

$$T(1) = 1, \quad T(x) = 2, \quad T(x^2) = 4, \quad T(x^3) = 8.$$ 

Therefore,

$$[T(1)]_C = [1], \quad [T(x)]_C = [2], \quad [T(x^2)]_C = [4], \quad [T(x^3)]_C = [8].$$ 

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix}.$$ 

Now,

$$[p(x)]_B = [2x - 3x^2]_B = \begin{bmatrix} 0 \\ 2 \\ -3 \\ 0 \end{bmatrix},$$

and therefore,

$$[T(p(x))]_C = [T]_B^C[p(x)]_B = \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -3 \\ 0 \end{bmatrix} = [-8].$$

Therefore, $T(p(x)) = -8$.

(b): We have $p(2) = 2 \cdot 2 - 3 \cdot 2^2 = -8$.

17. The linear transformation $T_2T_1 : P_1 \to \mathbb{R}^2$ is given by

$$(T_2T_1)(a + bx) = (0, 0).$$

Let $A$ denote the standard basis on $P_1$, let $B$ denote the standard basis on $M_2(\mathbb{R})$, and let $C$ denote the standard basis on $\mathbb{R}^2$.

(a): To determine $[T_2T_1]_A^C$, we compute

$$(T_2T_1)(1) = (0, 0) \quad \text{and} \quad (T_2T_1)(x) = (0, 0).$$

Therefore, we obtain $[T_2T_1]_A^C = 0_2$.

(b): We have

$$[T_2]_B^C[T_1]_A^B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & -2 \\ -1 & 1 \end{bmatrix} = 0_2 = [T_2T_1]_A^C.$$ 

(c): The component vector of $p(x) = -3 + 8x$ relative to the standard basis $A$ is $[p(x)]_A = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$. Thus,

$$[(T_2T_1)(p(x))]_C = [T_2T_1]_A^C[p(x)]_A = 0_2[p(x)]_A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
Therefore,
\[(T_2T_1)(-3 + 8x) = (0, 0).\]

Of course
\[(T_2T_1)(-3 + 8x) = T_1 \begin{bmatrix} -11 & 0 \\ -16 & 11 \end{bmatrix} = (0, 0)\]

by direct calculation as well.

(d): **YES.** Since the matrix \([T_2T_1]_C\) computed in part (a) is invertible, \(T_2T_1\) is invertible.

19. **NO.** The matrices \([T]_B^C\) obtained in Problem 2 are not invertible (they contain rows of zeros), and therefore, the corresponding linear transformation \(T\) is not invertible.

21. Note that
\[w\] belongs to \(\text{Rng}(T) \iff w = T(v)\] for some \(v\) in \(V\)
\[\iff [w]_C = [T(v)]_C \text{ for some } v \text{ in } V\]
\[\iff [w]_C = [T]_B^C[v]_B \text{ for some } v \text{ in } V.\]

The right-hand side of this last expression can be expressed as a linear combination of the columns of \([T]_B^C\), and therefore, \(w\) belongs to \(\text{Rng}(T)\) if and only if \([w]_C\) can be expressed as a linear combination of the columns of \([T]_B^C\). That is, if and only if \([w]_C\) belongs to \(\text{colspace}([T]_B^C)\).

**Solutions to Section 5.6**

**True-False Review:**

1. **FALSE.** If \(v = 0\), then \(Av = \lambda v = 0\), but by definition, an eigenvector must be a **nonzero** vector.

3. **TRUE.** The eigenvalues of a matrix are precisely the set of roots of its characteristic equation. Therefore, two matrices \(A\) and \(B\) that have the same characteristic equation will have the same eigenvalues.

5. **TRUE.** Geometrically, all nonzero points \(v = (x, y)\) in \(\mathbb{R}^2\) are oriented in a different direction from the origin after a \(90^\circ\) rotation than they are initially. Therefore, the vectors \(v\) and \(Av\) are not parallel.

7. **FALSE.** This is not true, in general, when the linear combination formed involves eigenvectors corresponding to **different** eigenvalues. For example, let \(A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\), with eigenvalues \(\lambda = 1\) and \(\lambda = 2\). It is easy to see that corresponding eigenvectors to these eigenvalues are, respectively, \(v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\) and \(v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\). However, note that
\[A(v_1 + bfv_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\]
which is not of the form \(\lambda(v_1 + v_2)\), and therefore \(v_1 + v_2\) is not an eigenvector of \(A\). As a more trivial illustration, note that if \(v\) is an eigenvector of \(A\), then \(0v\) is a linear combination of \(\{v\}\) that is no longer an eigenvector of \(A\).

9. **TRUE.** If \(\lambda\) is an eigenvalue of \(A\), then we have \(Av = \lambda v\) for some eigenvector \(v\) of \(A\) corresponding to \(\lambda\). Then
\[A^2v = A(\lambda v) = \lambda(Av) = \lambda^2v,\]
which shows that \(v\) is also an eigenvector of \(A^2\), this time corresponding to the eigenvalue \(\lambda^2\).

**Problems:**
1. \[ Av = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda v. \]

2. Since \( v = c_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + 4c_2 \\ -3c_2 \\ -3c_1 \end{bmatrix}, \) it follows that \[ A v = \begin{bmatrix} 1 & 4 & 1 \\ 3 & 2 & 1 \\ 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} c_1 + 4c_2 \\ -3c_2 \\ -3c_1 \end{bmatrix} = \begin{bmatrix} -2c_1 - 8c_2 \\ 6c_2 \\ 6c_1 \end{bmatrix} = -2 \begin{bmatrix} c_1 + 4c_2 \\ -3c_2 \\ -3c_1 \end{bmatrix} = \lambda v. \]

5. The only vectors that are mapped into a scalar multiple of themselves under a rotation in the \( x \)-axis are those vectors that either point along the \( x \)-axis, or that point along the \( y \)-axis. Hence, the eigenvectors are of the form \((a, 0)\) or \((0, b)\) where \(a\) and \(b\) are arbitrary nonzero real numbers. A vector that points along the \( x \)-axis will have neither its magnitude nor its direction altered by a reflection in the \( x \)-axis. Hence, the eigenvectors of the form \((a, 0)\) correspond to the eigenvalue \(\lambda = 1\). A vector of the form \((0, b)\) will be mapped into the vector \((0, -b) = -1(0, b)\) under a reflection in the \(x\)-axis. Consequently, the eigenvectors of the form \((0, b)\) correspond to the eigenvalue \(\lambda = -1\).

7. If \(\theta \neq 0, \pi\), there are no vectors that are mapped into scalar multiples of themselves under the rotation, and consequently, there are no real eigenvalues and eigenvectors in this case. If \(\theta = 0\), then every vector is mapped onto itself under the rotation, therefore \(\lambda = 1\), and every nonzero vector in \(\mathbb{R}^2\) is an eigenvector. If \(\theta = \pi\), then every vector is mapped onto its negative under the rotation, therefore \(\lambda = -1\), and once again, every nonzero vector in \(\mathbb{R}^2\) is an eigenvector.

9. \[ \det(A - \lambda I) = 0 \iff \begin{vmatrix} 3 - \lambda & -1 \\ -5 & -1 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 2\lambda - 8 = 0 \]
   \[ \iff (\lambda + 2)(\lambda - 4) = 0 \iff \lambda = -2 \text{ or } \lambda = 4. \]
   If \(\lambda = -2\) then \((A - \lambda I)v = 0\) assumes the form \[ \begin{bmatrix} 5 & -1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
   so the eigenvectors corresponding to \(\lambda = -2\) are \(v = t(1, 1)\) where \(t \in \mathbb{R}\).
   If \(\lambda = 4\) then \((A - \lambda I)v = 0\) assumes the form \[ \begin{bmatrix} -1 & -1 \\ -5 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
   so the eigenvectors corresponding to \(\lambda = 4\) are \(v = r(1, -1)\) where \(r \in \mathbb{R}\).

11. \[ \det(A - \lambda I) = 0 \iff \begin{vmatrix} 7 - \lambda & 4 \\ -1 & 3 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 10\lambda + 25 = 0 \]
   \[ \iff (\lambda - 5)^2 = 0 \iff \lambda = 5 \text{ of multiplicity two}. \]
   If \(\lambda = 5\) then \((A - \lambda I)v = 0\) assumes the form \[ \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
   so the eigenvectors corresponding to \(\lambda = 5\) are \(v = t(-2, 1)\) where \(t \in \mathbb{R}\).

13. \[ \det(A - \lambda I) = 0 \iff \begin{vmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 2\lambda + 5 = 0 \iff \lambda = 1 \pm 2i. \]
   If \(\lambda = 1 - 2i\) then \((A - \lambda I)v = 0\) assumes the form \[ \begin{bmatrix} 2 + 2i & -2 \\ 4 & -2 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
   so the eigenvectors corresponding to \(\lambda = 1 - 2i\) are \(v = s(1, 1 + i)\) where \(s \in \mathbb{C}\). By Theorem 5.6.8, since the entries of \(A\) are real, \(\lambda = 1 + 2i\) has corresponding eigenvectors of the form \(v = t(1, 1 - i)\) where \(t \in \mathbb{C}\).
15. \( \det(A - \lambda I) = 0 \iff \begin{vmatrix} 10 - \lambda & -12 & 8 \\ 0 & 2 - \lambda & 0 \\ -8 & 12 & -6 - \lambda \end{vmatrix} = 0 \iff (\lambda - 2)^3 = 0 \iff \lambda = 2 \) of multiplicity three.

If \( \lambda = 2 \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{bmatrix} 8 & -12 & 8 \\ 0 & 0 & 0 \\ -8 & 12 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
\[\implies 2v_1 - 3v_2 + 2v_3 = 0. \]
Thus, if we let \( v_2 = 2s \) and \( v_3 = t \) where \( s, t \in \mathbb{R} \), then the solution set of this system is \( \{(3s - t, 2s, t) : s, t \in \mathbb{R}\} \) so the eigenvectors corresponding to \( \lambda = 2 \) are \( v = s(3, 2, 0) + t(-1, 0, 1) \) where \( s, t \in \mathbb{R} \).

17. \( \det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & 3 - \lambda & 2 \\ 2 & -2 & -1 - \lambda \end{vmatrix} = 0 \iff (\lambda - 1)^3 = 0 \iff \lambda = 1 \) of multiplicity three.

If \( \lambda = 1 \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
\[\implies v_1 = 0 \text{ and } v_2 + v_3 = 0. \]
Thus, if we let \( v_3 = s \) where \( s \in \mathbb{R} \), then the solution set of this system is \( \{(0, -s, s) : s \in \mathbb{R}\} \) so the eigenvectors corresponding to \( \lambda = 1 \) are \( v = s(0, -1, 1) \) where \( s \in \mathbb{R} \).

19. \( \det(A - \lambda I) = 0 \iff \begin{vmatrix} 7 - \lambda & -8 & 6 \\ 8 & -9 - \lambda & 6 \\ 0 & 0 & -1 - \lambda \end{vmatrix} = 0 \iff (\lambda + 1)^3 = 0 \iff \lambda = -1 \) of multiplicity three.

If \( \lambda = -1 \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{bmatrix} 8 & -8 & 6 \\ 8 & -8 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
\[\implies 4v_1 - 4v_2 + 3v_3 = 0. \]
Thus, if we let \( v_2 = r \) and \( v_3 = 4s \) where \( r, s \in \mathbb{R} \), then the solution set of this system is \( \{(r - 3s, r, 4s) : r, s \in \mathbb{R}\} \) so the eigenvectors corresponding to \( \lambda = -1 \) are \( v = r(1, 1, 0) + s(-3, 0, 4) \) where \( r, s \in \mathbb{R} \).

21. \( \det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \iff (1 - \lambda)(1 + \lambda^2) = 0 \iff \lambda = 1 \text{ or } \lambda = \pm i. \)

If \( \lambda = 1 \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
\[\implies -v_2 + v_3 = 0 \text{ and } -v_2 - v_3 = 0. \]
The solution set of this system is \( \{(r, 0, 0) : r \in \mathbb{C}\} \) so the eigenvectors corresponding to \( \lambda = 1 \) are \( v = r(1, 0, 0) \) where \( r \in \mathbb{C} \).

If \( \lambda = -i \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{bmatrix} 1 + i & 0 & 0 \\ 0 & i & 1 \\ 0 & -1 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
\[\implies v_1 = 0 \text{ and } -v_2 + iv_3 = 0. \]
The solution set of this system is \( \{(0, si, s) : s \in \mathbb{C}\} \) so the eigenvectors corresponding to \( \lambda = -i \) are \( v = s(0, i, 1) \) where \( s \in \mathbb{C} \). By Theorem 5.6.8, since the entries of \( A \) are real, \( \lambda = i \) has corresponding eigenvectors of the form \( v = t(0, -i, 1) \) where \( t \in \mathbb{C} \).

23. \( \det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & -1 & 3 \\ 3 & 1 - \lambda & 0 \\ 2 & -1 & 3 - \lambda \end{vmatrix} = 0 \iff \lambda(\lambda - 2)(\lambda - 4) = 0 \iff \lambda = 0, \lambda = 2, \text{ or } \lambda = 4. \)

If \( \lambda = 0 \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

If \( \lambda = 2 \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
\[ v_1 + 2v_2 - 3v_3 = 0 \] and \[-5v_2 + 9v_3 = 0. \] Thus, if we let \( v_3 = 5r \) where \( r \in \mathbb{R} \), then the solution set of this system is \( \{-3r, 9r, 5r \} : r \in \mathbb{R} \) so the eigenvectors corresponding to \( \lambda = 0 \) are \( \mathbf{v} = r(-3, 9, 5) \) where \( r \in \mathbb{R} \).

If \( \lambda = 2 \) then \( (A - \lambda I)\mathbf{v} = \mathbf{0} \) assumes the form

\[
\begin{bmatrix}
0 & -1 & 3 \\
3 & -1 & 0 \\
2 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[ \Rightarrow v_1 - v_3 = 0 \] and \( v_2 - 3v_3 = 0. \) Thus, if we let \( v_3 = s \) where \( s \in \mathbb{R} \), then the solution set of this system is \( \{(s, 3s, s) : s \in \mathbb{R} \} \) so the eigenvectors corresponding to \( \lambda = 2 \) are \( \mathbf{v} = s(1, 3, 1) \) where \( s \in \mathbb{R} \).

If \( \lambda = 4 \) then \( (A - \lambda I)\mathbf{v} = \mathbf{0} \) assumes the form

\[
\begin{bmatrix}
-2 & -1 & 3 \\
3 & -3 & 0 \\
2 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[ \Rightarrow v_1 - v_3 = 0 \] and \( v_2 - v_3 = 0. \) Thus, if we let \( v_3 = t \) where \( t \in \mathbb{R} \), then the solution set of this system is \( \{(t, t, t) : t \in \mathbb{R} \} \) so the eigenvectors corresponding to \( \lambda = 4 \) are \( \mathbf{v} = t(1, 1, 1) \) where \( t \in \mathbb{R} \).

25. \[ \det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = 0 \iff (\lambda - 4)(\lambda + 2)^2 = 0 \iff \lambda = 4 \text{ or } \lambda = -2 \] of multiplicity two.

If \( \lambda = 4 \) then \( (A - \lambda I)\mathbf{v} = \mathbf{0} \) assumes the form

\[
\begin{bmatrix}
-4 & 2 & 2 \\
2 & -4 & 2 \\
2 & 2 & -4
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[ \Rightarrow v_1 - v_3 = 0 \] and \( v_2 - v_3 = 0. \) Thus, if we let \( v_3 = r \) where \( r \in \mathbb{R} \), then the solution set of this system is \( \{(r, r, r) : r \in \mathbb{R} \} \) so the eigenvectors corresponding to \( \lambda = 4 \) are \( \mathbf{v} = r(1, 1, 1) \) where \( r \in \mathbb{R} \).

If \( \lambda = -2 \) then \( (A - \lambda I)\mathbf{v} = \mathbf{0} \) assumes the form

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[ \Rightarrow v_1 + v_2 + v_3 = 0. \] Thus, if we let \( v_2 = s \) and \( v_3 = t \) where \( s, t \in \mathbb{R} \), then the solution set of this system is \( \{(-s - t, s, t) : s, t \in \mathbb{R} \} \) so the eigenvectors corresponding to \( \lambda = -2 \) are \( \mathbf{v} = s(-1, 1, 0) + t(-1, 0, 1) \) where \( s, t \in \mathbb{R} \).

27. \[ \det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ -1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = 0 \iff (\lambda^2 + 1)^2 = 0 \iff \lambda = \pm i, \] where each root is of multiplicity two.

If \( \lambda = -i \) then \( (A - \lambda I)\mathbf{v} = \mathbf{0} \) assumes the form

\[
\begin{bmatrix}
i & 1 & 0 & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & 1 & i
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[ \Rightarrow v_1 - iv_2 = 0 \] and \( v_3 + iv_4 = 0. \) Thus, if we let \( v_2 = r \) and \( v_4 = s \) where \( r, s \in \mathbb{C} \), then the solution set of this system is \( \{(ir, r, -is, s) : r, s \in \mathbb{C} \} \) so the eigenvectors corresponding to \( \lambda = -i \) are \( \mathbf{v} = r(i, 1, 0, 0) + s(0, 0, -i, 1) \) where \( r, s \in \mathbb{C} \). By Theorem 5.6.8, since the entries of \( A \) are real, \( \lambda = i \) has corresponding eigenvectors \( \mathbf{v} = a(-i, 1, 0, 0) + b(0, 0, i, 1) \) where \( a, b \in \mathbb{C} \).

29. (a) \( p(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 1 - \lambda & -1 \\ 2 & 4 - \lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6. \)

(b)

\[
A^2 - 5A + 6I_2 =
\begin{bmatrix}
1 & -1 \\ 2 & 4
\end{bmatrix}
\begin{bmatrix}
1 & -1 \\ 2 & 4
\end{bmatrix}
- 5
\begin{bmatrix}
1 & -1 \\ 2 & 4
\end{bmatrix}
+ 6
\begin{bmatrix}
1 & 0 \\ 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-1 & 5 \\ 10 & -20
\end{bmatrix}
+ \begin{bmatrix}
-5 & 5 \\ 0 & 6
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\ 0 & 0
\end{bmatrix}
= I_2.
\]
(c) Using part (b) of this problem:
\[ A^2 - 5A + 6I_2 = 0_2 \iff A^{-1}(A^2 - 5A + 6I_2) = A^{-1} \cdot 0_2 \]
\[ \iff A - 5I_2 + 6A^{-1} = 0_2 \]
\[ \iff 6A^{-1} = 5I_2 - A \]
\[ \iff A^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -1 \\ -2 & 1 \end{bmatrix}, \text{ or } A^{-1} = \left[ -\frac{1}{3} 1 \right] \frac{1}{6}. \]

(31.)
\[ A(3v_1 - v_2) = 3Av_1 - Av_2 = 3(2v_1) - (-3v_2) = 6v_1 + 3v_2 \]
\[ = 6 \left[ \begin{array}{c} 1 \\ -1 \end{array} \right] + 3 \left[ \begin{array}{c} 2 \\ 1 \end{array} \right] = \begin{bmatrix} 6 \\ -6 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ -3 \end{bmatrix}. \]

(33.)
\[ A(c_1v_1 + c_2v_2 + c_3v_3) = A(c_1v_1) + A(c_2v_2) + A(c_3v_3) \]
\[ = c_1(Av_1) + c_2(Av_2) + c_3(Av_3) = c_1(\lambda v_1) + c_2(\lambda v_2) + c_3(\lambda v_3) \]
\[ = \lambda (c_1v_1 + c_2v_2 + c_3v_3). \]

Thus, \( c_1v_1 + c_2v_2 + c_3v_3 \) is an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda \).

35. Any scalar \( \lambda \) such that \( \det(A - \lambda I) = 0 \) is an eigenvalue of \( A \). Therefore, if 0 is an eigenvalue of \( A \), then \( \det(A - 0 \cdot I) = 0 \), which implies that \( A \) is not invertible. On the other hand, if 0 is not an eigenvalue of \( A \), then \( \det(A - 0 \cdot I) \neq 0 \), or \( \det(A) \neq 0 \), which implies that \( A \) is invertible.

37. By assumption, we have \( Av = \lambda v \) and \( Bv = \mu v \).

(a) Therefore,
\[ (AB)v = A(Bv) = A(\mu v) = \mu(Av) = \mu(\lambda v) = (\lambda \mu)v, \]
which shows that \( v \) is an eigenvector of \( AB \) with corresponding eigenvalue \( \lambda \mu \).

(b) Also,
\[ (A + B)v = Av + Bv = \lambda v + \mu v = (\lambda + \mu)v, \]
which shows that \( v \) is an eigenvector of \( A + B \) with corresponding eigenvalue \( \lambda + \mu \).

39. (a) \( v = r + is \) is an eigenvector with eigenvalue \( \lambda = a + bi \), \( b \neq 0 \iff Av = \lambda v \)
\[ \iff A(r + is) = (a + bi)(r + is) = (ar - bs) + i(as + br) \iff Ar = ar - bs \text{ and } As = as + br. \]

Now if \( r = 0 \), then \( A0 = a0 - bs \iff 0 = 0 - bs \iff 0 = bs \iff s = 0 \) since \( b \neq 0 \). This would mean that \( v = 0 \) so \( v \) could not be an eigenvector. Thus, it must be that \( r \neq 0 \). Similarly, if \( s = 0 \), then \( r = 0 \), and again, this would contradict the fact that \( v \) is an eigenvector. Hence, it must be the case that \( r \neq 0 \) and \( s \neq 0 \).

(b) As in part (a), \( Ar = ar - bs \) and \( As = as + br \).

Let \( c_1, c_2 \in \mathbb{R} \). Then if
\[ c_1r + c_2s = 0, \]
we have \( A(c_1r + c_2s) = 0 \iff c_1Ar + c_2As = 0 \iff c_1(ar - bs) + c_2(as + br) = 0. \]
Hence, \( (c_1a + c_2b)r + (c_2a - c_1b)s = 0 \iff a(c_1r + c_2s) + b(c_2r - c_1s) = 0 \iff b(c_2r - c_1s) = 0 \) where we have used (39.1). Since \( b \neq 0 \), we must have \( c_2r - c_1s = 0 \). Combining this with (39.1) yields \( c_1 = c_2 = 0. \) Therefore, it follows that \( r \) and \( s \) are linearly independent vectors.
41. $\lambda_1 = -2$ (multiplicity two), $\mathbf{v} = r(1,1,1)$. $\lambda_2 = -5$, $\mathbf{v} = s(20,11,14)$.

43. $\lambda_1 = 3 - \sqrt{6}$, $\mathbf{v} = r(\sqrt{6}, -1 + \sqrt{6}, -5 + \sqrt{6})$. $\lambda_2 = 3 + \sqrt{6}$, $\mathbf{v} = s(\sqrt{6}, 1 + \sqrt{6}, 5 + \sqrt{6})$, $\lambda_3 = -2$, $\mathbf{v} = t(-1,3,0)$.

45. $\lambda_1 = -1$ (multiplicity four), $\mathbf{v} = a(-1,0,0,1,0) + b(-1,0,1,0,0) + c(-1,0,0,0,1) + d(-1,1,0,0,0)$.

**Solutions to Section 5.7**

**True-False Review:**

1. **TRUE.** This is the definition of a nondefective matrix.

3. **TRUE.** The dimension of an eigenspace never exceeds the algebraic multiplicity of the corresponding eigenvalue.

5. **TRUE.** Since each eigenvalue of the matrix $A$ occurs with algebraic multiplicity 1, we can simply choose one eigenvector from each eigenspace to obtain a basis of eigenvectors for $A$. Thus, $A$ is nondefective.

7. **TRUE.** Eigenvectors corresponding to distinct eigenvalues are always linearly independent, as proved in the text in this section.

**Problems:**

1. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 4\lambda - 5 = 0 \iff (\lambda - 5)(\lambda + 1) = 0 \iff \lambda = 5$ or $\lambda = -1$.

   If $\lambda_1 = 5$ then $(A - \lambda I)\mathbf{v} = 0$ assumes the form $\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $v_1 = v_2 = 0$. The solution set of this system is $\{(r,r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = 5$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = r(1,1), r \in \mathbb{R}\}$. A basis for $E_1$ is $\{(1,1)\}$, and $\dim[E_1] = 1$.

   If $\lambda_2 = -1$ then $(A - \lambda I)\mathbf{v} = 0$ assumes the form $\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $v_1 = 2v_2 = 0$. The solution set of this system is $\{(-2s,s) : s \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_2 = -1$ is $E_2 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = s(-2,1), s \in \mathbb{R}\}$. A basis for $E_2$ is $\{(-2,1)\}$, and $\dim[E_2] = 1$.

   A complete set of eigenvectors for $A$ is given by $\{(1,1),(-2,1)\}$, so $A$ is nondefective.

3. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 2 \\ -2 & 5 - \lambda \end{vmatrix} = 0 \iff (\lambda - 3)^2 = 0 \iff \lambda = 3$ of multiplicity two.

   If $\lambda_1 = 3$ then $(A - \lambda I)\mathbf{v} = 0$ assumes the form $\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $v_1 = v_2 = 0$. The solution set of this system is $\{(r,r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = 3$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = r(1,1), r \in \mathbb{R}\}$. A basis for $E_1$ is $\{(1,1)\}$, and $\dim[E_1] = 1$.

4. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3 - \lambda & -4 & -1 \\ 0 & -1 - \lambda & -1 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = 0 \iff (\lambda + 2)(\lambda - 3)^2 = 0 \iff \lambda = -2$ or $\lambda = 3$ of multiplicity two.

   If $\lambda_1 = -2$ then $(A - \lambda I)\mathbf{v} = 0$ assumes the form $\begin{pmatrix} 5 & -4 & -1 \\ 0 & 1 & -1 \\ 0 & -4 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

   The solution set of this system is $\{(r,r,r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = -2$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(1,1,1), r \in \mathbb{R}\}$. A basis for $E_1$ is $\{(1,1,1)\}$, and $\dim[E_1] = 1$. 
If $\lambda_2 = 3$ then $(A - \lambda I)v = 0$ assumes the form

$$
\begin{bmatrix}
0 & -4 & -1 \\
0 & -4 & -1 \\
0 & -4 & -1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\implies 4v_2 + v_3 = 0.
$$

The solution set of this system is $\{(s, t, -4t) : s, t \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_2 = 3$ is $E_2 = \{v \in \mathbb{R}^3 : v = s(1, 0, 0) + t(0, 1, -4), s, t \in \mathbb{R}\}$. A basis for $E_2$ is $\{(1, 0, 0), (0, 1, -4)\}$, and $\dim[E_2] = 2$. A complete set of eigenvectors for $A$ is given by $\{(1, 1, 1), (1, 0, 0), (0, 1, -4)\}$, so $A$ is nondefective.

7. $\det(A - \lambda I) = 0 \iff
\begin{bmatrix}
3 - \lambda & 1 & 0 \\
-1 & 5 - \lambda & 0 \\
0 & 0 & 4 - \lambda
\end{bmatrix} = 0 \iff (\lambda - 4)^3 = 0 \iff \lambda = 4$ of multiplicity three.

If $\lambda_1 = 4$ then $(A - \lambda I)v = 0$ assumes the form

$$
\begin{bmatrix}
-1 & 1 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\implies v_1 - v_2 = 0$ and $v_3 \in \mathbb{R}$. The solution set of this system is $\{(r, r, s) : r, s \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = 4$ is $E_1 = \{v \in \mathbb{R}^3 : v = r(1, 1, 0) + s(0, 0, 1), r, s \in \mathbb{R}\}$. A basis for $E_1$ is $\{(1, 1, 0), (0, 0, 1)\}$, and $\dim[E_1] = 2$.

A is defective since it does not have a complete set of eigenvectors.

9. $\det(A - \lambda I) = 0 \iff
\begin{bmatrix}
4 - \lambda & 1 & 6 \\
-4 & -\lambda & -7 \\
0 & 0 & -3 - \lambda
\end{bmatrix} = 0 \iff (\lambda + 3)(\lambda - 2)^2 = 0 \iff \lambda = -3$ or $\lambda = 2$ of multiplicity two.

If $\lambda_1 = -3$ then $(A - \lambda I)v = 0$ assumes the form

$$
\begin{bmatrix}
7 & 1 & 6 \\
-4 & 3 & -7 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\implies v_1 + v_2 = 0$ and $v_3 = 0$. The solution set of this system is $\{(r, r, s) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = -3$ is $E_1 = \{v \in \mathbb{R}^3 : v = r(-1, 1, 1), r \in \mathbb{R}\}$. A basis for $E_1$ is $\{(-1, 1, 1)\}$, and $\dim[E_1] = 1$.

If $\lambda_2 = 2$ then $(A - \lambda I)v = 0$ assumes the form

$$
\begin{bmatrix}
2 & 1 & 6 \\
-4 & -2 & -7 \\
0 & 0 & 5
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\implies 2v_1 + v_2 = 0$ and $v_3 = 0$. The solution set of this system is $\{(-s, 2s, 0) : s \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_2 = 2$ is $E_2 = \{v \in \mathbb{R}^3 : v = s(-1, 2, 0), s \in \mathbb{R}\}$. A basis for $E_2$ is $\{(-1, 2, 0)\}$, and $\dim[E_2] = 1$.

A is defective because it does not have a complete set of eigenvectors.

11. $\det(A - \lambda I) = 0 \iff
\begin{bmatrix}
7 - \lambda & -8 & 6 \\
8 & -9 - \lambda & 6 \\
0 & 0 & -1 - \lambda
\end{bmatrix} = 0 \iff (\lambda + 1)^3 = 0 \iff \lambda = -1$ of multiplicity three.

If $\lambda_1 = -1$ then $(A - \lambda I)v = 0$ assumes the form

$$
\begin{bmatrix}
8 & -8 & 6 \\
8 & -8 & 6 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\implies 4v_1 - 4v_2 + 3v_3 = 0.$$

The solution set of this system is $\{(r - 3s, r, 4s) : r, s \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = -1$ is $E_1 = \{v \in \mathbb{R}^3 : v = r(1, 1, 0) + s(-3, 0, 4), r, s \in \mathbb{R}\}$. A basis for $E_1$ is $\{(1, 1, 0), (-3, 0, 4)\}$, and $\dim[E_1] = 2$.

A is defective since it does not have a complete set of eigenvectors.

13. $\det(A - \lambda I) = 0 \iff
\begin{bmatrix}
1 - \lambda & -1 & 2 \\
1 & -1 - \lambda & 2 \\
1 & -1 & 2 - \lambda
\end{bmatrix} = 0 \iff (\lambda - 2)^2 = 0 \iff \lambda = 2$ or $\lambda = 0$ of multiplicity two.
If \( \lambda_1 = 2 \) then \((A - \lambda I)v = 0\) assumes the form 
\[
\begin{bmatrix}
-1 & -1 & 2 \\
1 & -3 & 2 \\
1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]
\(\implies v_1 - v_3 = 0\) and \(v_2 - v_3 = 0\). The solution set of this system is \(\{(r, r, r) : r \in \mathbb{R}\}\), so the eigenspace corresponding to \(\lambda_1 = 2\) is \(E_1 = \{v \in \mathbb{R}^3 : v = r(1,1,1), r \in \mathbb{R}\}\). A basis for \(E_1\) is \(\{(1,1,1)\}\), and \(\dim[E_1] = 1\).

If \(\lambda_2 = 0\) then \((A - \lambda I)v = 0\) assumes the form
\[
\begin{bmatrix}
1 & -1 & 2 \\
1 & -1 & 2 \\
1 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\implies v_1 - v_2 + 2v_3 = 0.
\]
The solution set of this system is \(\{(s - 2t, s, t) : s, t \in \mathbb{R}\}\), so the eigenspace corresponding to \(\lambda_2 = 0\) is \(E_2 = \{v \in \mathbb{R}^3 : v = s(1,1,0) + t(-2,0,1), s, t \in \mathbb{R}\}\). A basis for \(E_2\) is \(\{(1,1,0), (-2,0,1)\}\), and \(\dim[E_2] = 2\). 
\(A\) is nondefective because it has a complete set of eigenvectors.

15. \(\det(A - \lambda I) = 0 \iff \begin{vmatrix}
-\lambda & -1 & -1 \\
-1 & -\lambda & -1 \\
-1 & -1 & -\lambda
\end{vmatrix}
= 0 \iff (\lambda + 2)(\lambda - 1)^2 = 0 \iff \lambda = -2\) or \(\lambda = 1\) of multiplicity two.

If \(\lambda_1 = -2\) then \((A - \lambda I)v = 0\) assumes the form 
\[
\begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\implies v_1 - v_3 = 0\) and \(v_2 - v_3 = 0\). The solution set of this system is \(\{(r, r, r) : r \in \mathbb{R}\}\), so the eigenspace corresponding to \(\lambda_1 = -2\) is \(E_1 = \{v \in \mathbb{R}^3 : v = r(1,1,1), r \in \mathbb{R}\}\). A basis for \(E_1\) is \(\{(1,1,1)\}\), and \(\dim[E_1] = 1\).

If \(\lambda_2 = 1\) then \((A - \lambda I)v = 0\) assumes the form 
\[
\begin{bmatrix}
-1 & -1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\implies v_1 + v_2 + v_3 = 0.
\]
The solution set of this system is \(\{(-s - t, s, t) : s, t \in \mathbb{R}\}\), so the eigenspace corresponding to \(\lambda_2 = 1\) is \(E_2 = \{v \in \mathbb{R}^3 : v = s(-1,1,0) + t(-1,0,1), s, t \in \mathbb{R}\}\). A basis for \(E_2\) is \(\{(-1,1,0), (-1,0,1)\}\), and \(\dim[E_2] = 2\).
\(A\) is nondefective because it has a complete set of eigenvectors.

17. \((\lambda - 1)^2 = 0 \iff \lambda = 1\) of multiplicity two.

If \(\lambda_1 = 1\) then \((A - \lambda I)v = 0\) assumes the form 
\[
\begin{bmatrix}
5 & 5 \\
-5 & -5
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\implies v_1 + v_2 = 0.\]The solution set of this system is \(\{(-r, r) : r \in \mathbb{R}\}\), so the eigenspace corresponding to \(\lambda_1 = 1\) is \(E_1 = \{v \in \mathbb{R}^2 : v = r(-1,1), r \in \mathbb{R}\}\). A basis for \(E_1\) is \(\{(1,1)\}\), and \(\dim[E_1] = 1\).
\(A\) is defective since it does not have a complete set of eigenvectors.

19. \((\lambda - 2)^2(\lambda + 1) = 0 \iff \lambda = -1\) or \(\lambda = 2\) of multiplicity two. To determine whether \(A\) is nondefective, all we require is the dimension of the eigenspace corresponding to \(\lambda = 2\).

If \(\lambda = 2\) then \((A - \lambda I)v = 0\) assumes the form 
\[
\begin{bmatrix}
-1 & -3 & 1 \\
-1 & -3 & 1 \\
-1 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\implies -v_1 - 3v_2 + v_3 = 0.
\]
The solution set of this system is \(\{(-3s + t, s, t) : s, t \in \mathbb{R}\}\), so the eigenspace corresponding to \(\lambda = 2\) is \(E = \{v \in \mathbb{R}^3 : v = s(-3,1,0) + t(1,0,1), s, t \in \mathbb{R}\}\). Since \(\dim[E] = 2\), \(A\) is nondefective.

21. \(\det(A - \lambda I) = 0 \iff \begin{vmatrix}
2 - \lambda & 1 \\
3 & 4 - \lambda
\end{vmatrix}
= 0 \iff (\lambda - 1)(\lambda - 5) = 0 \iff \lambda = 1\) or \(\lambda = 5\).

If \(\lambda_1 = 1\) then \((A - \lambda I)v = 0\) assumes the form 
\[
\begin{bmatrix}
1 & 1 \\
3 & 3
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\implies v_1 + v_2 = 0.\]The eigenspace corresponding to \(\lambda_1 = 1\) is \(E_1 = \{v \in \mathbb{R}^2 : v = r(-1,1), r \in \mathbb{R}\}\). A basis for \(E_1\) is \(\{(-1,1)\}\).
If $\lambda_2 = 5$ then $(A - \lambda I)v = 0$ assumes the form \[
\begin{bmatrix}
-3 & 1 \\
3 & -1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix} \implies 3v_1 - v_2 = 0.\] The eigenspace corresponding to $\lambda_2 = 5$ is $E_2 = \{v \in \mathbb{R}^2 : v = s(1, 3), s \in \mathbb{R}\}$. A basis for $E_2$ is $\{(1, 3)\}$.

**Figure 0.0.40:** Figure for Exercise 21

23. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 5 - \lambda & 0 \\ 0 & 5 - \lambda \end{vmatrix} = 0 \iff (5 - \lambda)^2 = 0 \iff \lambda = 5$ of multiplicity two.

If $\lambda_1 = 5$ then $(A - \lambda I)v = 0$ assumes the form \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix} \implies v_1, v_2 \in \mathbb{R}.\] The eigenspace corresponding to $\lambda_1 = 5$ is $E_1 = \{v \in \mathbb{R}^2 : v = r(1, 0) + s(0, 1), r, s \in \mathbb{R}\}$. A basis for $E_1$ is $\{(1, 0), (0, 1)\}$.

**Figure 0.0.41:** Figure for Exercise 23

25. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -3 - \lambda & 1 & 0 \\ -1 & -1 - \lambda & 2 \\ 0 & 0 & -2 - \lambda \end{vmatrix} = 0 \iff (\lambda + 2)^3 = 0 \iff \lambda = -2$ of multiplicity three.

If $\lambda_1 = -2$ then $(A - \lambda I)v = 0$ assumes the form \[
\begin{bmatrix}
-1 & 1 & 0 \\
-1 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix} \implies v_1 - v_2 = 0 \text{ and } v_3 = 0.\] The eigenspace corresponding to $\lambda_1 = -2$ is $E_1 = \{v \in \mathbb{R}^3 : v = r(1, 1, 0), r \in \mathbb{R}\}$. A basis for $E_1$ is $\{(1, 1, 0)\}$.

27. (a) If $\lambda_1 = 2$ then $(A - \lambda I)v = 0$ assumes the form

\[
\begin{bmatrix}
-1 & -1 & 1 \\
-1 & -1 & 1 \\
1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix} \implies v_1 + v_2 - v_3 = 0.\] The eigenspace corresponding to $\lambda_1 = 2$ is
\[ E_1 = \{v \in \mathbb{R}^3 : v = r(-1, 1, 0) + s(0, 1, 1), r, s \in \mathbb{R} \}. \] A basis for \( E_1 \) is \{(-1, 1, 0), (1, 0, 1)\}.

Now apply the Gram-Schmidt process where \( v_1 = (1, 0, 1) \), and \( v_2 = (-1, 1, 0) \). Let \( u_1 = v_1 \) so that \( \langle v_2, u_1 \rangle = \langle (-1, 1, 0), (1, 0, 1) \rangle = -1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = -1 \) and \( ||u_1||^2 = 1^2 + 0^2 + 1^2 = 2 \).

\[ u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{||u_1||^2} u_1 = (-1, 1, 0) + \frac{1}{2}(1, 0, 1) = \frac{1}{2}(-1, 2, 1). \]

Thus, \{\((1, 0, 1), (-1, 2, 1)\)\} is an orthogonal basis for \( E_1 \).

(b) If \( \lambda_2 = -1 \) then \((A - \lambda I)v = 0 \) assumes the form

\[
\begin{bmatrix}
2 & -1 & 1 \\
-1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

The eigenspace corresponding to \( \lambda_2 = -1 \) is \( E_2 = \{v \in \mathbb{R}^3 : v = r(-1, -1, 1), r \in \mathbb{R} \} \). A basis for \( E_2 \) is \{\((-1, -1, 1)\)\}.

To determine the orthogonality of the vectors, consider the following inner products:

\[ \langle (1, 0, 1), (-1, -1, 1) \rangle = -1 + 0 + 1 = 0 \]

Thus, the vectors in \( E_1 \) are orthogonal to the vectors in \( E_2 \).

29. **(a)** Setting \( \lambda = 0 \) in (5.7.4), we have \( p(\lambda) = \det(A - \lambda I) = \det(A) \), and in (5.7.5), we have \( p(\lambda) = p(0) = b_n \). Thus, \( b_n = \det(A) \).

The value of \( \det(A - \lambda I) \) is the sum of products of its elements, one taken from each row and each column. Expanding \( \det(A - \lambda I) \) yields equation (5.7.5). The expression involving \( \lambda^n \) in \( p(\lambda) \) comes from the product,

\[
\prod_{i=1}^{n} (a_{ii} - \lambda),
\]

of the diagonal elements. All the remaining products of the determinant have degree not higher than \( n - 2 \), since, if one of the factors of the product is \( a_{ij} \), where \( i \neq j \), then this product cannot contain the factors \( \lambda - a_{ii} \) and \( \lambda - a_{jj} \). Hence,

\[
p(\lambda) = \prod_{i=1}^{n} (a_{ii} - \lambda) + \text{(terms of degree not higher than \( n - 2 \))}
\]

so,

\[
p(\lambda) = (-1)^n \lambda^n + (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn}) \lambda^{n-1} + \cdots + a_n.
\]

Equating like coefficients from (5.7.5), it follows that \( b_1 = (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn}) \).

(b) Letting \( \lambda = 0 \), we have from (5.7.6) that \( p(0) = \prod_{i=1}^{n} (\lambda_i - 0) \) or \( p(0) = \prod_{i=1}^{n} \lambda_i \), but from (5.7.5), \( p(0) = b_n \),
Closure under Scalar Multiplication:

Let $b = \prod_{i=1}^{n} \lambda_i$. Letting $\lambda = 1$, we have from (5.7.6) that

$$p(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i) = (-1)^n \prod_{i=1}^{n} (\lambda - \lambda_i) = (-1)^n [\lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n) \lambda^{n-1} + \cdots + b_n].$$

Equating like coefficients with (5.7.5), it follows that $\lambda^n = (-1)^{n-1} (\lambda_1 + \lambda_2 + \cdots + \lambda_n)$.

(c) From (a), $b_n = \det(A)$, and from (b), $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$, so $\det(A)$ is the product of the eigenvalues of $A$.

From (a), $b_1 = (-1)^{n-1} (a_{11} + a_{22} + \cdots + a_{nn})$ and from (b), $b_1 = (-1)^{n-1} (\lambda_1 + \lambda_2 + \cdots + \lambda_n)$, thus $a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. That is, $\text{tr}(A)$ is the sum of the eigenvalues of $A$.

31. Note that $E_i \neq \emptyset$ since $0$ belongs to $E_i$.

Closure under Addition: Let $v_1, v_2 \in E_i$. Then $A(v_1 + v_2) = Av_1 + Av_2 = \lambda_1 v_1 + \lambda_1 v_2 = \lambda_1 (v_1 + v_2) \implies v_1 + v_2 \in E_i$.

Closure under Scalar Multiplication: Let $c \in \mathbb{C}$ and $v_1 \in E_i$. Then $A(cv_1) = c(Av_1) = c(\lambda_1 v_1) = \lambda_1 (cv_1) \implies cv_1 \in E_i$.

Thus, by Theorem 4.3.2, $E_i$ is a subspace of $C^n$.

33. Consider

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0. \quad (33.1)$$

If $c_1 \neq 0$, then the preceding equation can be written as

$$w_1 + w_2 = 0,$$

where $w_1 = c_1 v_1$ and $w_2 = c_2 v_2 + c_3 v_3$. But this would imply that $\{w_1, w_2\}$ is linearly dependent, which would contradict Theorem 5.7.5 since $w_1$ and $w_2$ are eigenvectors corresponding to different eigenvalues. Consequently, we must have $c_1 = 0$. But then (33.1) implies that $c_2 = c_3 = 0$ since $\{v_1, v_2\}$ is a linearly independent set by assumption. Hence $\{v_1, v_2, v_3\}$ is linearly independent.

35. $\lambda_1 = 0$ (multiplicity 2), basis: $\{(-1, 1, 0), (-1, 0, 1)\}$. $\lambda_2 = 3$, basis: $\{(1, 1, 1)\}$.

37. $\lambda_1 = -2$, basis: $\{(2, 1, -4)\}$. $\lambda_2 = 3$ (multiplicity 2), basis: $\{(0, 2, 1), (3, 11, 0)\}$.

39. $A$ has eigenvalues:

$$\lambda_1 = \frac{3}{2} a + \frac{1}{2} \sqrt{a^2 + 8b^2},$$

$$\lambda_2 = \frac{3}{2} a - \frac{1}{2} \sqrt{a^2 + 8b^2},$$

$$\lambda_3 = 0.$$

Provided $a \neq \pm b$, these eigenvalues are distinct, and therefore the matrix is nondefective.

If $a = b \neq 0$, then the eigenvalue $\lambda = 0$ has multiplicity two. A basis for the corresponding eigenspace is $\{(-1, 0, 1), (-1, 1, 0)\}$. Since this is two-dimensional, the matrix is nondefective in this case.

If $a = -b \neq 0$, then the eigenvalue $\lambda = 0$ once more has multiplicity two. A basis for the corresponding eigenspace is $\{(0, 1, 1), (1, 1, 0)\}$, therefore the matrix is nondefective in this case also.

If $a = b = 0$, then $A = 0_2$, so that $\lambda = 0$ (multiplicity three), and the corresponding eigenspace is all of $\mathbb{R}^3$. Hence $A$ is nondefective.
Solutions to Section 5.8

True-False Review:

1. **TRUE.** The terms “diagonalizable” and “nondefective” are synonymous. The diagonalizability of a matrix $A$ hinges on the ability to form an invertible matrix $S$ with a full set of linearly independent eigenvectors of the matrix as its columns. This, in turn, requires the original matrix to be nondefective.

2. **FALSE.** For instance, the matrices $A = I_2$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ both have eigenvalue $\lambda = 1$ (with multiplicity 2). However, $A$ and $B$ are not similar. [**Reason:** If $A$ and $B$ were similar, then $S^{-1}AS = B$ for some invertible matrix $S$, but since $A = I_2$, this would imply that $B = I_2$, contrary to our choice of $B$ above.]

3. **TRUE.** Assume $A$ is an $n \times n$ matrix such that $p(\lambda) = \det(A - \lambda I)$ has no repeated roots. This implies that $A$ has $n$ distinct eigenvalues. Corresponding to each eigenvalue, we can select an eigenvector. Since eigenvectors corresponding to distinct eigenvalues are linearly independent, this yields $n$ linearly independent eigenvectors for $A$. Therefore, $A$ is nondefective, and hence, diagonalizable.

4. **TRUE.** Since $I_n^{-1}AI_n = A$, $A$ is similar to itself.

Problems:

1. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -1 - \lambda & -2 \\ -2 & 2 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - \lambda - 6 = 0 \iff (\lambda - 3)(\lambda + 2) = 0 \iff \lambda = 3$ or $\lambda = -2$. $A$ is diagonalizable because it has two distinct eigenvalues.

   If $\lambda_1 = 3$ then $(A - \lambda I)v = 0$ assumes the form $\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2v_1 + v_2 = 0$. If we let $v_1 = r \in \mathbb{R}$, then the solution set of this system is $\{(r, 2r) : r \in \mathbb{R}\}$, so the eigenvectors corresponding to $\lambda_1 = 3$ are $v_1 = r(-1, 2)$ where $r \in \mathbb{R}$.

   If $\lambda_2 = -2$ then $(A - \lambda I)v = 0$ assumes the form $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 - 2v_2 = 0$. If we let $v_2 = s \in \mathbb{R}$, then the solution set of this system is $\{(2s, s) : s \in \mathbb{R}\}$, so the eigenvectors corresponding to $\lambda_2 = -2$ are $v_2 = s(2, 1)$ where $s \in \mathbb{R}$.

   Thus, the matrix $S = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ satisfies $S^{-1}AS = \text{diag}(3, -2)$.

2. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & -8 \\ 2 & -7 - \lambda \end{vmatrix} = 0 \iff \lambda^2 + 6\lambda + 9 = 0 \iff (\lambda + 3)^2 = 0 \iff \lambda = -3$ of multiplicity two.

   If $\lambda = -3$ then $(A - \lambda I)v = 0$ assumes the form $\begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 - 2v_2 = 0$. If we let $v_2 = r \in \mathbb{R}$, then the solution set of this system is $\{(2r, r) : r \in \mathbb{R}\}$, so the eigenvectors corresponding to $\lambda = -3$ are $v = r(2, 1)$ where $r \in \mathbb{R}$.

   A has only one linearly independent eigenvector, so by Theorem 5.8.4, $A$ is not diagonalizable.

3. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & 3 - \lambda & 7 \\ 1 & 1 & -3 - \lambda \end{vmatrix} = 0 \iff (1 - \lambda)(\lambda + 4)(\lambda - 4) = 0 \iff \lambda = 1$, $\lambda = -4$ or $\lambda = 4$.

   If $\lambda = 1$ then $(A - \lambda I)v = 0$ assumes the form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 7 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
\[ 2v_1 - 15v_3 = 0 \text{ and } 2v_2 + 7v_3 = 0. \] If we let \( v_3 = 2r \) where \( r \in \mathbb{R} \), then the solution set of this system is \( \{(15r, -7r, 2r) : r \in \mathbb{R}\} \) so the eigenvectors corresponding to \( \lambda = 1 \) are \( v_1 = r(15, -7, 2) \) where \( r \in \mathbb{R} \).

If \( \lambda = -4 \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{pmatrix}
5 & 0 & 0 \\
0 & 7 & 7 \\
1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
\end{pmatrix}
= 0.
\]

\[ v_1 = 0 \text{ and } v_2 + v_3 = 0. \] If we let \( v_2 = s \in \mathbb{R} \), then the solution set of this system is \( \{(0, s, -s) : s \in \mathbb{R}\} \) so the eigenvectors corresponding to \( \lambda = -4 \) are \( v_2 = s(0, 1, -1) \) where \( s \in \mathbb{R} \).

If \( \lambda = 4 \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{pmatrix}
-3 & 0 & 0 \\
0 & -1 & 7 \\
1 & 1 & -7 \\
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
\end{pmatrix}
= 0.
\]

\[ v_1 = 0 \text{ and } v_2 - 7v_3 = 0. \] If we let \( v_3 = t \in \mathbb{R} \), then the solution set of this system is \( \{(0, 7t, t) : t \in \mathbb{R}\} \) so the eigenvectors corresponding to \( \lambda = 4 \) are \( v_3 = t(0, 7, 1) \) where \( t \in \mathbb{R} \).

Thus, the matrix \( S = \begin{pmatrix} 15 & 0 & 0 \\ -7 & 7 & 1 \\ 2 & 1 & -1 \end{pmatrix} \) satisfies \( S^{-1}AS = \text{diag}(1, -4, 4) \).

7. \( \det(A - \lambda I) = 0 \iff \begin{vmatrix}
-\lambda & -2 & -2 \\
-2 & -\lambda & -2 \\
-2 & -2 & -\lambda \\
\end{vmatrix} = 0 \iff (\lambda - 2)(\lambda + 4) = 0 \iff \lambda = -4 \text{ or } \lambda = 2 \) of multiplicity two.

If \( \lambda = -4 \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{pmatrix}
4 & -2 & -2 \\
-2 & 4 & -2 \\
-2 & -2 & 4 \\
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
\end{pmatrix}
= 0.
\]

\[ v_1 - v_3 = 0 \text{ and } v_2 - v_3 = 0. \] If we let \( v_3 = r \in \mathbb{R} \), then the solution set of this system is \( \{(r, r, r) : r \in \mathbb{R}\} \) so the eigenvectors corresponding to \( \lambda = -4 \) are \( v_1 = r(1, 1, 1) \) where \( r \in \mathbb{R} \).

If \( \lambda = 2 \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{pmatrix}
-2 & -2 & -2 \\
-2 & -2 & -2 \\
-2 & -2 & -2 \\
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
\end{pmatrix}
= 0 \iff v_1 + v_2 + v_3 = 0. \]

If we let \( v_2 = s \in \mathbb{R} \) and \( v_3 = t \in \mathbb{R} \), then the solution set of this system is \( \{(-s, t, s) : s, t \in \mathbb{R}\} \), so two linearly independent eigenvectors corresponding to \( \lambda = 2 \) are \( v_2 = s(-1, 1, 0) \) and \( v_3 = t(-1, 0, 1) \).

Thus, the matrix \( S = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \) satisfies \( S^{-1}AS = \text{diag}(-4, 2, 2) \).

9. \( \det(A - \lambda I) = 0 \iff \begin{vmatrix}
2 - \lambda & 0 & 0 \\
0 & 1 - \lambda & 0 \\
2 & -1 & 1 - \lambda \\
\end{vmatrix} = 0 \iff (\lambda - 2)(\lambda - 1)^2 = 0 \iff \lambda = 2 \text{ or } \lambda = 1 \) of multiplicity two.

If \( \lambda = 1 \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
2 & -1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
\end{pmatrix}
= 0 \iff v_1 = v_2 = 0 \text{ and } v_3 \in \mathbb{R}. \] If we let \( v_3 = r \in \mathbb{R} \) then the solution set of this system is \( \{(0, 0, r) : r \in \mathbb{R}\} \), so there is only one corresponding linearly independent eigenvector. Hence, by Theorem 5.8.4, \( A \) is not diagonalizable.

11. \( \det(A - \lambda I) = 0 \iff \begin{vmatrix}
-\lambda & 2 & -1 \\
-2 & -\lambda & -2 \\
1 & 2 & -\lambda \\
\end{vmatrix} = 0 \iff \lambda(\lambda^2 + 9) = 0 \iff \lambda = 0, \text{ or } \lambda = \pm 3i. \)

If \( \lambda = 0 \) then \( (A - \lambda I)v = 0 \) assumes the form
\[
\begin{pmatrix}
0 & 2 & -1 \\
-2 & 0 & -2 \\
1 & 2 & 0 \\
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
\end{pmatrix}
= 0 \]
13. \( \lambda_1 = 2 \) (multiplicity 2), basis for eigenspace: \( \{(-3, 1, 0), (3, 0, 1)\} \).

14. \( \lambda_2 = 1 \), basis for eigenspace: \( \{(1, 2, 2)\} \).

Set \( S = \begin{bmatrix} -3 & 3 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \). Then \( S^{-1}AS = \text{diag}(2, 2, 1) \).

15. The given system can be written as \( x' = Ax \), where \( A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \).

\( A \) has eigenvalues \( \lambda_1 = -1, \lambda_2 = 5 \) with corresponding linearly independent eigenvectors \( v_1 = (-2, 1) \) and \( v_2 = (1, 1) \). If we set \( S = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \), then \( S^{-1}AS = \text{diag}(-1, 5) \), therefore, under the transformation \( x = Sy \), the given system of differential equations simplifies to

\[
\begin{bmatrix}
y'_1 \\ y'_2
\end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.
\]

Hence, \( y'_1 = -y_1 \) and \( y'_2 = 5y_2 \). Integrating these equations, we obtain

\[
y_1(t) = c_1e^{-t}, \quad y_2(t) = c_2e^{5t}.
\]

Returning to the original variables, we have

\[
x = Sy = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1e^{-t} \\ c_2e^{5t} \end{bmatrix} = \begin{bmatrix} -2c_1e^{-t} + c_2e^{5t} \\ c_1e^{-t} + c_2e^{5t} \end{bmatrix}.
\]

Consequently, \( x_1(t) = -2c_1e^{-t} + c_2e^{5t} \) and \( x_2(t) = c_1e^{-t} + c_2e^{5t} \).

17. The given system can be written as \( x' = Ax \), where \( A = \begin{bmatrix} 9 & 6 \\ -10 & -7 \end{bmatrix} \).

\( A \) has eigenvalues \( \lambda_1 = -1, \lambda_2 = 3 \) with corresponding linearly independent eigenvectors \( v_1 = (3, -5) \) and \( v_2 = (-1, 1) \). If we set \( S = \begin{bmatrix} 3 & -1 \\ -5 & 1 \end{bmatrix} \), then \( S^{-1}AS = \text{diag}(-1, 3) \), therefore, under the transformation \( x = Sy \), the given system of differential equations simplifies to

\[
\begin{bmatrix}
y'_1 \\ y'_2
\end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.
\]

Hence, \( y'_1 = -y_1 \) and \( y'_2 = 3y_2 \). Integrating these equations, we obtain

\[
y_1(t) = c_1e^{-t}, \quad y_2(t) = c_2e^{3t}.
\]
Returning to the original variables, we have

\[
x = Sy = \begin{bmatrix} 3 & -1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{3t} \end{bmatrix} = \begin{bmatrix} 3c_1 e^{-t} - c_2 e^{3t} \\ -5c_1 e^{-t} + c_2 e^{3t} \end{bmatrix}.
\]

Consequently, \(x_1(t) = 3c_1 e^{-t} - c_2 e^{3t}\) and \(x_2(t) = -5c_1 e^{-t} + c_2 e^{3t}\).

19. The given system can be written as \(\mathbf{x}' = A \mathbf{x}\), where \(A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\).

\(A\) has eigenvalues \(\lambda_1 = i\), \(\lambda_2 = -i\) with corresponding linearly independent eigenvectors \(\mathbf{v}_1 = (1, i)\) and \(\mathbf{v}_2 = (1, -i)\). If we set \(S = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\), then \(S^{-1}AS = \text{diag}(i, -i)\), therefore, under the transformation \(\mathbf{x} = Sy\), the given system of differential equations simplifies to

\[
\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.
\]

Hence, \(y_1' = iy_1\) and \(y_2' = -iy_2\). Integrating these equations, we obtain

\[
y_1(t) = c_1 e^{it}, \quad y_2(t) = c_2 e^{-it}.
\]

Returning to the original variables, we have

\[
x = Sy = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} c_1 e^{it} \\ c_2 e^{-it} \end{bmatrix} = \begin{bmatrix} c_1 e^{it} + c_2 e^{-it} \\ i(c_1 e^{it} - c_2 e^{-it}) \end{bmatrix}.
\]

Consequently, \(x_1(t) = c_1 e^{it} + c_2 e^{-it}\) and \(x_2(t) = i(c_1 e^{it} - c_2 e^{-it})\). Using Eulers formula, these expressions can be written as

\[
x_1(t) = (c_1 + c_2) \cos t + i(c_1 - c_2) \sin t, \\
x_2(t) = i(c_1 - c_2) \cos t - (c_1 + c_2) \sin t,
\]

or equivalently,

\[
x_1(t) = a \cos t + b \sin t, \quad x_2(t) = b \cos t - a \sin t,
\]

where \(a = c_1 + c_2\), and \(b = i(c_1 - c_2)\).

21. The given system can be written as \(\mathbf{x}' = A \mathbf{x}\), where \(A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\).

\(A\) has eigenvalue \(\lambda_1 = -1\) with corresponding eigenvector \(\mathbf{v}_1 = (-1, 1, -1)\), and eigenvalue \(\lambda_2 = 2\) with corresponding linearly independent eigenvectors \(\mathbf{v}_2 = (0, 1, 1)\) and \(\mathbf{v}_3 = (1, 0, -1)\). If we set \(S = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix}\), then \(S^{-1}AS = \text{diag}(-1, 2, 2)\), therefore, under the transformation \(\mathbf{x} = Sy\), the given system of differential equations simplifies to

\[
\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.
\]

Hence, \(y_1' = -y_1\), \(y_2' = 2y_2\), and \(y_3' = 2y_3\). Integrating these equations, we obtain

\[
y_1(t) = c_1 e^{-t}, \quad y_2(t) = c_2 e^{2t}, \quad y_3(t) = c_3 e^{2t}.
\]
Returning to the original variables, we have
\[
\mathbf{x} = \mathbf{S} \mathbf{y} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{2t} \\ c_3 e^{2t} \end{bmatrix} = \begin{bmatrix} -c_1 e^{-t} + c_3 e^{2t} \\ c_1 e^{-t} + c_2 e^{2t} \\ -c_1 e^{-t} + c_2 e^{2t} - c_3 e^{2t} \end{bmatrix}.
\]
Consequently, \(x_1(t) = -c_1 e^{-t} + c_3 e^{2t}\), \(x_2(t) = c_1 e^{-t} + c_2 e^{2t}\), and \(-c_1 e^{-t} + (c_2 - c_3)e^{2t}\).

23. Let \(A = \text{diag}(a_1, a_2, \ldots, a_n)\) and let \(B = \text{diag}(b_1, b_2, \ldots, b_n)\). Then from the index form of the matrix product,
\[
(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{ii} b_{jj} = \begin{cases} 0, & \text{if } i \neq j, \\ a_i b_i, & \text{if } i = j. \end{cases}
\]

Consequently, \(AB = \text{diag}(a_1 b_1, a_2 b_2, \ldots, a_n b_n)\). Applying this result to the matrix \(D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)\), it follows directly that \(D^k = \text{diag}(\lambda_1^k, \lambda_2^k, \ldots, \lambda_k^k)\).

25.

(a) This is self-evident from matrix multiplication. Another perspective on this is that when we multiply a matrix \(B\) on the left by a diagonal matrix \(D\), the \(i\)th row of \(B\) gets multiplied by the \(i\)th diagonal element of \(D\). Thus, if we multiply the \(i\)th row of \(\sqrt{D}\), \(\sqrt{\lambda_i}\), by the \(i\)th diagonal element of \(\sqrt{D}\), \(\sqrt{\lambda_i}\), the result in the \(i\)th row of the product is \(\sqrt{\lambda_i} \sqrt{\lambda_i} = \lambda_i\). Therefore, \(\sqrt{D} \sqrt{D} = D\), which means that \(\sqrt{D}\) is a square root of \(D\).

(b) We have
\[
(S \sqrt{DS^{-1}})^2 = (S \sqrt{DS^{-1}})(S \sqrt{DS^{-1}}) = S(\sqrt{D} \sqrt{D})S^{-1} = SDS^{-1} = A,
\]
as required.

(c) We begin by diagonalizing \(A\). We have
\[
\det(A - \lambda I) = \det \begin{bmatrix} 6 - \lambda & -2 \\ -3 & 7 - \lambda \end{bmatrix} = (6 - \lambda)(7 - \lambda) - 6 = \lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9),
\]
so the eigenvalues of \(A\) are \(\lambda = 4\) and \(\lambda = 9\). An eigenvector of \(A\) corresponding to \(\lambda = 4\) is \([-1, 1]^T\), and an eigenvector of \(A\) corresponding to \(\lambda = 9\) is \([2, -3]^T\). Thus, we can form
\[
S = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.
\]

We take \(\sqrt{D} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\). A fast computation shows that \(S^{-1} = \begin{bmatrix} 3/5 & 2/5 \\ 1/5 & -1/5 \end{bmatrix}\). By part (b), one square root of \(A\) is given by
\[
\sqrt{A} = S \sqrt{DS} = \begin{bmatrix} 12/5 & -2/5 \\ -3/5 & 13/5 \end{bmatrix}.
\]

Directly squaring this result confirms this matrix as a square root of \(A\).

27. Let \(A \sim B\) mean that \(A\) is similar to \(B\). Show: \(A \sim B \iff A^T \sim B^T\).

\(A \sim B \implies\) there exists an invertible matrix \(S\) such that \(B = S^{-1}AS\)
\(\implies B^T = (S^{-1}AS)^T = S^T A^T (S^{-1})^T = S^T A^T (S^T)^{-1}. \) \(S^T\) is invertible because \(S\) is invertible, [since \(\det(S) = \det(S^T)\)]. Thus, \(A^T \sim B^T\).
29. (a) $S^{-1}AS = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \implies \det(S^{-1}AS) = \lambda_1 \lambda_2 \cdots \lambda_n$
\implies \det(A) \det(S^{-1}) \det(S) = \lambda_1 \lambda_2 \cdots \lambda_n \implies \det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$. Since all eigenvalues are nonzero, it follows that $\det(A) \neq 0$. Consequently, $A$ is invertible.

(b) $S^{-1}AS = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \implies [S^{-1}AS]^{-1} = \text{diag}(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n})$
\implies S^{-1}A^{-1}S = \text{diag}(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}).$

31. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -2 - \lambda & 4 \\ 1 & 1 - \lambda \end{vmatrix} = 0 \iff (\lambda + 3)(\lambda - 2) = 0 \iff \lambda = -3$ or $\lambda = 2$. If $\lambda = -3$ the corresponding eigenvectors are of the $v_1 = r(-4, 1)$ where $r \in \mathbb{R}$. If $\lambda = 2$ the corresponding eigenvectors are of the $v_2 = s(1, 1)$ where $s \in \mathbb{R}$.

Thus, a complete set of eigenvectors is $\{(4, 1), (1, 1)\}$ so that $S = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$. From problem 17, if $M_C$ denotes the matrix of cofactors of $S$, then $M_C = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$. Consequently, $(1, -1)$ is an eigenvector corresponding to $\lambda = -3$ and $(-1, 4)$ is an eigenvector corresponding to $\lambda = 2$ for the matrix $A^T$.

33. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 1 \\ -1 & 4 - \lambda \end{vmatrix} = 0 \iff (\lambda - 3)^2 = 0 \iff \lambda = 3$ of multiplicity two. If $\lambda = 3$ the corresponding eigenvectors are of the $v_1 = r(1, 1)$ where $r \in \mathbb{R}$. Consequently, $A$ does not have a complete set of eigenvectors, so it is a defective matrix.

By the preceding problem, $J_3 = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is similar to $A$. Hence, there exists $S = [v_1, v_2]$ such that $S^{-1}AS = J_3$. From the first part of the problem, we can let $v_1 = (1, 1)$. Now consider $(A - \lambda I)v_2 = v_1$ where $v_1 = (a, b)$ for $a, b \in \mathbb{R}$. Upon substituting, we obtain

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies -a + b = 1.$$ 

Thus, $S$ takes the form $S = \begin{bmatrix} 1 & b - 1 \\ 1 & b \end{bmatrix}$ where $b \in \mathbb{R}$, and if $b = 0$, then $S = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

35. (a) From (5.8.15), $c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0 \iff \sum_{i=1}^{n} \left[ c_i \sum_{j=1}^{n} s_{ji}e_j \right] = 0$
\iff \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} s_{ji}c_i \right] e_j = 0 \iff \sum_{i=1}^{n} s_{ji}c_i = 0, j = 1, 2, \ldots, n$, since $\{e_j\}$ is a linearly independent set. The latter equation is just the component form of the linear system $Sc = 0$. Since $\{e_j\}$ is a linearly independent set, the only solution to this system is the trivial solution $c = 0$, so $\det(S) \neq 0$. Consequently, $S$ is invertible.

(b) From (5.8.14) and (5.8.15), we have

$$T(f_k) = \sum_{i=1}^{n} b_{ik} \sum_{j=1}^{n} s_{ji}e_j = \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} s_{ji}b_{ik} \right] e_j, k = 1, 2, \ldots, n.$$
Replacing $i$ with $j$ and $j$ with $i$ yields

$$T(f_k) = \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} s_{ij}b_{jk} \right] e_i, \quad k = 1, 2, \ldots, n. \quad (\ast)$$

(c) From (5.8.15) and (5.8.13), we have

$$T(f_k) = \sum_{j=1}^{n} s_{jk}T(e_j) = \sum_{j=1}^{n} s_{jk} \sum_{i=1}^{n} a_{ij} e_i,$$

that is,

$$T(f_k) = \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} a_{ij}s_{jk} \right] e_i, \quad k = 1, 2, \ldots, n. \quad (\ast\ast)$$

(d) Subtracting (\ast) from (\ast\ast) yields

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (s_{ij}b_{jk} - a_{ij}s_{jk}) e_i = 0.$$ 

Thus, since $\{e_1\}$ is a linearly independent set,

$$\sum_{j=1}^{n} s_{ij}b_{jk} = \sum_{j=1}^{n} a_{ij}s_{jk}, \quad i = 1, 2, \ldots, n.$$

But this is just the index form of the matrix equation $SB = AS$. Multiplying both sides of the preceding equation on the left by $S^{-1}$ yields $B = S^{-1}AS$.

Solutions to Section 5.9

True-False Review:

1. TRUE. In the definition of the matrix exponential function $e^{At}$, we see that powers of the matrix $A$ must be computed:

$$e^{At} = I_n + (At) + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots.$$

In order to do this, $A$ must be a square matrix.

3. FALSE. The inverse of the matrix exponential function $e^{At}$ is the matrix exponential function $e^{-At}$, and this will exist for all square matrices $A$, not just invertible ones.

5. FALSE. The correct statement is

$$(SDS^{-1})^k = SD^kS^{-1}.$$ 

The matrices $S$ and $S^{-1}$ on the right-hand side of this equation do not get raised to the power $k$.

Problems:

1. \(\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 2 \\ 2 & -1 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - \lambda - 6 = 0 \iff (\lambda + 2)(\lambda - 3) = 0 \iff \lambda = -2 \text{ or } \lambda = 3.\)
If $\lambda = -2$ then $(A - \lambda I)v = 0$ assumes the form \[ \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2v_1 + v_2 = 0. \] $v_1 = (1, -2)$, is an eigenvector corresponding to $\lambda = -2$ and $w_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$ is a unit eigenvector corresponding to $\lambda = -2$.

If $\lambda = 3$ then $(A - \lambda I)v = 0$ assumes the form \[ \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 + 2v_2 = 0. \] $v_2 = (2, 1)$, is an eigenvector corresponding to $\lambda = 3$ and $w_2 = \frac{v_2}{\|v_2\|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ is a unit eigenvector corresponding to $\lambda = 3$.

Thus, $S = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ and $S^TAS = \text{diag}(-2, 3)$.

3. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix} = 0 \iff (\lambda - 1)^2 - 4 = 0 \iff (\lambda + 1)(\lambda - 3) = 0 \iff \lambda = -1$ or $\lambda = 3$.

If $\lambda = -1$ then $(A - \lambda I)v = 0$ assumes the form \[ \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 + v_2 = 0. \] $v_1 = (-1, 1)$, is an eigenvector corresponding to $\lambda = -1$ and $w_1 = \frac{v_1}{\|v_1\|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a unit eigenvector corresponding to $\lambda = -1$.

If $\lambda = 3$ then $(A - \lambda I)v = 0$ assumes the form \[ \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 - v_2 = 0. \] $v_2 = (1, 1)$, is an eigenvector corresponding to $\lambda = 3$ and $w_2 = \frac{v_2}{\|v_2\|} = \left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a unit eigenvector corresponding to $\lambda = 3$.

Thus, $S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ and $S^TAS = \text{diag}(-1, 3)$.

5. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 2 & 1 \\ 2 & 4 - \lambda & 2 \\ 1 & 2 & 1 - \lambda \end{vmatrix} = 0 \iff \lambda^3 - 6\lambda^2 = 0 \iff \lambda^2(\lambda - 6) = 0 \iff \lambda = 6$ or $\lambda = 0$ of multiplicity two.

If $\lambda = 0$ then $(A - \lambda I)v = 0$ assumes the form \[ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 + 2v_2 + v_3 = 0. \] $v_1 = (-1, 0, 1)$ and $v_2 = (-2, 1, 0)$ are linearly independent eigenvectors corresponding to $\lambda = 0$. $v_1$ and $v_2$ are not orthogonal since $\langle v_1, v_2 \rangle = 2 \neq 0$, so we will use the Gram-Schmidt procedure.

Let $u_1 = v_1 = (-1, 0, 1)$, so

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2}u_1 = (-2, 1, 0) - \frac{2}{2}(-1, 0, 1) = (-1, 1, -1).$$

Now $w_1 = \frac{u_1}{\|u_1\|} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and $w_2 = \frac{u_2}{\|u_2\|} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ are orthonormal eigenvectors corresponding to $\lambda = 0$. 
If $\lambda = 6$ then $(A - \lambda I)v = 0$ assumes the form
\[
\begin{bmatrix}
-5 & 2 & 1 \\
2 & -2 & 2 \\
1 & 2 & -5
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 - v_3 = 0 \text{ and } v_2 - 2v_3 = 0.
\]
v_3 = (1, 2, 1), is an eigenvector corresponding to $\lambda = 6$ and $w_3 = \displaystyle \frac{v_3}{||v_3||}$ = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ is a unit eigenvector corresponding to $\lambda = 6$.

Thus, $S = \begin{bmatrix}
-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\
0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}
\end{bmatrix}$ and $S^TAS = \text{diag}(0, 0, 6)$.

If $\lambda = 1$ then $(A - \lambda I)v = 0$ assumes the form
\[
\begin{bmatrix}
-1 & 1 & 0 \\
1 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 - v_2 = 0 \text{ and } v_1 = (1, 1, 0) \text{ and } v_2 = (0, 0, 1) \text{ are linearly independent eigenvectors corresponding to } \lambda = 1, \text{ and }
\]
w_1 = \frac{v_1}{||v_1||} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \text{ and } w_2 = \frac{v_2}{||v_2||} = (0, 0, 1) \text{ are unit eigenvectors corresponding to } \lambda = 1. w_1 \text{ and } w_2 \text{ are also orthogonal because } \langle w_1, w_2 \rangle = 0.

If $\lambda = -1$ then $(A - \lambda I)v = 0$ assumes the form
\[
\begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 + v_2 = 0 \text{ and } v_3 = 0.
\]
v_3 = (-1, 1, 0), is an eigenvector corresponding to $\lambda = -1$ and $w_3 = \frac{v_3}{||v_3||}$ = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is a unit eigenvector corresponding to $\lambda = -1$.

Thus, $S = \begin{bmatrix}
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0
\end{bmatrix}$ and $S^TAS = \text{diag}(-1, 1, 1)$.

If $\lambda = 2$, then $(A - \lambda I)v = 0$ assumes the form
\[
\begin{bmatrix}
2 & 0 & -1 \\
0 & 2 & 1 \\
-1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 2v_1 - v_3 = 0 \text{ and } 2v_2 + v_3 = 0.
\]
v_1 = (1, -1, 2), is an eigenvector corresponding to $\lambda = -1$ and $w_1 = \frac{v_1}{||v_1||}$ = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$ is a unit eigenvector corresponding to $\lambda = -1$.

If $\lambda = -1$ then $(A - \lambda I)v = 0$ assumes the form
\[
\begin{bmatrix}
0 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 - v_2 = 0 \text{ and }
let $v_3 = 0$. $v_2 = (1, 1, 0)$, is an eigenvector corresponding to $\lambda = 1$ and $w_2 = \frac{v_2}{||v_2||} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is a unit eigenvector corresponding to $\lambda = 1$.

If $\lambda = 2$ then $(A - \lambda I)v = 0$ assumes the form
$$
\begin{pmatrix}
-1 & 0 & -1 \\
0 & -1 & 1 \\
-1 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}
= 0
\implies v_1 + v_3 = 0 and v_2 - v_3 = 0.
$$
$v_2 - v_3 = 0$. $v_3 = (-1, 1, 1)$, is an eigenvector corresponding to $\lambda = 2$ and $w_3 = \frac{v_3}{||v_3||} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a unit eigenvector corresponding to $\lambda = 2$.

Thus, $S = \begin{bmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}}
\end{bmatrix}$ and $S^TAS = \text{diag}(-1, 1, 2)$.

11. $\det(A - \lambda I) = 0 \iff \begin{vmatrix}
-3 - \lambda & 2 & 2 \\
2 & -3 - \lambda & 2 \\
2 & 2 & -3 - \lambda
\end{vmatrix} = 0 \iff (\lambda + 5)^2(\lambda - 1) = 0 \iff \lambda = -5 \text{ or } \lambda = 1$.

If $\lambda = 1$ then $(A - \lambda I)v = 0$ assumes the form
$$
\begin{pmatrix}
-4 & 2 & 2 \\
2 & -4 & 2 \\
2 & 2 & -4
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}
= 0
\implies v_1 - v_3 = 0 and v_2 - v_3 = 0.
$$
$v_1 = (1, 1, 1)$ is an eigenvector corresponding to $\lambda = 1$ and $w_1 = \frac{v_1}{||v_1||} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a unit eigenvector corresponding to $\lambda = 1$.

If $\lambda = -5$ then $(A - \lambda I)v = 0$ assumes the form
$$
\begin{pmatrix}
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}
= 0
\implies v_1 + v_2 + v_3 = 0.
$$
$v_2 = (-1, 1, 0)$ and $v_3 = (-1, 0, 1)$ are linearly independent eigenvectors corresponding to $\lambda = -5$. $v_2$ and $v_3$ are not orthogonal since $\langle v_2, v_3 \rangle = -1 \neq 0$, so we will use the Gram-Schmidt procedure.

Let $u_2 = v_2 = (-1, 1, 0)$, so
$$
u_3 = v_3 - \langle v_3, u_2 \rangle \frac{u_2}{||u_2||^2} = (-1, 0, 1) - \frac{1}{2}(-1, 1, 0) = \left(-\frac{1}{2}, -\frac{1}{2}, 1\right).
$$

Now $w_2 = \frac{u_2}{||u_2||} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, 0\right)$ and $w_3 = \frac{u_3}{||u_3||} = \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ are orthonormal eigenvectors corresponding to $\lambda = -5$.

Thus, $S = \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0
\end{bmatrix}$ and $S^TAS = \text{diag}(1, -5, -5)$.

13. $A$ has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -2$ with corresponding eigenvectors $v_1 = (1, 1)$ and $v_2 = (-1, 1)$. Therefore, a set of principal axes is $\left\{\frac{1}{\sqrt{2}}(1, 1), \frac{1}{\sqrt{2}}(-1, -1)\right\}$. Relative to these principal axes, the quadratic form reduces to $4y_1^2 - 2y_2^2$.

15. $A$ has eigenvalue $\lambda = 2$ of multiplicity two with corresponding linearly independent eigenvectors
\( \mathbf{v}_1 = (1, 0, -1) \) and \( \mathbf{v}_2 = (0, 1, 1) \). Using the Gram-Schmidt procedure, an orthogonal basis in this eigenspace is \( \{ \mathbf{u}_1, \mathbf{u}_2 \} \) where
\[
\mathbf{u}_1 = (1, 0, -1), \quad \mathbf{u}_2 = (0, 1, 1) + \frac{1}{2}(1, 0, -1) = \frac{1}{2}(1, 2, 1).
\]

An orthonormal basis for the eigenspace is \( \left\{ \frac{1}{\sqrt{2}}(1, 0, -1), \frac{1}{\sqrt{6}}(1, 2, 1) \right\} \). The remaining eigenvalue of \( A \) is \( \lambda = -1 \), with eigenvector \( \mathbf{v}_3 = (-1, 1, -1) \). Consequently, a set of principal axes for the given quadratic form is \( \left\{ \frac{1}{\sqrt{2}}(1, 0, -1), \frac{1}{\sqrt{6}}(1, 2, 1), \frac{1}{\sqrt{3}}(-1, 1, -1) \right\} \). Relative to these principal axes, the quadratic form reduces to \( 2y_1^2 + 2y_2^2 - y_3^2 \).

17. \( A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \) where \( a, b, c \in \mathbb{R} \).
Consider \( \det(A - \lambda I) = 0 \iff \begin{vmatrix} a - \lambda & c \\ c & b - \lambda \end{vmatrix} = 0 \iff \lambda^2 - (a + c)\lambda + (ac - b^2) = 0 \iff \lambda = \frac{(a + c) \pm \sqrt{(a - c)^2 + 4b^2}}{2} \). Now \( A \) has repeated eigenvalues \( \iff (a - c)^2 + 4b^2 = 0 \iff a = c \) and \( b = 0 \) (since \( a, b, c \in \mathbb{R} \)) \( \iff A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \) \( \iff A = aI_2 \iff A \) is a scalar matrix.

19. Since real eigenvectors of \( A \) that correspond to distinct eigenvalues are orthogonal, it must be the case that if \( \mathbf{y} = (y_1, y_2) \) corresponds to \( \lambda_2 \) where \( A\mathbf{y} = \lambda_2\mathbf{y} \), then
\[
\langle \mathbf{x}, \mathbf{y} \rangle = 0 \implies \langle (1, 2), (y_1, y_2) \rangle = 0 \implies y_1 + 2y_2 = 0 \implies y_1 = -2y_2 \implies \mathbf{y} = (-2y_2, y_2) = y_2(-2, 1).
\]
Consequently, \((-2, 1)\) is an eigenvector corresponding to \( \lambda_2 \).

21. \( A \) is a real symmetric 3 \( \times \) 3 matrix with eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of multiplicity two.
(a) Let \( \mathbf{v}_1 = (1, -1, 1) \) be an eigenvector of \( A \) that corresponds to \( \lambda_1 \). Since real eigenvectors of \( A \) that correspond to distinct eigenvalues are orthogonal, it must be the case that if \( \mathbf{v} = (a, b, c) \) corresponds to \( \lambda_2 \) where \( A\mathbf{v} = \lambda_2\mathbf{v} \), then \( \langle \mathbf{v}_1, \mathbf{v} \rangle = 0 \implies \langle (1, -1, 1), (a, b, c) \rangle = 0 \implies a - b + c = 0 \implies \mathbf{v} = r(1, 1, 0) + s(-1, 0, 1) \) where \( r \) and \( s \) are free variables. Consequently, \( \mathbf{v}_2 = (1, 1, 0) \) and \( \mathbf{v}_3 = (-1, 0, 1) \) are linearly independent eigenvectors corresponding to \( \lambda_2 \). Thus, \( \{ (1, 1, 0), (-1, 0, 1) \} \) is a basis for \( E_{\lambda_2} \). \( \mathbf{v}_2 \) and \( \mathbf{v}_3 \) are not orthogonal since \( \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = -1 \neq 0 \), so we will apply the Gram-Schmidt procedure to \( \mathbf{v}_2 \) and \( \mathbf{v}_3 \). Let \( \mathbf{u}_2 = \mathbf{v}_2 = (1, 1, 0) \) and
\[
\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{||\mathbf{u}_2||^2} \mathbf{u}_2 = (-1, 0, 1) + \frac{1}{2}(1, 1, 0) = \left(-\frac{1}{2}, \frac{1}{2}, 1\right).
\]
Now, \( \mathbf{w}_1 = \frac{\mathbf{v}_1}{||\mathbf{v}_1||} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \) is a unit eigenvector corresponding to \( \lambda_1 \), and
\[
\mathbf{w}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \quad \mathbf{w}_3 = \frac{\mathbf{u}_3}{||\mathbf{u}_3||} = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) \text{ are orthonormal eigenvectors corresponding to } \lambda_1.
\]
Consequently, \( S \) is
\[
S = \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}
\end{bmatrix}
\]
and \( S^T A S = \text{diag}(\lambda_1, \lambda_2, \lambda_2) \).
(b) Since $S$ is an orthogonal matrix, $S^TAS = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \implies AS = S \text{diag}(\lambda_1, \lambda_2, \lambda_3)$

$\implies A = S \text{diag}(\lambda_1, \lambda_2, \lambda_3)S^T \implies$

$$A = \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{2} & 0 \\
\frac{6}{\sqrt{3}} & 0 & \sqrt{6}
\end{bmatrix}
\begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{2} & 0 \\
\frac{6}{\sqrt{3}} & 0 & \sqrt{6}
\end{bmatrix}$$

23. Let $A$ be an $n \times n$ real skew-symmetric matrix where $n$ is odd. Since $A$ is real, the characteristic equation, $\det(A - \lambda I) = 0$, has real coefficients, so its roots come in conjugate pairs. By problem 20, all nonzero solutions of $\det(A - \lambda I) = 0$ are pure imaginary, hence when $n$ is odd, zero will be one of the eigenvalues of $A$.

25. $\det(A - \lambda I) = 0 \iff \begin{vmatrix}
-\lambda & -1 & -6 \\
1 & -\lambda & 5 \\
6 & -5 & -\lambda
\end{vmatrix} = 0 \iff -\lambda^3 - 62\lambda = 0 \iff \lambda = 0 \text{ or } \lambda = \pm \sqrt{62}i$.

If $\lambda = 0$ then $(A - \lambda I)v = 0$ assumes the form
$$\begin{bmatrix}
0 & -1 & -6 \\
1 & 0 & 5 \\
6 & -5 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \implies v_3 = t, v_2 = -6t, v_1 = -5t, \text{ where } t \in \mathbb{C}. \text{ Thus, the solution set of the system is } \{(5t, -6t, t) : t \in \mathbb{C}\}, \text{ so the eigenvectors corresponding to } \lambda = 0 \text{ are } v = t(-5, -6, 1), \text{ where } t \in \mathbb{C}.$$

For the other eigenvalues, it is best to use technology to generate the corresponding eigenvectors.

True-False Review:

1. TRUE. See Remark 1 following Definition 5.11.1.

3. FALSE. For example, in a diagonalizable $n \times n$ matrix, the $n$ linearly independent eigenvectors can be arbitrarily placed in the columns of the matrix $S$. Thus, an ample supply of invertible matrices $S$ can be constructed.

5. TRUE. This is simply a restatement of the definition of a generalized eigenvector.

7. TRUE. This is the content of Theorem 5.11.8.

9. TRUE. If we place the vectors in a cycle of generalized eigenvectors of $A$ (see Equation (5.11.3)) in the columns of the matrix $S$ formulated in this section in the order they appear in the cycle, then the corresponding columns of the matrix $S^{-1}AS$ will form a Jordan block.

11. TRUE. Suppose that $S^{-1}AS = B$ and that $J$ is a Jordan canonical form of $A$. So there exists an invertible matrix $T$ such that $T^{-1}AT = J$. Then $B = S^{-1}AS = S^{-1}(TJT^{-1})S = (T^{-1}S)^{-1}J(T^{-1}S)$, and hence,

$$(T^{-1}S)B(T^{-1}S)^{-1} = J,$$

which shows that $J$ is also a Jordan canonical form of $B$. 

Solutions to Section 5.11
Problems:

1. There are 3 possible Jordan canonical forms:
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}.
\]

3. There are 7 possible Jordan canonical forms:
\[
\begin{bmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2
\end{bmatrix},
\begin{bmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2
\end{bmatrix},
\begin{bmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2
\end{bmatrix},
\begin{bmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2
\end{bmatrix}.
\]

5. Since \( \lambda = 2 \) occurs with multiplicity 4, it can give rise to the following possible Jordan block sizes:
   (a) 4
   (b) 3,1
   (c) 2,2
   (d) 2,1,1
   (e) 1,1,1,1

   Likewise, \( \lambda = 6 \) occurs with multiplicity 4, so it can give rise to the same five possible Jordan block sizes.

Finally, \( \lambda = 8 \) occurs with multiplicity 3, so it can give rise to three possible Jordan block sizes:
   (a) 3
   (b) 2,1
   (c) 1,1,1

Since the block sizes for each eigenvalue can be independently determined, we have \( 5 \cdot 5 \cdot 3 = 75 \) possible Jordan canonical forms.

7. Since \((A - 5I)^2 = 0\), no cycles of generalized eigenvectors corresponding to \( \lambda = 5 \) can have length greater than 2, and hence, only Jordan block sizes 2 or less are possible. Thus, the possible block sizes under this restriction (corresponding to \( \lambda = 5 \)) are:
   2,2
   2,2,1,1
   2,1,1,1,1
   1,1,1,1,1,1

   There are four such. There are still five possible block size lists corresponding to \( \lambda = 2 \). Multiplying these results, we have \( 5 \cdot 4 = 20 \) possible Jordan canonical forms under this restriction.

9. The assumption that \((A - \lambda I)^3 = 0\) implies no Jordan blocks of size greater than \(3 \times 3\) are possible. The fact that \((A - \lambda I)^2 \neq 0\) implies that there is at least one Jordan block of size \(3 \times 3\). Thus, the possible block size combinations for a \(6 \times 6\) matrix with eigenvalue \( \lambda \) of multiplicity 6 and no blocks of size greater than \(3 \times 3\) with at least one \(3 \times 3\) block are:
Thus, there are 3 possible Jordan canonical forms. (We omit the list itself; it can be produced simply from the list of block sizes above.)

11. The eigenvalues of the matrix with this characteristic polynomial are $\lambda = 4, 4, 4, -1, -1$. The possible Jordan canonical forms in this case are therefore:

$$
\begin{pmatrix}
4 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}, \quad
\begin{pmatrix}
4 & 1 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}, \quad
\begin{pmatrix}
4 & 1 & 0 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}, \quad
\begin{pmatrix}
4 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}, \quad
\begin{pmatrix}
4 & 1 & 0 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}, \quad
\begin{pmatrix}
4 & 1 & 0 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}.
$$

13. The eigenvalues of the matrix with this characteristic polynomial are $\lambda = -2, -2, 6, 6, 6, 6, 6$. The possible Jordan canonical forms in this case are therefore:

$$
\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6
\end{pmatrix}, \quad
\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6
\end{pmatrix}, \quad
\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6
\end{pmatrix}, \quad
\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6
\end{pmatrix}, \quad
\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 6
\end{pmatrix}.
$$
15. Many examples are possible here. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The only eigenvalue of $A$ is 0. The vector $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not an eigenvector since $Av = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \lambda v$. However $A^2 = 0$, so every vector is a generalized eigenvector of $A$ corresponding to $\lambda = 0$.

17. The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 1 \\ -1 & 3 - \lambda \end{bmatrix} = (1 - \lambda)(3 - \lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2,$$

with roots $\lambda = 2, 2$. We have

$$A - 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$ 

Because there is only one unpivoted column in this latter matrix, we only have one eigenvector for $A$. Hence, $A$ is not diagonalizable, and therefore

$$\text{JCF}(A) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$ 

To determine the matrix $S$, we must find a cycle of generalized eigenvectors of length 2. Therefore, it suffices to find a vector $v$ in $\mathbb{R}^2$ such that $(A - 2I)v \neq 0$. Many choices are possible here. We take $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$$(A - 2I)v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

Thus, we have

$$S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$
19. We can get the characteristic polynomial by using cofactor expansion along the second column as follows:

\[
\text{nullspace}(A - \lambda I) = \det \begin{bmatrix}
5 - \lambda & 0 & -1 \\
1 & 4 - \lambda & -1 \\
1 & 0 & 3 - \lambda
\end{bmatrix} = (4 - \lambda) \left( (5 - \lambda)(3 - \lambda) + 1 \right) = (4 - \lambda)(\lambda^2 - 8\lambda + 16) = (4 - \lambda)(\lambda - 4)^2,
\]

with roots \( \lambda = 4, 4, 4 \).

We have \( A - 4I = \begin{bmatrix}
1 & 0 & -1 \\
1 & 0 & -1 \\
1 & 0 & -1
\end{bmatrix} \), and so vectors \((x, y, z)\) in the nullspace of this matrix must satisfy \( x - z = 0 \). Setting \( z = t \) and \( y = s \), we have \( x = t \). Hence, we obtain two linearly independent eigenvectors of \( A \) corresponding to \( \lambda = 4 \): \( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \). Therefore, \( \text{JCF}(A) \) contains exactly two Jordan blocks.

This uniquely determines \( \text{JCF}(A) \), up to a rearrangement of the Jordan blocks:

\[
\text{JCF}(A) = \begin{bmatrix}
4 & 1 & 0 \\
0 & 4 & 0 \\
0 & 0 & 4
\end{bmatrix}.
\]

To determine the matrix \( S \), we must seek a generalized eigenvector. It is easy to verify that \((A - 4I)^2 = 0_3\), so every nonzero vector \( v \) is a generalized eigenvector. We must choose one such that \((A - 4I)v \neq 0\) in order to form a cycle of length 2. There are many choices here, but let us choose \( v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \). Then \((A - 4I)v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \). Notice that this is an eigenvector of \( A \) corresponding to \( \lambda = 4 \). To complete the matrix \( S \), we will need a second linearly independent eigenvector. Again, there are a multitude of choices. Let us choose the eigenvector \( \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \) found above. Thus,

\[
S = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}.
\]

21. We are given that \( \lambda = -5 \) occurs with multiplicity 2 as a root of the characteristic polynomial of \( A \). To search for corresponding eigenvectors, we consider

\[
\text{nullspace}(A + 5I) = \text{nullspace} \begin{bmatrix}
-1 & 1 & 0 \\
-1 & 1 & 1 \\
-2 & 1 & -2
\end{bmatrix},
\]

and this matrix row-reduces to \( \begin{bmatrix}
1 & -1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \). Since there is only one unpivoted column in this row-echelon form of \( A \), the eigenspace corresponding to \( \lambda = -5 \) is only one-dimensional. Thus, based on the eigenvalues
\[ \lambda = -5, -5, -6, \] we already know that
\[
\text{JCF}(A) = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{bmatrix}.
\]

Next, we seek a cycle of generalized eigenvectors of length 2 corresponding to \( \lambda = -5 \). The cycle takes the form \( \{(A + 5I)v, v\} \), where \( v \) is a vector such that \((A + 5I)^2v = 0\). We readily compute that 
\[
(A + 5I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\]
An obvious vector that is killed by \((A + 5I)^2\) (although other choices are also possible) is 
\[
v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.
\]
Then \((A + 5I)v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\). Hence, we have a cycle of generalized eigenvectors corresponding to \( \lambda = -5 \):
\[
\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.
\]

Now consider the eigenspace corresponding to \( \lambda = -6 \). We need only find one eigenvector \((x, y, z)\) in this eigenspace. To do so, we must compute
\[
\text{nullspace}(A + 6I) = \text{nullspace} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix},
\]
and this matrix row-reduces to 
\[
\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
We see that \( y = 0 \) and \( x - 3y - z = 0 \), which is equivalent to \( x - z = 0 \). Setting \( z = t \), we have \( x = t \). With \( t = 1 \), we obtain the eigenvector \((1, 0, 1)\). Hence, we can form the matrix
\[
S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.
\]

23. We use the characteristic polynomial to determine the eigenvalues of \( A \):
\[
\det(A - \lambda I) = \det \begin{bmatrix} 7 - \lambda & -2 & 2 \\ 0 & 4 - \lambda & -1 \\ -1 & 1 & 4 - \lambda \end{bmatrix}
= (4 - \lambda)[(7 - \lambda)(4 - \lambda) + 2] + (7 - \lambda - 2)
= (4 - \lambda)(\lambda^2 - 11\lambda + 30) + (5 - \lambda)
= (4 - \lambda)(\lambda - 5)(\lambda - 6) + (5 - \lambda)
= (5 - \lambda)(\lambda - 5)^2
= - (\lambda - 5)^3.
\]
Hence, the eigenvalues are \( \lambda = 5, 5, 5 \). Let us consider the eigenspace corresponding to \( \lambda = 5 \). We consider
\[
\text{nullspace}(A - 5I) = \text{nullspace} \begin{bmatrix} 2 & -2 & 2 \\ 0 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix},
\]
and this latter matrix row-reduces to \[
\begin{bmatrix}
1 & -1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix},
\]
which contains one unpivoted column. Therefore, the eigenspace corresponding to \(\lambda = 5\) is one-dimensional. Therefore, the Jordan canonical form of \(A\) consists of one Jordan block:

\[
\text{JCF}(A) = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.
\]

A corresponding invertible matrix \(S\) in this case must have columns that consist of one cycle of generalized eigenvectors, which will take the form \(\{(A - 5I)^2v, (A - 5I)v, v\}\), where \(v\) is a generalized eigenvector. Now, we can verify quickly that

\[
A - 5I = \begin{bmatrix} 2 & -2 & 2 \\ 0 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix}, \quad (A - 5I)^2 = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 0 & 2 \\ -1 & 0 & -2 \end{bmatrix}, \quad (A - 5I)^3 = 0_3.
\]

The fact that \((A - 5I)^3 = 0_3\) means that every nonzero vector \(v\) is a generalized eigenvector. Hence, we simply choose \(v\) such that \((A - 5I)^2v \neq 0\). There are many choices. Let us take \(v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\). Then

\[
(A - 5I)v = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad (A - 5I)^2v = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.
\]

Thus, we have the cycle of generalized eigenvectors

\[
\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.
\]

Hence, we have

\[
S = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}.
\]

25. We use the characteristic polynomial to determine the eigenvalues of \(A\):

\[
\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & -1 & 0 & 1 \\ 0 & 3 - \lambda & -1 & 0 \\ 0 & 1 & 1 - \lambda & 0 \\ 0 & -1 & 0 & 3 - \lambda \end{bmatrix}
\]

\[
= (2 - \lambda)(3 - \lambda) [(3 - \lambda)(1 - \lambda) + 1]
\]

\[
= (2 - \lambda)(3 - \lambda)(\lambda^2 - 4\lambda + 4) = (2 - \lambda)(3 - \lambda)(\lambda - 2)^2,
\]

and so the eigenvalues are \(\lambda = 2, 2, 2, 3\).

First, consider the eigenspace corresponding to \(\lambda = 2\). We consider

\[
nullspace(A - 2I) = nullspace \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},
\]
and this latter matrix can be row-reduced to
\[
\begin{bmatrix}
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
There are two unpivoted columns, and therefore two linearly independent eigenvectors corresponding to \( \lambda = 2 \). Thus, we will obtain two Jordan blocks corresponding to \( \lambda = 2 \), and they necessarily will have size \( 2 \times 2 \) and \( 1 \times 1 \). Thus, we are already in a position to write down the Jordan canonical form of \( A \):
\[
\text{JCF}(A) = \begin{bmatrix}
2 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}.
\]

We continue in order to obtain an invertible matrix \( S \) such that \( S^{-1}AS \) is in Jordan canonical form. To this end, we see a generalized eigenvector \( v \) such that \((A - 2I)v \neq 0\) and \((A - 2I)^2v = 0\). Note that
\[
A - 2I = \begin{bmatrix}
0 & -1 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
(A - 2I)^2 = \begin{bmatrix}
0 & -2 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -2 & 1 & 1
\end{bmatrix}.
\]

By inspection, we see that by taking \( v = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \) (there are many other valid choices, of course), then
\[
(A - 2I)v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad (A - 2I)^2v = 0.
\]

We also need a second eigenvector corresponding to \( \lambda = 2 \) that is linearly independent from \((A - 2I)v\) just obtained. From the row-echelon form of \( A - 2I \), we see that all eigenvectors corresponding to \( \lambda = 2 \) take the form \( \begin{bmatrix} s \\ t \\ t \\ t \end{bmatrix} \), so for example, we can take \( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \).

Next, we consider the eigenspace corresponding to \( \lambda = 3 \). We consider
\[
\text{nullspace}(A - 3I) = \text{nullspace} \begin{bmatrix}
-1 & -1 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & -2 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}.
\]

Now, if \((x, y, z, w)\) is an eigenvector corresponding to \( \lambda = 3 \), the last three rows of the matrix imply that \( y = z = 0 \). Thus, the first row becomes \(-x + w = 0\). Setting \( w = t \), then \( x = t \), so we obtain eigenvectors in the form \((t, 0, 0, t)\). Setting \( t = 1 \) gives the eigenvector \((1, 0, 0, 1)\). Thus, we can now form the matrix \( S \) such that \( S^{-1}AS \) is the Jordan canonical form we obtained above:
\[
S = \begin{bmatrix}
1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 \\
1 & 1 & 0 & 1
\end{bmatrix}.
\]
Since \( A \) is upper triangular, the eigenvalues appear along the main diagonal: \( \lambda = 2, 2, 2, 2 \). Looking at

\[
\text{nullspace}(A - 2I) = \text{nullspace} \begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

we see that the row-echelon form of this matrix will contain two pivots, and therefore, three unpivoted columns. That means that the eigenspace corresponding to \( \lambda = 2 \) is three-dimensional. Therefore, \( \text{JCF}(A) \) consists of three Jordan blocks. The only list of block sizes for a \( 5 \times 5 \) matrix with three blocks are (a) 3,1,1 and (b) 2,2,1. In this case, note that

\[
(A - 2I)^2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \neq 0_5,
\]

so that it is possible to find a vector \( v \) that generates a cycle of generalized eigenvectors of length 3: \( \{(A - 2I)^2v, (A - 2I)v, v\} \). Thus, \( \text{JCF}(A) \) contains a Jordan block of size \( 3 \times 3 \). We conclude that the correct list of block sizes for this matrix is 3,1,1:

\[
\text{JCF}(A) = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2
\end{bmatrix}.
\]

Since \( A \) is upper triangular, the eigenvalues appear along the main diagonal: \( \lambda = 1, 1, 1, 1, 1, 1 \). Looking at

\[
\text{nullspace}(A - I) = \text{nullspace} \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

we see that if \( (a, b, c, d, e, f, g, h) \) is an eigenvector of \( A \), then \( b = d = f = h = 0 \), and \( a, c, e, \) and \( g \) are free variables. Thus, we have four linearly independent eigenvectors of \( A \), and hence we expect four Jordan blocks. Now, an easy calculation shows that \( (A - I)^2 = 0 \), and thus, no Jordan blocks of size greater than \( 2 \times 2 \) are permissible. Thus, it must be the case that \( \text{JCF}(A) \) consists of four Jordan blocks, each of which
is a $2 \times 2$ matrix:

$$JCF(A) = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$ 

31. SIMILAR. We will compute $JCF(A)$ and $JCF(B)$. If they are the same (up to a rearrangement of the Jordan blocks), then $A$ and $B$ are similar; otherwise they are not. Both matrices have eigenvalues $\lambda = 5, 5, 5$. (For $A$, this is easiest to compute by expanding $\det(A - \lambda I)$ along the middle row, and for $B$, this is easiest to compute by expanding $\det(B - \lambda I)$ along the second column.) In Problem 23, we computed

$$JCF(A) = \begin{bmatrix}
5 & 1 & 0 \\
0 & 5 & 1 \\
0 & 0 & 5
\end{bmatrix}.$$ 

Next, consider

$$\text{nullspace}(B - 5I) = \text{nullspace}\begin{bmatrix}
-2 & -1 & -2 \\
1 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix},$$

and this latter matrix can be row-reduced to $\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}$, which has only one unpivoted column. Thus, the eigenspace of $B$ corresponding to $\lambda = 5$ is only one-dimensional, and so

$$JCF(B) = \begin{bmatrix}
5 & 1 & 0 \\
0 & 5 & 1 \\
0 & 0 & 5
\end{bmatrix}.$$ 

Since $A$ and $B$ each had the same Jordan canonical form, they are similar matrices.

33. The eigenvalues of $A$ are $\lambda = -1, -1, 1$. The eigenspace corresponding to $\lambda = -1$ is

$$\text{nullspace}(A + I) = \text{nullspace}\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix},$$

which is only one-dimensional, spanned by the vector $(1, -1, 1)$. Therefore, we seek a generalized eigenvector $v$ of $A$ corresponding to $\lambda = -1$ such that $\{(A + I)v, v\}$ is a cycle of generalized eigenvectors. Note that

$$A + I = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix} \quad \text{and} \quad (A + I)^2 = \begin{bmatrix}
1 & 2 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1
\end{bmatrix}.$$ 

In order that $v$ be a generalized eigenvector of $A$ corresponding to $\lambda = -1$, we should choose $v$ such that $(A + I)^2v = 0$ and $(A + I)v \neq 0$. There are many valid choices; let us choose $v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Then
\((A + I)v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\). Hence, we obtain the cycle of generalized eigenvectors corresponding to \(\lambda = -1\):

\[
\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.
\]

Next, consider the eigenspace corresponding to \(\lambda = 1\). For this, we compute

\[
\text{nullspace}(A - I) = \text{nullspace} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.
\]

This can be row-reduced to \(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}\). We find the eigenvector \(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\) as a basis for this eigenspace.

Hence, we are ready to form the matrices \(S\) and \(J\):

\[
S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Via the substitution \(x = Sy\), the system \(x' = Ax\) is transformed into \(y' = Jy\). The corresponding equations are

\[
y'_1 = -y_1 + y_2, \quad y'_2 = -y_2, \quad y'_3 = y_3.
\]

The third equation has solution \(y_3(t) = c_3e^t\), the second equation has solution \(y_2(t) = c_2e^{-t}\), and so the first equation becomes

\[
y'_1 + y_1 = c_2e^{-t}.
\]

This is a first-order linear equation with integrating factor \(I(t) = e^t\). When we multiply the differential equation for \(y_1(t)\) by \(I(t)\), it becomes \((y_1 \cdot e^t)' = c_2\). Integrating both sides yields \(y_1 \cdot e^t = c_2t + c_1\). Thus,

\[
y_1(t) = c_2te^{-t} + c_1e^{-t}.
\]

Thus, we have

\[
y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} c_2te^{-t} + c_1e^{-t} \\ c_2e^{-t} \\ c_3e^t \end{bmatrix}.
\]

Finally, we solve for \(x(t)\):

\[
x(t) = Sy(t) = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_2te^{-t} + c_1e^{-t} \\ c_2e^{-t} \\ c_3e^t \end{bmatrix} = \begin{bmatrix} c_2te^{-t} + c_1e^{-t} + c_2e^{-t} + c_3e^t \\ -c_2te^{-t} - c_1e^{-t} + c_3e^t \\ c_2te^{-t} + c_1e^{-t} - c_2e^{-t} + c_3e^t \end{bmatrix} = c_1e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2e^{-t} \begin{bmatrix} t + 1 \\ -t \\ t - 1 \end{bmatrix} + c_3e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\]
35. The eigenvalues of $A$ are $\lambda = 4, 4, 4$. The eigenspace corresponding to $\lambda = 4$ is

$$\text{nullspace}(A - 4I) = \text{nullspace} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and there is only one eigenvector. Therefore, the Jordan canonical form of $A$ is

$$J = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$ 

Next, we need to find an invertible matrix $S$ such that $S^{-1}AS = J$. To do this, we must find a cycle of generalized eigenvectors $\{(A - 4I)^2v, (A - 4I)v, v\}$ of length 3 corresponding to $\lambda = 4$. We have

$$A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad (A - 4I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (A - 4I)^3 = 0_3.$$ 

From $(A - 4I)^3 = 0_3$, we know that every nonzero vector is a generalized eigenvector corresponding to $\lambda = 4$. We choose $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (any multiple of the chosen vector $v$ would be acceptable as well). Thus,

$$(A - 4I)v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad (A - 4I)^2v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$ 

Thus, we have the cycle of generalized eigenvectors

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$ 

Thus, we can form the matrix

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

Via the substitution $x = Sy$, the system $x' = Ax$ is transformed into $y' = Jy$. The corresponding equations are

$$y_1' = 4y_1 + y_2, \quad y_2' = 4y_2 + y_3, \quad y_3' = 4y_3.$$ 

The third equation has solution $y_3(t) = c_3e^{4t}$, and the second equation becomes

$$y_2' - 4y_2 = c_3e^{4t}.$$ 

This is a first-order linear equation with integrating factor $I(t) = e^{-4t}$. When we multiply the differential equation for $y_2(t)$ by $I(t)$, it becomes $(y_2 \cdot e^{-4t})' = c_3$. Integrating both sides yields $y_2 \cdot e^{-4t} = c_3t + c_2$. Thus,

$$y_2(t) = c_3te^{4t} + c_2e^{4t} = e^{4t}(c_3t + c_2).$$

Therefore, the differential equation for $y_1(t)$ becomes

$$y_1' - 4y_1 = e^{4t}(c_3t + c_2).$$
This equation is first-order linear with integrating factor \( I(t) = e^{-4t} \). When we multiply the differential equation for \( y_1(t) \) by \( I(t) \), it becomes
\[
(y_1 \cdot e^{-4t})' = c_3t + c_2.
\]
Integrating both sides, we obtain
\[
y_1 \cdot e^{-4t} = \frac{t^2}{2} + c_2t + c_1.
\]
Hence,
\[
y_1(t) = e^{4t} \left( \frac{t^2}{2} + c_2t + c_1 \right).
\]
Thus, we have
\[
y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} e^{4t} \left( \frac{t^2}{2} + c_2t + c_1 \right) \\ e^{4t}(c_3t + c_2) \\ c_3e^{4t} \end{bmatrix}.
\]
Finally, we solve for \( x(t) \):
\[
x(t) = S y(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{4t} \left( \frac{t^2}{2} + c_2t + c_1 \right) \\ e^{4t}(c_3t + c_2) \\ c_3e^{4t} \end{bmatrix} = \begin{bmatrix} c_1e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2e^{4t} \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} + c_3e^{4t} \begin{bmatrix} 1 \\ t \\ t^2/2 \end{bmatrix} \end{bmatrix}.
\]

37. Let \( J = \text{JCF}(A) = \text{JCF}(B) \). Thus, there exist invertible matrices \( S \) and \( T \) such that
\[
S^{-1}AS = J \quad \text{and} \quad T^{-1}BT = J.
\]
Thus,
\[
S^{-1}AS = T^{-1}BT,
\]
and so
\[
B = TS^{-1}AST^{-1} = (ST^{-1})^{-1}A(ST^{-1}),
\]
which implies by definition that \( A \) and \( B \) are similar matrices.

39. (a): Let \( J \) be an \( n \times n \) Jordan block with eigenvalue \( \lambda \). Then the eigenvalues of \( J^T \) are \( \lambda \) (with multiplicity \( n \)). The matrix \( J^T - \lambda I \) consists of 1's on the subdiagonal (the diagonal parallel and directly beneath the main diagonal) and zeros elsewhere. Hence, the null space of \( J^T - \lambda I \) is one-dimensional (with a free variable corresponding to the right-most column of \( J^T - \lambda I \)). Therefore, the Jordan canonical form of \( J^T \) consists of a single Jordan block, since there is only one linearly independent eigenvector corresponding to the eigenvalue \( \lambda \). However, a single Jordan block with eigenvalue \( \lambda \) is precisely the matrix \( J \). Therefore,
\[
\text{JCF}(J^T) = J.
\]
(b): Let JCF(A) = J. Then there exists an invertible matrix S such that S^{-1}AS = J. Transposing both sides, we obtain (S^{-1}AS)^T = J^T, or S^T A^T (S^{-1})^T = J^T, or S^T A^T (S^{-1})^{-1} = J^T. Hence, the matrix A^T is similar to J^T. However, by applying part (a) to each block in J^T, we find that JCF(J^T) = J. Hence, J^T is similar to J. By Problem 26 in Section 5.8, we conclude that A^T is similar to J. Hence, A^T and J have the same Jordan canonical form. However, since JCF(J) = J, we deduce that JCF(A^T) = J = JCF(A), as required.

**Solutions to Section 5.12**

**Problems:**

1. **NO.** Note that T(1,1) = (2,0,0,1) and T(2,2) = (4,0,0,4) \( \neq 2T(1,1) \). Thus, T is not a linear transformation.

3. **YES.** The function T can be represented by the matrix function

\[ T(x) = Ax, \]

where

\[ A = \begin{bmatrix} 0 & 0 & -3 \\ 2 & -1 & 5 \end{bmatrix} \]

and \( x \) is a vector in \( \mathbb{R}^3 \). Every matrix transformation of the form \( T(x) = Ax \) is linear. Since the domain of T has larger dimension than the codomain of T, T cannot be one-to-one. However, since

\[ T(1,1/3,-1/3) = (1,0) \quad \text{and} \quad T(1,1,0) = (0,1), \]

we see that T is onto. Thus, \( \text{Rng}(T) = \mathbb{R}^2 \), and so a basis for \( \text{Rng}(T) \) is \{(1,0),(0,1)\}, and \( \text{Rng}(T) \) is 2-dimensional. The kernel of T consists of vectors of the form \((t,2t,0)\), where \( t \in \mathbb{R} \), and hence, a basis for \( \text{Ker}(T) \) is \{(1,2,0)\}. We have that \( \text{Ker}(T) \) is 1-dimensional.

5. **YES.** The function T can be represented by the matrix function

\[ T(x) = Ax, \]

where

\[ A = \begin{bmatrix} 1/5 & 1/5 \end{bmatrix} \]

and \( x \) is a vector in \( \mathbb{R}^2 \). Every such matrix transformation is linear. Since the domain of T has larger dimension than the codomain of T, T cannot be one-to-one. However, \( T(5,0) = 1 \), so we see that T is onto. Thus, \( \text{Rng}(T) = \mathbb{R} \), a 1-dimensional space with basis \{1\}. The kernel of T consists of vectors of the form \( t(1,-1) \), and hence, a basis for \( \text{Ker}(T) \) is \{(1,-1)\}. We have that \( \text{Ker}(T) \) is 1-dimensional.

7. **YES.** We can verify that T respects addition and scalar multiplication as follows:

\( T \) respects addition: Let \( a_1 + b_1x + c_1x^2 \) and \( a_2 + b_2x + c_2x^2 \) belong to \( P_2 \). Then

\[ T((a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2)) = T((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2) \]

\[ = \begin{bmatrix} -(a_1 + a_2) - (b_1 + b_2) \\ 3(c_1 + c_2) - (a_1 + a_2) \\ -2(b_1 + b_2) \end{bmatrix} \]

\[ = \begin{bmatrix} -a_1 - b_1 \\ 3c_1 - a_1 - 2b_1 \end{bmatrix} + \begin{bmatrix} -a_2 - b_2 \\ 3c_2 - a_2 - 2b_2 \end{bmatrix} \]

\[ = T(a_1 + b_1x + c_1x^2) + T(a_2 + b_2x + c_2x^2). \]
Let $a + bx + cx^2$ belong to $P_2$ and let $k$ be a scalar. Then we have

$$T(k(a + bx + cx^2)) = T((ka) + (kb)x + (kc)x^2) = \begin{bmatrix} -ka - kb & 0 \\ 3kc - (ka) & -2kb \end{bmatrix}$$

Next, observe that $a + bx + cx^2$ belongs to Ker$(T)$ if and only if $-a - b = 0$, $3c - a = 0$, and $-2b = 0$. These equations require that $b = 0$, $a = 0$, and $c = 0$. Thus, Ker$(T) = \{0\}$, which implies that Ker$(T)$ is 0-dimensional (with basis $\emptyset$), and that $T$ is one-to-one. However, since $M_2(\mathbb{R})$ is 4-dimensional and $P_2$ is only 3-dimensional, we see immediately that $T$ cannot be onto. By the Rank-Nullity Theorem, in fact, Rng$(T)$ must be 3-dimensional, and a basis is given by

$$\text{Basis for Rng}(T) = \{T(1), T(x), T(x^2)\} = \left\{\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}\right\}.$$

9. YES. We can verify that $T$ respects addition and scalar multiplication as follows:

$T$ respects addition: Let $(a_1, b_1, c_1)$ and $(a_2, b_2, c_2)$ belong to $\mathbb{R}^3$. Then

$$T((a_1, b_1, c_1) + (a_2, b_2, c_2)) = T(a_1 + a_2, b_1 + b_2, c_1 + c_2) = (a_1 + a_2)x^2 + (2(b_1 + b_2) - (c_1 + c_2))x + (a_1 + a_2 - 2(b_1 + b_2) + (c_1 + c_2))$$

$$= \left[a_1x^2 + (2b_1 - c_1)x + (a_1 - 2b_1 + c_1)\right] + \left[a_2x^2 + (2b_2 - c_2)x + (a_2 - 2b_2 + c_2)\right] = T((a_1, b_1, c_1)) + T((a_2, b_2, c_2)).$$

$T$ respects scalar multiplication: Let $(a, b, c)$ belong to $\mathbb{R}^3$ and let $k$ be a scalar. Then

$$T(k(a, b, c)) = T(ka, kb, kc) = (ka)x^2 + (2kb - kc)x + (ka - 2kb + kc) = k(ax^2 + (2b - c)x + (a - 2b + c)) = kT((a, b, c)).$$

Thus, $T$ is a linear transformation. Now, $(a, b, c)$ belongs to Ker$(T)$ if and only if $a = 0$, $2b - c = 0$, and $a - 2b + c = 0$. These equations collectively require that $a = 0$ and $2b = c$. Setting $c = 2t$, we find that $b = t$. Hence, $(a, b, c)$ belongs to Ker$(T)$ if and only if $(a, b, c)$ has the form $(0, t, 2t) = t(0, 1, 2)$. Hence, $\{(0, 1, 2)\}$ is a basis for Ker$(T)$, which is therefore 1-dimensional. Hence, $T$ is not one-to-one.

By Proposition 5.4.13, $T$ is also not onto. In fact, the Rank-Nullity Theorem implies that Rng$(T)$ must be 2-dimensional. It is spanned by

$$\{T(1, 0, 0), T(0, 1, 0), T(0, 0, 1)\} = \{x^2 + 1, 2x - 2, -x + 1\},$$

but the last two polynomials are proportional to each other. Omitting the polynomial $2x - 2$ (this is an arbitrary choice; we could have omitted $-x + 1$ instead), we arrive at a basis for Rng$(T)$: $\{x^2 + 1, -x + 1\}$.

11. We have

$$T(x, y, z) = (-x + 8y, 2x - 2y - 5z).$$
13. We have
\[ T(x) = \frac{x}{2} T(2) = \frac{x}{2} (-1, 5, 0, -2) = \left( \frac{x}{2}, \frac{5x}{2}, 0, -x \right). \]

15. For an arbitrary element \( ax^2 + bx + c \) in \( P_2 \), if we write
\[ ax^2 + bx + c = r(x^2 - x - 3) + s(2x + 5) + 6t = rx^2 + (-r + 2s)x + (-3r + 5s + 6t), \]
we can solve for \( r, s, t \) to find \( r = a, s = \frac{1}{2}(a + b), \) and \( t = \frac{1}{12}a - \frac{5}{12}b + \frac{1}{6}c. \) Thus,
\[ T(ax^2 + bx + c) = T \left( a(x^2 - x - 3) + \frac{1}{2}(a + b)(2x + 5) + \frac{1}{2}a - \frac{5}{2}b + c \right) \]
\[ = aT(x^2 - x - 3) + \frac{1}{2}(a + b)T(2x + 5) + \left( \frac{1}{12}a - \frac{5}{12}b + \frac{c}{6} \right)T(6) \]
\[ = a \begin{bmatrix} -2 & 1 \\ -4 & -1 \end{bmatrix} + \frac{1}{2}(a + b) \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} + \left( \frac{1}{12}a - \frac{5}{12}b + \frac{c}{6} \right) \begin{bmatrix} 12 & 6 \\ 6 & 18 \end{bmatrix} \]
\[ = \begin{bmatrix} -a - 5b + 2c \\ -\frac{5}{2}a - \frac{3}{2}b + c \end{bmatrix} - 2a - 2b + c \]
\[ = \begin{bmatrix} -a - 5b + 2c \\ -\frac{5}{2}a - \frac{3}{2}b + c \end{bmatrix}. \]

17. Since \( T \) is one-to-one, \( \dim[\ker(T)] = 0 \), so the Rank-Nullity Theorem gives
\[ \dim[\text{Rng}(T)] = \dim[M_{2 \times 3}(\mathbb{R})] = 6. \]

19. To compute the eigenvalues, we find the characteristic equation
\[ \det(A - \lambda I) = \det \begin{bmatrix} 13 - \lambda & -9 \\ 25 & -17 - \lambda \end{bmatrix} = (13 - \lambda)(-17 - \lambda) + 225 = \lambda^2 + 4\lambda + 4, \]
and the roots of this equation are \( \lambda = -2, -2. \)

**Eigenvalue \( \lambda = 2. \)** We compute
\[ \text{nullspace}(A - 2I) = \text{nullspace} \begin{bmatrix} 15 & -9 \\ 25 & -15 \end{bmatrix}, \]
but since there is only one linearly independent solution to the corresponding system (one free variable), the eigenvalue \( \lambda = 2 \) does not have two linearly independent solutions. Hence, \( A \) is not diagonalizable.

21. To compute the eigenvalues, we find the characteristic equation
\[ \det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 1 & 0 \\ -4 & 5 - \lambda & 0 \\ 17 & -11 & -2 - \lambda \end{bmatrix} = (1 - \lambda)(5 - \lambda)(-2 - \lambda) \]
\[ = (1 - \lambda)(\lambda^2 - 6\lambda + 9) = (1 - \lambda)(\lambda - 3)^2, \]
so the eigenvalues are \( \lambda_1 = 3 \) and \( \lambda_2 = -2. \)

**Eigenvalue \( \lambda_1 = 3. \)** To get eigenvectors, we consider
\[ \text{nullspace}(A - 3I) = \text{nullspace} \begin{bmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ 17 & -11 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 0 \\ 17 & -11 & -5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -5 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}. \]
The latter matrix contains only one unpivoted column, so that only one linearly independent eigenvector can be obtained. However, $\lambda_1 = 3$ occurs with multiplicity 2 as a root of the characteristic equation for the matrix. Therefore, the matrix is not diagonalizable.

23. We are given that the eigenvalues of $A$ are $\lambda_1 = 4$ and $\lambda_2 = -1$.

**Eigenvalue $\lambda_1 = 4$:** We consider

$$\text{nullspace}(A - 4I) = \text{nullspace} \begin{bmatrix} 5 & 5 & -5 \\ 0 & -5 & 0 \\ 10 & 5 & -10 \end{bmatrix}.$$  

The middle row tells us that nullspace vectors $(x, y, z)$ must have $y = 0$. From this information, the first and last rows of the matrix tell us the same thing: $x = z$. Thus, an eigenvector corresponding to $\lambda_1 = 4$ may be chosen as

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

**Eigenvalue $\lambda_2 = -1$:** We consider

$$\text{nullspace}(A + I) = \text{nullspace} \begin{bmatrix} 10 & 5 & -5 \\ 0 & 0 & 0 \\ 10 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 10 & 5 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

From the first row, a vector $(x, y, z)$ in the nullspace must satisfy $10x + 5y - 5z = 0$. Setting $z = t$ and $y = s$, we get $x = \frac{1}{2}t - \frac{1}{2}s$. Hence, the eigenvectors corresponding to $\lambda_2 = -1$ take the form $(\frac{1}{2}t - \frac{1}{2}s, s, t)$, and so a basis for this eigenspace is

$$\left\{ \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Putting the above results together, we form

$$S = \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

29. We will compute the dimension of each of the two eigenspaces associated with the matrix $A$.

For $\lambda_1 = -1$, we compute as follows:

$$\text{nullspace}(A + I) = \text{nullspace} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & -2 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix},$$

which has one unpivoted column. Thus, this eigenspace is 1-dimensional.

For $\lambda_2 = 3$, we compute as follows:

$$\text{nullspace}(A - 3I) = \text{nullspace} \begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & -2 \\ -1 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix},$$
which has one unpivoted column. Thus, this eigenspace is also one-dimensional.

Since we have only generated two linearly independent eigenvectors from the eigenvalues of $A$, we know that $A$ is not diagonalizable, and hence, the Jordan canonical form of $A$ is not a diagonal matrix. We must have one $1 \times 1$ Jordan block and one $2 \times 2$ Jordan block. To determine which eigenvalue corresponds to the $1 \times 1$ block and which corresponds to the $2 \times 2$ block, we must determine the multiplicity of the eigenvalues as roots of the characteristic equation of $A$.

A short calculation shows that $\lambda_1 = -1$ occurs with multiplicity 2, while $\lambda_2 = 3$ occurs with multiplicity 1. Thus, the Jordan canonical form of $A$ is

$$J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$ 

31. There are 7 different possible Jordan canonical forms, up to a rearrangement of the Jordan blocks:

**Case 1:**

$$J = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$ 

In this case, the matrix has five linearly independent eigenvectors, and because all Jordan blocks have size $1 \times 1$, the maximum length of a cycle of generalized eigenvectors for this matrix is 1.

**Case 2:**

$$J = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$ 

In this case, the matrix has four linearly independent eigenvectors, and because the largest Jordan block is of size $2 \times 2$, the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

**Case 3:**

$$J = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$ 

In this case, the matrix has three linearly independent eigenvectors, and because the largest Jordan block is of size $2 \times 2$, the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

**Case 4:**

$$J = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$ 

In this case, the matrix has three linearly independent eigenvectors, and because the largest Jordan block is of size $3 \times 3$, the maximum length of a cycle of generalized eigenvectors for this matrix is 3.
Case 5:

\[ J = \begin{bmatrix}
4 & 1 & 0 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 4 & 1 \\
0 & 0 & 0 & 0 & 4
\end{bmatrix}. \]

In this case, the matrix has two linearly independent eigenvectors, and because the largest Jordan block is of size 3 \( \times \) 3, the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

Case 6:

\[ J = \begin{bmatrix}
4 & 1 & 0 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 4
\end{bmatrix}. \]

In this case, the matrix has two linearly independent eigenvectors, and because the largest Jordan block is of size 4 \( \times \) 4, the maximum length of a cycle of generalized eigenvectors for this matrix is 4.

Case 7:

\[ J = \begin{bmatrix}
4 & 1 & 0 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 \\
0 & 0 & 4 & 1 & 0 \\
0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 4
\end{bmatrix}. \]

In this case, the matrix has only one linearly independent eigenvector, and because the largest Jordan block is of size 5 \( \times \) 5, the maximum length of a cycle of generalized eigenvectors for this matrix is 5.

33. There are 15 different possible Jordan canonical forms, up to a rearrangement of Jordan blocks:

Case 1:

\[ J = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}. \]

In this case, the matrix has seven linearly independent eigenvectors (four corresponding to \( \lambda = 2 \) and three corresponding to \( \lambda = -4 \)), and because all Jordan blocks are size 1 \( \times \) 1, the maximum length of a cycle of generalized eigenvectors for this matrix is 1.

Case 2:

\[ J = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}. \]
In this case, the matrix has six linearly independent eigenvalues (three corresponding to $\lambda = 2$ and three corresponding to $\lambda = -4$), and because the largest Jordan block is of size $2 \times 2$, the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

**Case 3:**

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$  

In this case, the matrix has five linearly independent eigenvalues (two corresponding to $\lambda = 2$ and three corresponding to $\lambda = -4$), and because the largest Jordan block is of size $2 \times 2$, the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

**Case 4:**

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$  

In this case, the matrix has five linearly independent eigenvalues (two corresponding to $\lambda = 2$ and three corresponding to $\lambda = -4$), and because the largest Jordan block is of size $3 \times 3$, the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

**Case 5:**

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$  

In this case, the matrix has four linearly independent eigenvalues (one corresponding to $\lambda = 2$ and three corresponding to $\lambda = -4$), and because the largest Jordan block is of size $4 \times 4$, the maximum length of a cycle of generalized eigenvectors for this matrix is 4.

**Case 6:**

$$J = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$
In this case, the matrix has six linearly independent eigenvectors (four corresponding to \( \lambda = 2 \) and two corresponding to \( \lambda = -4 \)), and because the largest Jordan block is of size \( 2 \times 2 \), the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

**Case 7:**

\[
J = \begin{bmatrix}
  2 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 2 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 2 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 2 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -4 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & -4 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -4 \\
\end{bmatrix}.
\]

In this case, the matrix has five linearly independent eigenvectors (three corresponding to \( \lambda = 2 \) and two corresponding to \( \lambda = -4 \)), and because the largest Jordan block is of size \( 2 \times 2 \), the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

**Case 8:**

\[
J = \begin{bmatrix}
  2 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 2 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 2 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 2 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -4 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & -4 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -4 \\
\end{bmatrix}.
\]

In this case, the matrix has four linearly independent eigenvectors (two corresponding to \( \lambda = 2 \) and two corresponding to \( \lambda = -4 \)), and because the largest Jordan block is of size \( 2 \times 2 \), the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

**Case 9:**

\[
J = \begin{bmatrix}
  2 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 2 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 2 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 2 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -4 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & -4 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -4 \\
\end{bmatrix}.
\]

In this case, the matrix has four linearly independent eigenvectors (two corresponding to \( \lambda = 2 \) and two corresponding to \( \lambda = -4 \)), and because the largest Jordan block is of size \( 3 \times 3 \), the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

**Case 10:**

\[
J = \begin{bmatrix}
  2 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 2 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 2 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 2 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -4 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & -4 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -4 \\
\end{bmatrix}.
\]
In this case, the matrix has three linearly independent eigenvectors (one corresponding to \( \lambda = 2 \) and two corresponding to \( \lambda = -4 \)), and because the largest Jordan block is of size \( 4 \times 4 \), the maximum length of a cycle of generalized eigenvectors for this matrix is 4.

**Case 11:**

\[
J = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -4 \\
\end{bmatrix}
\]

In this case, the matrix has five linearly independent eigenvectors (four corresponding to \( \lambda = 2 \) and one corresponding to \( \lambda = -4 \)), and because the largest Jordan block is of size \( 3 \times 3 \), the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

**Case 12:**

\[
J = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -4 \\
\end{bmatrix}
\]

In this case, the matrix has four linearly independent eigenvectors (three corresponding to \( \lambda = 2 \) and one corresponding to \( \lambda = -4 \)), and because the largest Jordan block is of size \( 3 \times 3 \), the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

**Case 13:**

\[
J = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -4 \\
\end{bmatrix}
\]

In this case, the matrix has three linearly independent eigenvectors (two corresponding to \( \lambda = 2 \) and one corresponding to \( \lambda = -4 \)), and because the largest Jordan block is of size \( 3 \times 3 \), the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

**Case 14:**

\[
J = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -4 \\
\end{bmatrix}
\]
In this case, the matrix has three linearly independent eigenvectors (two corresponding to \( \lambda = 2 \) and one corresponding to \( \lambda = -4 \)), and because the largest Jordan block is of size \( 3 \times 3 \), the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

**Case 15:**

\[
J = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -4
\end{bmatrix}
\]

In this case, the matrix has two linearly independent eigenvectors (one corresponding to \( \lambda = 2 \) and one corresponding to \( \lambda = -4 \)), and because the largest Jordan block is of size \( 4 \times 4 \), the maximum length of a cycle of generalized eigenvectors for this matrix is 4.

35. **FALSE.** For instance, if \( A = I_2 \) and \( B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \), then \( A^2 = B^2 = I_2 \), but the matrices \( A \) and \( B \) are not similar. (Otherwise, there would exist an invertible matrix \( S \) such that \( S^{-1}AS = B \). But since \( A = I_2 \) this reduces to \( I_2 = B \), which is clearly not the case. Thus, no such invertible matrix \( S \) exists.)

37. **FALSE.** For instance, consider \( T_1 : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( T_1(x) = x \), and consider \( T_2 : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( T_2(x) = -x \). Both \( T_1 \) and \( T_2 \) are linear transformations, and both of them are onto. However, \( (T_1 + T_2)(x) = T_1(x) + T_2(x) = x + (-x) = 0 \), so \( \text{Rng}(T_1 + T_2) = \{0\} \), which implies that \( T_1 + T_2 \) is not onto.

39. Assume that

\[
c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) = 0.
\]

We wish to show that

\[
c_1 = c_2 = \cdots = c_n = 0.
\]

To do this, use the linearity of \( T \) to rewrite the above equation as

\[
T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = 0.
\]

Now, since \( \text{Ker}(T) = \{0\} \), we conclude that

\[
c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0.
\]

Since \( \{v_1, v_2, \ldots, v_n\} \) is a linearly independent set, we conclude that \( c_1 = c_2 = \cdots = c_n = 0 \), as required.

41. We have

\[
A_1v = \lambda_1v
\]

for each \( i = 1, 2, \ldots, k \). Thus,

\[
(A_1A_2 \ldots A_k)v = (A_1A_2 \ldots A_{k-1})(A_kv) = (A_1A_2 \ldots A_{k-1})(\lambda_kv) = \lambda_k(A_1A_2 \ldots A_{k-1})v = \lambda_k(A_1A_2 \ldots A_{k-2})(A_{k-1}v) = \lambda_k(A_1A_2 \ldots A_{k-2})(\lambda_{k-1}v) = \lambda_{k-1}\lambda_k(A_1A_2 \ldots A_{k-2})v = \cdots = \lambda_2\lambda_3 \ldots \lambda_k(A_1v) = \lambda_2\lambda_3 \ldots \lambda_k(\lambda_1v) = (\lambda_1\lambda_2 \ldots \lambda_k)v,
\]

which shows that \( v \) is an eigenvector of \( A_1A_2 \ldots A_k \) with corresponding eigenvalue \( \lambda_1\lambda_2 \ldots \lambda_k \).
Solutions to Section 6.1

True-False Review:

1. TRUE. This is essentially the statement of Theorem 6.1.3.

3. FALSE. Many counterexamples are possible. Note that

\[(xD - Dx)(x) = xD(x) - D(x^2) = x - 2x = (-1)(x).\]

Therefore, \(xD - Dx = -1\). Setting \(L_1 = x\) and \(L_2 = D\), we therefore see that \(L_1L_2 \neq L_2L_1\) in this example.

5. TRUE. By assumption \(L(y_1 + y_2) = L(y_1) + L(y_2)\) and \(L(ky) = kL(y)\) for all scalars \(k\). Therefore, for all constants \(c\), we have

\[cL(y_1 + y_2) = cL(y_1) + cL(y_2) = (cL)(y_1) + (cL)(y_2)\]

and

\[(cL)(ky) = cL(ky) = c(kL(y)) = k(cL(y)).\]

Therefore, \(cL\) is a linear differential operator.

7. TRUE. We have

\[L(y_1 + y_2) = L(y_1) + L(y_2) = F_1 + F_2.\]

Problems:

1.

(a)

\[L(y(x)) = (D - x)(2x - 3e^{2x}) = D(2x - 3e^{2x}) - x(2x - 3e^{2x})\]

\[= (2 - 6e^{2x}) - 2x^2 + 3xe^{2x}\]

\[= 2(1 - x^2) + 3e^{2x}(x - 2).\]

(b)

\[L(y(x)) = (D - x)(3\sin^2 x) = D(3\sin^2 x) - x(3\sin^2 x)\]

\[= 6\sin x \cos x - 3x \sin^2 x\]

\[= 3\sin x(2x \cos x - x \sin x).\]

3.

(a)

\[L(y(x)) = (D^3 - 2xD^2)(2x - 3e^{2x}) = D^3(2x - 3e^{2x}) - 2xD^2(2x - 3e^{2x})\]

\[= D^3(-3e^{2x}) - 2xD^2(-3e^{2x})\]

\[= -24e^{2x} + 24xe^{2x}\]

\[= 24e^{2x}(x - 1).\]

(b)

\[L(y(x)) = (D^3 - 2xD^2)(3\sin^2 x) = D^3(3\sin^2 x) - 2xD^2(3\sin^2 x)\]

\[= D^2(6\sin x \cdot \cos x) - 2x D(6\sin x \cdot \cos x)\]

\[= D(12 \cos^2 x - 6) - 2x(12 \cos^2 x - 6)\]

\[= -24 \sin x \cdot \cos x - 24x \cos^2 x + 12x.\]
5. \[ L(y(x)) = (x^2 D^2 + 2x D - 2)(x^{-2}) = x^2 D^2(x^{-2}) + 2x D(x^{-2}) - 2x^{-2} \]
\[ = x^2 D(-2x^{-3}) - 4x^{-2} - 2x^{-2} \]
\[ = 6x^{-2} - 6x^{-2} = 0. \]

Thus, \( f \in \text{Ker}(L) \).

7. \[ L(y(x)) = (D^3 + D^2 + D + 1)(\sin x + \cos x) \]
\[ = D^3(\sin x + \cos x) + D^2(\sin x + \cos x) + D(\sin x + \cos x) + (\sin x + \cos x) \]
\[ = D^2(\cos x - \sin x) + D(\cos x - \sin x) + \cos x - \sin x + \sin x + \cos x \]
\[ = -\cos x + \sin x - \sin x - \cos x + 2 \cos x \]
\[ = 0. \]

Thus, \( f \in \text{Ker}(L) \).

9. \[ L(y(x)) = 0 \iff (D - 2x)y = 0 \iff y' - 2xy = 0. \] This linear equation has integrating factor \( I = e^{-\int 2x \, dx} = e^{-x^2} \), so that the differential equation can be written as \( \frac{d}{dx}(e^{-x^2} y) = 0 \), which has a general solution \( e^{-x^2} y = c \), that is, \( y(x) = ce^{x^2} \). Consequently,
\[ \text{Ker}(L) = \{ y \in C^1(\mathbb{R}) : y(x) = ce^{x^2}, c \in \mathbb{R} \}. \]

11. \[ L(y(x)) = 0 \iff (D^2 + 2D - 15)y = 0 \iff y'' + 2y' - 15y = 0. \] This is a second-order homogeneous linear differential equation, so by Theorem 6.1.3, the solution set to this equation forms a 2-dimensional vector space. Following the hint given in the text, we try for solutions of the form \( y(x) = e^{rx} \). Substituting this into the differential equation, we get \( r^2 e^{rx} + 2r e^{rx} - 15 e^{rx} = 0 \), or \( e^{rx}(r^2 + 2r - 15) = 0 \). It follows that \( r^2 + 2r - 15 = 0 \). That is \( (r + 5)(r - 3) = 0 \), and hence, \( r = -5 \) and \( r = 3 \) are the solutions. Therefore, we obtain the solutions \( y_1 = e^{-5x} \) and \( y_2 = e^{3x} \). Since they are linearly independent (by computing the Wronskian, for instance), these two functions form a basis for the solution set. Therefore,
\[ \text{Ker}(L) = \{ ae^{-5x} + be^{3x} : a, b \in \mathbb{R} \}. \]

13. We have
\[ (L_1 L_2)(f) = L_1(f' - 2x^2 f) \]
\[ = (D + 1)(f' - 2x^2 f) \]
\[ = f'' + f' - 4x f - 2x^2 f' - 2x^2 f \]
\[ = f'' + (1 - 2x^2) f' - (4x + 2x^2) f, \]
so that
\[ L_1 L_2 = D^2 + (1 - 2x^2)D - 2x(2 + x). \]

Furthermore,
\[ (L_2 L_1)(f) = L_2(f' + f) \]
\[ = (D - 2x^2)(f' + f) \]
\[ = f'' + f' - 2x^2 f' - 2x^2 f, \]
so that

\[ L_2L_1 = D^2 + (1 - 2x^2)D - 2x^2. \]

Therefore, \( L_1L_2 \neq L_2L_1 \).

15. We have

\[
(L_1L_2)(f) = L_1(D + b_1)f \\
= (D + a_1)(f' + b_1f) \\
= f'' + [b_1 + a_1]f' + (b_1' + a_1b_1)f.
\]

Thus

\[ L_1L_2 = D^2 + (b_1 + a_1)D + (b_1' + a_1b_1). \]

Similarly,

\[ L_2L_1 = D^2 + (a_1 + b_1)D + (a_1' + b_1a_1). \]

Thus \( L_1L_2 - L_2L_1 = b_1' - a_1' \), which is the zero operator if and only if \( b_1' = a_1' \) which can be integrated directly to obtain \( b_1 = a_1 + c_2 \), where \( c_2 \) is an arbitrary constant. Consequently we must have \( L_1 = D + [a_1(x) + c_2] \).

17. \( (D^2 + 4xD - 6x^2)y = x^2 \sin x \) and \( y'' + 4xy' - 6x^2y = 0 \).

19. Let \( a_1, ..., a_n \) be functions that are continuous on the interval I. Then, for any \( x_0 \) in I, the initial-value problem \( y^{(n)} + a_1(x)y^{(n-1)} + ... + a_{n-1}(x)y' + a_n(x)y = 0 \), \( y(x_0) = 0 \), \( y'(x_0) = 0 \), ..., \( y^{(n-1)}(x_0) = 0 \), has only the trivial solution \( y(x) = 0 \).

Proof:

All of the conditions of the existence-uniqueness theorem are satisfied and \( y(x) = 0 \) is a solution; consequently, it is the only solution.

21. Given \( y'' + 7y' + 10y = 0 \) then \( r^2 + 7r + 10 = 0 \) \( \implies r \in \{-5, -2\} \implies y(x) = c_1e^{-5x} + c_2e^{-2x} \).

23. Given \( y'' + 4y' = 0 \) then \( r^2 + 4r = 0 \) \( \implies r \in \{-4, 0\} \implies y(x) = c_1e^{-4x} + c_2 \).

25. Substituting \( y(x) = e^{rx} \) into the given differential equation yields \( e^{rx}(r^3 + 3r^2 - 4r - 12) = 0 \), so that we will have a solution provided that \( r \) satisfies \( r^3 + 3r^2 - 4r - 12 = 0 \), that is, \( (r - 2)(r + 2)(r + 3) = 0 \). Consequently, three solutions to the given differential equation are \( y_1(x) = e^{2x}, \ y_2(x) = e^{-2x}, \ y_3(x) = e^{-3x} \).

Further, the Wronskian of these solutions is

\[
W[y_1, y_2, y_3] = \begin{vmatrix} e^{2x} & e^{-2x} & e^{-3x} \\
2e^{2x} & -2e^{-2x} & -3e^{-3x} \\
4e^{2x} & 4e^{-2x} & 9e^{-3x} \end{vmatrix} = -20e^{-3x}.
\]

Since the Wronskian is never zero, the solutions are linearly independent on any interval. Hence the general solution to the differential equation is \( y(x) = c_1e^{2x} + c_2e^{-2x} + c_3e^{-3x} \).

27. Given \( y'' - y'' - 2y' = 0 \) then \( r^3 - r^2 - 2r = 0 \) \( \implies r(r^2 - r - 2) = 0 \) \( \implies r \in \{-1, 0, 2\} \implies y(x) = c_1e^{-x} + c_2 + c_3e^{2x} \).

29. Given \( y^{(iv)} - 2y''' = y'' + 2y' = 0 \) then \( r^4 - 3r^3 - 2r^2 - 2r = 0 \) \( \implies r(r - 2)(r + 1)(r + 3) = 0 \) \( \implies r \in \{-3, 0, 1, 2\} \implies y(x) = c_1e^{-x} + c_2e^{x} + c_3e^{2x} \).

31. Given \( x^2y'' + 3xy' - 8y = 0 \), the trial solution gives \( x^2r(r - 1)x^{r-2} + 3rxr^{r-1} - 8x^r = 0 \), or \( x'[r(r - 1) + 3r - 8] = 0 \). Therefore, \( r^2 + 2r - 8 = 0 \), which factors as \( (r + 4)(r - 2) = 0 \). Therefore, \( r = -4 \)
or \( r = 2 \). Hence, we obtain the solutions \( y_1(x) = x^{-4} \) and \( y_2(x) = x^2 \). Furthermore,

\[
W[x^{-4}, x^2] = (x^{-4})(2x) - (-4x^{-5})(x^2) = 6x^{-3} \neq 0,
\]

so that \( \{x^{-4}, x^2\} \) is a linearly independent set of solutions to the given differential equation on \((0, \infty)\). Consequently, from Theorem 6.1.3, the general solution is given by \( y(x) = c_1 x^{-4} + c_2 x^2 \).

33. Substituting \( y(x) = x^r \) into the given differential equation yields \( x^r[r(r-1)(r-2)+r(r-1)-2r+2] = 0 \), so that \( r \) must satisfy \( (r-1)(r-2)(r+1) = 0 \). If follows that three solutions to the differential equation are \( y_1(x) = x, \ y_2(x) = x^2, \ y_3(x) = x^{-1} \). Further, the Wronskian of these solutions is

\[
W[y_1, y_2, y_3] = \begin{vmatrix}
x & x^2 & x^{-1} \\
1 & 2x & -x^{-2} \\
0 & 2 & 2x^{-3}
\end{vmatrix} = 6x^{-1}.
\]

Since the Wronskian is nonzero on \((0, \infty)\), the solutions are linearly independent on this interval. Consequently, the general solution to the differential equation is \( y(x) = c_1 x + c_2 x^2 + c_3 x^{-1} \).

35. To determine a particular solution of the form \( y_p(x) = A_0 e^{3x} \), we substitute this solution into the differential equation:

\[
(A_0 e^{3x})'' + (A_0 e^{3x})' - 6(A_0 e^{3x}) = 18 e^{3x}.
\]

Therefore

\[
9A_0 e^{3x} + 3A_0 e^{3x} - 6A_0 e^{3x} = 18 e^{3x},
\]

which forces \( A_0 = 3 \). Therefore, \( y_p(x) = 3e^{3x} \).

To obtain the general solution, we need to find the complementary function \( y_c(x) \), the solution to the associated homogeneous differential equation: \( y'' + y' - 6y = 0 \). Seeking solutions of the form \( y(x) = e^{rx} \), we obtain \( r^2 e^{rx} + re^{rx} - 6 e^{rx} = 0 \), or \( e^{rx}(r^2 + r - 6) = 0 \). Therefore, \( r^2 + r - 6 = 0 \), which factors as \( (r+3)(r-2) = 0 \). Hence, \( r = -3 \) or \( r = 2 \). Therefore, we obtain the solutions \( y_1(x) = e^{-3x} \) and \( y_2(x) = e^{2x} \). Since

\[
W[e^{-3x}, e^{2x}] = (e^{-3x})(2e^{2x}) - (-3e^{-3x})(e^{2x}) = 2e^{-x} + 3e^{-x} = 5e^{-x} \neq 0,
\]

\( \{e^{-3x}, e^{2x}\} \) is linearly independent. Therefore, the complementary function is \( y_c(x) = c_1 e^{-3x} + c_2 e^{2x} \). By Theorem 6.1.7, the general solution to the differential equation is

\[
y(x) = c_1 e^{-3x} + c_2 e^{2x} + 3x.
\]

37. Substituting \( y(x) = A_0 e^{3x} \) into the given differential equation yields \( A_0 e^{3x}(27 + 18 - 2) = 4e^{3x} \), so that \( A_0 = \frac{1}{10} e^{3x} \). Hence, a particular solution to the differential equation is \( y_p(x) = \frac{1}{10} e^{3x} \). To determine the general solution we need to solve the associated homogeneous differential equation \( y'' + 2y' - 2y = 0 \). We try for solutions of the form \( y(x) = e^{rx} \). Substituting into the homogeneous differential equation gives \( e^{rx}(r^3 + 2r^2 - r - 2) = 0 \), so that we choose \( r \) to satisfy \( r^3 + 2r^2 - r - 2 = 0 \), or equivalently, \( (r+2)(r-1) = 0 \). It follows that three solutions to the differential equation are \( y_1(x) = e^{-2x}, \ y_2(x) = e^x, \ y_3(x) = e^{-x} \). Further, the Wronskian of these solutions is

\[
W[y_1, y_2, y_3] = \begin{vmatrix}
e^{-2x} & e^x & e^{-x} \\
-2e^{-2x} & e^x & -e^{-x} \\
4e^{-2x} & e^x & e^{-x}
\end{vmatrix} = -6e^{-2x}.
\]
By Theorem 6.1.7, the general solution to the differential equation is

39. Substituting \( y(x) = A_0e^{4x} \) into the differential equation yields

\[
64A_0e^{4x} + 80A_0e^{4x} + 24A_0e^{4x} = -3e^{4x}.
\]

Therefore, \( 168A_0 = -3 \). Hence, \( A_0 = -\frac{1}{56} \). Thus, \( y_p(x) = -\frac{1}{56}e^{4x} \).

To obtain the general solution, we need to find the complementary function \( y_c(x) \), the solution to the associated homogeneous differential equation: \( y'''' + 5y''' + 6y'' = 0 \). Seeking solutions of the form \( y(x) = e^{rx} \), we obtain \( r^4e^{rx} + 5r^3e^{rx} + 6re^{rx} = 0 \), or \( e^{rx}(r^4 + 5r^3 + 6r) = 0 \). Therefore, \( r^4 + 5r^3 + 6r = 0 \), which has roots \( r = 0, r = -2, \) and \( r = -3 \). Therefore, we obtain the solutions \( y_1(x) = 1, y_2(x) = e^{-2x}, \) and \( y_3(x) = e^{-3x} \). Since

\[
W[1, e^{-2x}, e^{-3x}] = \begin{vmatrix} 1 & e^{-2x} & e^{-3x} \\ 0 & -2e^{-2x} & -3e^{-3x} \\ 0 & 4e^{-2x} & 9e^{-3x} \end{vmatrix} = -6e^{-5x} \neq 0,
\]

\( \{1, e^{-2x}, e^{-3x}\} \) is linearly independent. Therefore, the complementary function is \( y_c(x) = c_1 + c_2e^{-2x} + c_3e^{-3x} \).

By Theorem 6.1.7, the general solution to the differential equation is

\[
y(x) = y_1 + y_2 + y_3 = c_1 + c_2e^{-2x} + c_3e^{-3x} - \frac{1}{56}e^{4x}.
\]

41. Consider the linear system

\[
c_1y_1(x_0) + c_2y_2(x_0) + \ldots + c_ny_n(x_0) = 0
\]

\[
c_1y_1'(x_0) + c_2y_2'(x_0) + \ldots + c_ny_n'(x_0) = 0
\]

\[
\vdots
\]

\[
c_1y_1^{(n-1)}(x_0) + c_2y_2^{(n-1)}(x_0) + \ldots + c_ny_n^{(n-1)}(x_0) = 0,
\]

where we are solving for \( c_1, c_2, \ldots, c_n \). The determinant of the matrix of coefficients of this system is \( W[y_1, y_2, \ldots, y_n](x_0) = 0 \), so that the system has non-trivial solutions. Let \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \) be one such non-trivial solution for \( (c_1, c_2, \ldots, c_n) \). Therefore, not all of the \( \alpha_i \) are zero. Define the function \( u(x) \) by

\[
u(x) = \alpha_1y_1(x) + \alpha_2y_2(x) + \ldots + \alpha_ny_n(x).
\]

It follows that \( y = u(x) \) satisfies the initial-value problem:

\[
an_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_ny = 0
\]

\[
y(x_0) = 0, \quad y'(x_0) = 0, \ldots, y^{(n-1)}(x_0) = 0.
\]

However, \( y(x) = 0 \) also satisfies the above initial value problem and hence, by uniqueness, we must have \( u(x) = 0 \), that is, \( \alpha_1y_1(x) + \alpha_2y_2(x) + \ldots + \alpha_ny_n(x) = 0 \), where not all of the \( \alpha_i \) are zero. Thus, the functions \( y_1, y_2, \ldots, y_n \) are linearly dependent on \( I \).

43. Let \( y_1 \) and \( y_2 \) belong to \( C^n(I) \), and let \( c \) be a scalar. We must show that

\[
L(y_1 + y_2) = L(y_1) + L(y_2) \quad \text{and} \quad L(cy_1) = cL(y_1).
\]
We have
\[ L(y_1 + y_2) = (D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)(y_1 + y_2) \]
\[ = D^n(y_1 + y_2) + a_1 D^{n-1}(y_1 + y_2) + \cdots + a_{n-1} D(y_1 + y_2) + a_n(y_1 + y_2) \]
\[ = (D^n y_1 + a_1 D^{n-1} y_1 + \cdots + a_{n-1} D y_1 + a_n y_1) + (D^n y_2 + a_1 D^{n-1} y_2 + \cdots + a_{n-1} D y_2 + a_n y_2) \]
\[ = (D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)(y_1) + (D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)(y_2) \]
\[ = L(y_1) + L(y_2), \]

and
\[ L(c y_1) = (D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)(c y_1) \]
\[ = D^n(c y_1) + a_1 D^{n-1}(c y_1) + \cdots + a_{n-1} D(c y_1) + a_n(c y_1) \]
\[ = c D^n(y_1) + c a_1 D^{n-1}(y_1) + \cdots + c a_{n-1} D(y_1) + c a_n(y_1) \]
\[ = c(D^n(y_1) + a_1 D^{n-1}(y_1) + \cdots + a_{n-1} D(y_1) + a_n(y_1)) \]
\[ = c(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)(y_1) \]
\[ = cL(y_1). \]

**Solutions to Section 6.2**

1. **FALSE.** Even if the auxiliary polynomial fails to have \( n \) distinct roots, the differential equation still has \( n \) linearly independent solutions. For example, if \( L = D^2 + 2D + 1 \), then the differential equation \( Ly = 0 \) has auxiliary polynomial with (repeated) roots \( r = -1, -1 \). Yet we do have two linearly independent solutions \( y_1(x) = e^{-x} \) and \( y_2(x) = xe^{-x} \) to the differential equation.

3. **TRUE.** This is really just the statement that a polynomial of degree \( n \) always has \( n \) roots, with multiplicities counted.

5. **FALSE.** Note that \( r = 0 \) is a root of the auxiliary polynomial, but only of multiplicity 1. The expression \( c_1 + c_2 x \) in the solution reflects \( r = 0 \) as a root of multiplicity 2.

7. **TRUE.** The roots of the auxiliary polynomial are \( r = 2 \pm i, 2 \pm i \). The terms corresponding to the first pair \( 2 \pm i \) are \( c_1 e^{2x} \cos x \) and \( c_2 e^{2x} \sin x \), and the repeated root gives two more terms: \( c_3 x e^{2x} \cos x \) and \( c_4 x e^{2x} \sin x \).

**Problems:**

1. The auxiliary polynomial is \( P(r) = r^2 + 2r - 3 = (r + 3)(r - 1) \). Therefore, the auxiliary equation has roots \( r = -3 \) and \( r = 1 \). Therefore, two linearly independent solutions to the given differential equation are
\[ y_1(x) = e^{-3x} \quad \text{and} \quad y_2(x) = e^x. \]

By Theorem 6.1.3, the solution space to this differential equation is 2-dimensional, and hence \( \{e^{-3x}, e^x\} \) forms a basis for the solution space.

3. The auxiliary polynomial is \( P(r) = r^2 - 6r + 25 \). According to the quadratic equation, the auxiliary equation has roots \( r = 3 \pm 4i \). Therefore, two linearly independent solutions to the given differential equation are
\[ y_1(x) = e^{3x} \cos 4x \quad \text{and} \quad y_2(x) = e^{3x} \sin 4x. \]
By Theorem 6.1.3, the solution space to this differential equation is 2-dimensional, and hence \( \{e^{3x} \cos 4x, e^{3x} \sin 4x\} \) forms a basis for the solution space.

5. We have \( r^2 - r - 2 = 0 \implies r \in \{-1, 2\} \implies y(x) = c_1 e^{-x} + c_2 e^{2x}. \)

7. We have \( r^2 + 6r + 25 = 0 \implies r \in \{-3 - 4i, -3 + 4i\} \implies y(x) = c_1 e^{-3x} \cos 4x + c_2 e^{-3x} \sin 4x. \)

9. We have \( (r + 2)^2 = 0 \implies r \in \{-2, -2\} \implies y(x) = c_1 e^{-2x} + c_2xe^{-2x}. \)

11. We have \( r^2 + 10r + 25 = 0 \implies r \in \{-5, -5\} \implies y(x) = c_1 e^{-5x} + c_2 e^{-5x}. \)

13. We have \( r^2 + 8r + 20 = 0 \implies r \in \{-4 - 2i, -4 + 2i\} \implies y(x) = c_1 e^{-4r} \cos 2x + c_2 e^{-4x} \sin 2x. \)

15. We have \( (r - 4)(r + 2) = 0 \implies r \in \{-2, 4\} \implies y(x) = c_1 e^{-2x} + c_2 e^{4x}. \)

17. We have \( r^3 - r^2 + r - 1 = 0 \implies r \in \{1, i, -i\} \implies y(x) = c_1 e^x + c_2 \cos x + c_3 \sin x. \)

19. We have \( (r - 2)(r + 2) = 0 \implies r \in \{2, 4, -4\} \implies y(x) = c_1 e^{2x} + c_2 e^{4x} + c_3 e^{-4x}. \)

21. We have \( (r^2 + 4)^2(r + 1) = 0 \implies r \in \{2i, 2i, -2i, -2i, -1\} \implies y(x) = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x + x(c_4 \cos 2x + c_5 \sin 2x). \)

23. We have \( r^2(r - 1) = 0 \implies r \in \{0, 0, 1\} \implies y(x) = c_1 + c_2x + c_3e^x. \)

25. We have \( r^4 - 16 = 0 \implies r \in \{2, -2, 2i, -2i\} \implies y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x. \)

27. We have \( r^4 - 16r^2 + 40r - 25 = 0 \implies r \in \{1, -5, 2 + i, 2 - i\} \implies y(x) = c_1 e^x + c_2 e^{-5x} + e^{2x}(c_3 \cos x + c_4 \sin x). \)

29. We have \( (r^2 - 2r + 2)^2(r^2 - 1) = 0 \implies r \in \{1 + i, 1 - i, 1 + i, 1 - i, 1, -1\} \implies y(x) = e^x(c_1 \cos x + c_2 \sin x) + xe^x(c_3 \cos x + c_4 \sin x) + c_5 e^{-x} + c_6 e^x. \)

31. We have \( (r^2 + 9)^3 = 0 \implies r \in \{3i, 3i, 3i, -3i, -3i, -3i\} \implies y(x) = c_1 \cos 3x + c_2 \sin 3x + x(c_3 \cos 3x + c_4 \sin 3x) + x^2(c_5 \cos 3x + c_6 \sin 3x). \)

33. We have \( r^2 - 4r + 5 = 0 \implies r \in \{2 + i, 2 - i\} \implies y(x) = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x. \) Now \( 3 = y(0) = c_1 \) and \( 5 = y'(0) = 2c_1 + c_2. \) Therefore, \( c_1 = 3 \) and \( c_2 = -1. \) Hence, the solution to this initial-value problem is \( y(x) = 3e^{2x} \cos x - e^{2x} \sin x. \)

35. We have \( r^3 + 2r^2 - 4r - 8 = (r - 2)(r + 2)^2 = 0 \implies r \in \{2, -2, -2\} \implies y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 xe^{-2x}. \) Then since \( y(0) = 0, \) we have \( c_1 + c_2 = 0. \) Moreover, since \( y'(0) = 6, \) we have \( 2c_1 - 2c_2 + c_3 = 6. \) Further, \( y''(0) = 8 \) implies that \( 4c_1 + 4c_2 - 4c_3 = 8. \) Solving these equations yields \( c_1 = 2, \) \( c_2 = c_3 = -2. \) Hence, the solution to this initial-value problem is \( y(x) = 2(e^{2x} - e^{-2x} - xe^{-2x}). \)

37. Given \( m > 0 \) and \( k > 0, \) we have auxiliary polynomial \( P(r) = r^2 - 2mr + (m^2 - k^2) = 0 \implies r = m \pm k. \) Therefore, the general solution to this differential equation is \( y(x) = ae^{(m+k)x} + be^{(m-k)x} = e^{mx}(ae^{kx} + be^{-kx}). \)
Letting $a = \frac{c_1 + c_2}{2}$ and $b = \frac{c_1 - c_2}{2}$ in the last equality gives

$$y(x) = e^{mx} \left( \frac{c_1 + c_2}{2} e^{kx} + \frac{c_1 - c_2}{2} e^{-kx} \right)$$

$$= e^{mx} \left( \frac{e^{kx} + e^{-kx}}{2} + \frac{e^{kx} - e^{-kx}}{2} \right)$$

$$= e^{mx} (c_1 \cosh kx + c_2 \sinh kx).$$

39. Let $u(x, y) = e^{x/\alpha} f(\xi)$, where $\xi = \beta x - \alpha y$, and assume that $\alpha > 0$ and $\beta > 0$.

(a) Let us compute the partial derivatives of $u$:

$$\frac{\partial u}{\partial x} = e^{x/\alpha} \frac{df}{dx} + \frac{1}{\alpha} e^{x/\alpha} f$$

$$= e^{x/\alpha} \frac{df}{d\xi} \frac{\partial \xi}{\partial x} + \frac{1}{\alpha} e^{x/\alpha} f$$

$$= e^{x/\alpha} \frac{df}{d\xi} \beta + \frac{1}{\alpha} e^{x/\alpha} f$$

and

$$\frac{\partial^2 u}{\partial x^2} = e^{x/\alpha} \frac{d^2 f}{dx^2} \beta + \frac{\beta}{\alpha} e^{x/\alpha} \frac{df}{d\xi} \frac{\partial \xi}{\partial x} + \frac{1}{\alpha} \frac{e^{x/\alpha}}{\alpha^2} f$$

Moreover,

$$\frac{\partial u}{\partial y} = e^{x/\alpha} \frac{df}{dy}$$

$$= e^{x/\alpha} \frac{df}{d\xi} \frac{\partial \xi}{\partial y}$$

$$= e^{x/\alpha} \frac{df}{d\xi} (-\alpha)$$

$$= -\alpha e^{x/\alpha} \frac{df}{d\xi}$$

and

$$\frac{\partial^2 u}{\partial y^2} = -\alpha e^{x/\alpha} \frac{\partial \xi}{\partial y} \left( \frac{df}{d\xi} \right)$$

$$= -\alpha e^{x/\alpha} \frac{d^2 f}{d\xi^2} \frac{\partial \xi}{\partial y}$$

$$= -\alpha e^{x/\alpha} \frac{d^2 f}{d\xi^2} (-\alpha)$$

$$= \alpha^2 e^{x/\alpha} \frac{d^2 f}{d\xi^2}.$$

From the formulas for $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{x/\alpha} \left( \frac{\beta^2 d^2 f}{d\xi^2} \alpha + \frac{\beta}{\alpha} \frac{df}{d\xi} \alpha \frac{f}{\alpha} + \alpha^2 \frac{d^2 f}{d\xi^2} \right).$$
Now if \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \), then the last equation becomes

\[
\beta^2 \frac{d^2 f}{d\xi^2} + 2\frac{\beta}{\alpha} \frac{df}{d\xi} + \frac{f}{\alpha^2} + \alpha^2 \frac{d^2 f}{d\xi^2} = 0
\]

or

\[
(\alpha^2 + \beta^2) \frac{d^2 f}{d\xi^2} + 2\frac{\beta}{\alpha} \frac{df}{d\xi} + \frac{f}{\alpha^2} = 0.
\]

Therefore,

\[
\frac{d^2 f}{d\xi^2} + \frac{2\beta}{\alpha(\alpha^2 + \beta^2)} \frac{df}{d\xi} + \frac{f}{\alpha^2(\alpha^2 + \beta^2)} = 0.
\]

Letting \( p = \frac{\beta}{\alpha(\alpha^2 + \beta^2)} \) and \( q = \frac{1}{\alpha^2 + \beta^2} \), the last equation reduces to

\[
\frac{d^2 f}{d\xi^2} + 2p \frac{df}{d\xi} + q \frac{f}{\alpha^2} = 0.
\]

(b) The auxiliary equation associated with Equation (6.2.10) is \( r^2 + 2pr + \frac{q}{\alpha^2} = 0 \). The quadratic formula yields the roots

\[
r = -p \pm \sqrt{p^2 - \frac{q}{\alpha^2}}.
\]

Therefore,

\[
r = -\frac{\beta}{\alpha(\alpha^2 + \beta^2)} \pm \left[ \frac{\beta^2}{\alpha^2(\alpha^2 + \beta^2)^2} - \frac{\alpha^2 + \beta^2}{\alpha^2(\alpha^2 + \beta^2)^2} \right]^{1/2}.
\]

Hence,

\[
r = -\frac{\beta \pm i\alpha}{\alpha(\alpha^2 + \beta^2)} = -p \pm iq.
\]

Therefore,

\[
f(\xi) = e^{-p\xi} \left[ A \sin q\xi + B \cos q\xi \right].
\]

Since \( u(x, y) = e^{x/\alpha} f(\xi) \), we conclude that

\[
u(x, y) = e^{x/\alpha} \left[ e^{-p\xi} (A \sin q\xi + B \cos q\xi) \right] = e^{x/\alpha - p\xi} (A \sin q\xi + B \cos q\xi).
\]

41. All the roots must have a negative real part.

45. \( y(x) = c_1 e^{-5x} + c_2 e^{-7x} + c_3 e^{19x} \).

47. \( y(x) = e^{-5x/2}[c_1 \cos (\sqrt{11}x/2) + c_2 \sin (\sqrt{11}x/2)] + e^{-3x/2}[c_3 \cos (\sqrt{7}x/2) + c_4 \sin (\sqrt{7}x/2)]. \)

Solutions to Section 6.3

True-False Review:

1. **FALSE.** Under the given assumptions, we have \( A_1(D)F_1(x) = 0 \) and \( A_2(D)F_2(x) = 0 \). However, this means that

\[
(A_1(D)+A_2(D))(F_1(x)+F_2(x)) = A_1(D)F_1(x)+A_1(D)F_2(x)+A_2(D)F_1(x)+A_2(D)F_2(x) = A_1(D)F_2(x)+A_2(D)F_1(x),
\]
which is not necessarily zero. As a specific example, if \( A_1(D) = D - 1 \) and \( A_2(D) = D - 2 \), then \( A_1(D) \) annihilates \( F_1(x) = e^x \) and \( A_2(D) \) annihilates \( F_2(x) = e^{2x} \). However, \( A_1(D) + A_2(D) = 2D - 3 \) does not annihilate \( e^x + e^{2x} \).

3. TRUE. We apply rule 1 in this section with \( k = 1 \), or we can compute directly that \((D - a)^2 \) annihilates \( xe^{ax} \).

5. FALSE. For instance, if \( F(x) = x \), \( A_1(D) = A_2(D) = D \), then although \( A_1(D)A_2(D)F(x) = 0 \), neither \( A_1(D) \) nor \( A_2(D) \) annihilates \( F(x) = x \).

7. FALSE. The annihilator of \( F(x) = x^4 \) is \( D^5 \), but since \( r = 0 \) already occurs three times as a root of the auxiliary equation, the appropriate trial solution here is \( y_p(x) = A_0x^3 + A_1x^4 + A_2x^5 + A_3x^6 + A_4x^7 \).

Problems:

Note: In Problems 1-15, we use the four boxed formulas on pages 473-474 of the text.

1. \( A(D) = D^2(D - 1); \quad (D - 1)(2e^x) = 0 \) and \( D^2(3x) = 0 \implies D^2(D - 1)(2e^x - 3x) = 0 \).

3. \( A(D) = (D - 7)^4(D^2 + 16); \quad (D - 7)^4(x^3e^{7x}) = 0 \) and \( (D^2 + 16)(5\sin 4x) = 0 \implies (D - 7)^4(D^2 + 16)(x^3e^{7x} + 5\sin 4x) = 0 \).

5. \( A(D) = (D^2 - 2D + 5)(D^2 + 4); \quad \text{In the expression } e^x \sin 2x, \quad a = 1 \) and \( b = 2 \), so that \( D^2 - 2aD + (a^2 + b^2) = D^2 - 2D + 5 \implies (D^2 - 2D + 5)(e^x \sin 2x) = 0 \). Moreover, in the expression \( 3x \cos 2x, \quad a = 0 \) and \( b = 2 \), so that \( D^2 - 2aD + (a^2 + b^2) = D^2 + 4 \implies (D^2 + 4)(3 \cos 2x) = 0 \). Thus we conclude that \( (D^2 - 2D + 5)(D^2 + 4)(e^x \sin 2x + 3 \cos 2x) = 0 \).

7. \( A(D) = (D^2 - 10D + 26)^3; \quad \text{In the expression } e^{5x} \cos x, \quad a = 5 \) and \( b = 1 \) so that \( D^2 - 2aD + (a^2 + b^2) = D^2 - 10D + 26 \implies (D^2 - 10D + 26)(e^{5x}(2 - x) \cos x) = 0 \).

9. \( A(D) = (D - 4)^2(D^2 - 8D + 41)D^2(D^2 + 4D + 5)^3; \quad \text{Note that } (D - 4)^2(xe^{4x}) = 0, \quad (D^2 - 8D + 41)(-2e^{4x} \sin 5x) = 0, \quad D^2(3x) = 0, \quad \text{an } (D^2 + 4D + 5)^3(x^2e^{-2x} \cos x) = 0 \). Therefore,

\[
(D - 4)^2(D^2 - 8D + 41)D^2(D^2 + 4D + 5)^3(e^{4x}(x - 2 \sin 5x) + 3x - x^2e^{-2x} \cos x) = 0.
\]

11. \( A(D) = (D^2 + 9)^2; \quad \text{Here we have applied Rule 3 (page 474) with } a = 0, \quad b = 3, \quad \text{and } k = 1. \)

13. First we simplify

\[
\sin^4 x = (\sin^2 x)^2 = \left( \frac{1 - \cos 2x}{2} \right)^2 = \frac{1}{4} \left[ 1 - 2\cos 2x + \frac{1 + \cos 4x}{2} \right] = \frac{3 - 4\cos 2x + \cos 4x}{8}.
\]

We use \( A_1(D) = D \) to annihilate \( \frac{3}{8} \), \( A_2(D) = D^2 + 4 \) to annihilate \( \frac{1}{2} \cos 2x \), and \( A_3(D) = D^2 + 16 \) to annihilate \( \frac{1}{8} \cos 4x \). Hence, \( A(D) = A_1(D)A_2(D)A_3(D) = D(D^2 + 4)(D^2 + 16) = D^5 + 20D^3 + 64D \).

15. To find the annihilator of \( F(x) \), let us use the identities

\[
\cos^2 x \equiv \frac{1 + \cos 2x}{2} \quad \text{and} \quad \sin^2 x \equiv \frac{1 - \cos 2x}{2}.
\]
to rewrite the formula for $F(x)$:

$$F(x) = \sin^2 x \cos^2 x \cos^2 2x$$

$$= \frac{1 - \cos 2x}{2} \cdot \frac{1 + \cos 2x}{2} \cdot \frac{1 + \cos 4x}{2}$$

$$= \frac{1}{8} \left[(1 - \cos^2 2x)(1 + \cos 4x)\right]$$

$$= \frac{1}{8} \sin^2 2x(1 + \cos 4x)$$

$$= \frac{1}{16}(1 - \cos 4x)(1 + \cos 4x)$$

$$= \frac{1}{16}(1 - \cos^2 4x)$$

$$= \frac{1}{16} \sin^2 4x = \frac{1}{32}(1 - \cos 8x) = \frac{1}{32} - \frac{1}{32} \cos 8x.$$ 

The annihilator for $\frac{1}{32}$ is $A_1(D) = D$, and the annihilator for $\frac{1}{32} \cos 8x$ is $A_2(D) = D^2 + 64$. Therefore, the annihilator for $F(x)$ is $A(D) = A_1(D)A_2(D) = D(D^2 + 64) = D^3 + 64D$.

17. We have $P(r) = r^2 + 4r + 4 \implies y_c(x) = c_1e^{-2x} + c_2xe^{-2x}$. Now, $A(D) = (D+2)^2$. Operating on the given differential equation with $A(D)$ yields the homogeneous differential equation $(D+2)^2 y = 0$, with solution $y(x) = c_1e^{-2x} + c_2xe^{-2x} + A_0xe^{-2x} + A_1x^3e^{-2x}$. Therefore, $y_p(x) = A_0xe^{-2x} + A_1x^3e^{-2x}$. Differentiating $y_p$ yields $y_p' = A_0e^{-2x}(-2x^2 + 2x) + A_1e^{-2x}(-2x^3 + 3x^2)$, $y_p'' = A_0e^{-2x}(4x^2 - 8x + 2) + A_1e^{-2x}(4x^3 - 12x^2 + 6x)$. Substituting into the given differential equation and simplifying yields $2A_0 + 6A_1x = 5x$, so that $A_0 = 0$, $A_1 = \frac{5}{6}$. Hence, $y_p(x) = \frac{5}{6}x^3e^{-2x}$, and therefore $y(x) = c_1e^{-2x} + x_2xe^{-2x} + \frac{5}{6}x^3e^{-2x}$.

19. We have $P(r) = (r - 2)(r + 1) \implies y_c(x) = c_1e^{2x} + c_2e^{-x}$. Now, $A(D) = D - 2$. Operating on the given differential equation with $A(D)$ yields $(D-2)^2(D+1) = 0$ with general solution $y(x) = c_1e^{2x} + c_2e^{-x} + A_0xe^{2x}$. We therefore choose $y_p(x) = A_0xe^{2x}$. Differentiating $y_p$ yields $y_p' = A_0e^{2x}(2x + 1)$, $y_p'' = A_0e^{2x}(4x + 4)$. Substituting into the given differential equation and simplifying we find $A_0 = \frac{5}{3}$. Hence, $y_p(x) = \frac{5}{3}x^2e^{2x}$, and so $y(x) = c_1e^{2x} + c_2e^{-x} + \frac{5}{3}x^2e^{2x}$.

21. We have $P(r) = (r - 2)(r - 3) \implies y_c(x) = c_1e^{2x} + c_2e^{3x}$. Moreover, the annihilator of $F(x) = 7e^{2x}$ is $A(D) = D - 2$. Operating on the differential equation with $A(D)$ gives

$$(D-2)^2(D-3)y = 0,$$

which has solution

$$y(x) = c_1e^{2x} + c_2e^{3x} + A_0xe^{2x}.$$ 

Therefore, our trial solution is $y_p(x) = A_0xe^{2x}$. We must solve for $A_0$. Substituting the expression for $y_p(x)$ into the differential equation and simplification yields $A_0 = -7$, so $y_p(x) = -7xe^{2x}$. Hence, the general solution to the given differential equation is $y(x) = c_1e^{2x} + c_2e^{3x} - 7xe^{2x}$.

23. We have $P(r) = r^2 + 6 \implies y_c(x) = c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x$. Next we must determine the annihilator of $F(x) = \sin^2 x \cos^2 x = \frac{1}{4}(1 - \cos 2x)(1 + \cos 2x) = \frac{1}{4}(1 - \cos^2 2x) = \frac{1}{8} \sin^2 2x = \frac{1}{8}(1 - \cos 4x) =$
\[ \frac{1}{8} - \frac{1}{8} \cos 4x, \text{ which is } A(D) = D(D^2 + 16). \] Operating on the given differential equation with \( A(D) \) yields \( D(D^2+16)(D^2+6)y = 0 \), which has general solution \( y(x) = c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x + A_0 + B_0 \cos 4x + C_0 \sin 4x \). We therefore choose \( y_p(x) = A_0 + B_0 \cos 4x + C_0 \sin 4x \). Substitution into the given differential equation and simplification yields \(-10B_0 \cos 4x - 10C_0 \sin 4x + 6A_0 = \frac{1}{8} - \frac{1}{8} \cos 4x \). Therefore, \( A_0 = \frac{1}{48}, B_0 = \frac{1}{80} \), and \( C_0 = 0 \). Therefore, \( y_p(x) = \frac{1}{48} + \frac{1}{80} \cos 4x \). Thus, the general solution to the given differential equation is

\[ y(x) = c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x + \frac{1}{48} + \frac{1}{80} \cos 4x. \]

25. We have \( P(r) = r^3 + 2r^2 - 5r - 6 = (r-2)(r+3)(r+1) \implies y_c(x) = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{-x} \). Now, \( A(D) = D^3 \). Operating on the given differential equation with \( A(D) \) yields \( D^3(D-2)(D+3)(D+1)y = 0 \), with general solution \( y(x) = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{-x} + A_0 + A_1 x + A_2 x^2 \). We therefore choose \( y_p(x) = A_0 + A_1 x + A_2 x^2 \). Substituting into the given differential equation and simplifying yields

\[ 4A_2 - 5(A_1 + 2A_2 x) - 6(A_0 + A_1 x + A_2 x^2) = 4x^2, \]

so that \( 4A_2 - 5A_1 - 6A_0 = 0, -10A_2 - 6A_1 = 0, -6A_2 = 4 \). This system has a solution \( A_0 = -\frac{37}{27}, A_1 = \frac{10}{9}, A_2 = -\frac{2}{3} \). Hence, \( y_p(x) = -\frac{37}{27} + \frac{10}{9} x - \frac{2}{3} x^2 \), and so \( y(x) = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{-x} - \frac{37}{27} + \frac{10}{9} x - \frac{2}{3} x^2 \).

27. We have \( P(r) = (r+1)^3 \implies y_c(x) = c_1 e^{-x} + c_2 xe^{-x} + c_3 x^2 e^{-x} \). Now, \( A(D) = (D-2)(D+1)^3 \). Operating on the given differential equation with \( A(D) \) yields \( (D-2)(D+1)^3y = 0 \), with general solution \( y(x) = c_1 e^{-x} + c_2 xe^{-x} + c_3 x^2 e^{-x} + A_0 x^2 e^{-x} + A_1 e^{2x} \). We therefore choose \( y_p(x) = A_0 x^2 e^{-x} + A_1 e^{2x} \). Differentiating \( y_p \) yields

\[ y_p' = A_0 e^{-x} (-x^3 + 3x^2) + 2A_1 e^{2x}, \quad y_p'' = A_0 e^{-x} (-x^3 - 6x^2 + 6x) + 4A_1 e^{2x}, \quad y_p''' = A_0 e^{-x} (-x^3 + 9x^2 - 18x + 6) + 8A_1 e^{2x}. \]

Substituting into the given differential equation and simplifying we find \( 6A_0 e^{-x} + 27A_1 e^{2x} = 2e^{-x} + 3e^{2x} \), so that \( A_0 = \frac{1}{3}, A_1 = \frac{1}{9} \). Hence, \( y_p(x) = \frac{1}{3} x^3 e^{-x} + \frac{1}{9} e^{2x} \), and therefore, the general solution to the differential equation is \( y(x) = c_1 e^{-x} + c_2 xe^{-x} + c_3 x^2 e^{-x} + \frac{1}{3} x^3 e^{-x} + \frac{1}{9} e^{2x} \).

29. We have \( P(r) = r^3 - 2r^2 + 2 = (r-1)(r+1)(r-2) \implies y_c(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} \). We take a trial solution of the form \( y_p(x) = A_0 e^{3x} \). Substitution into the differential equation yields \( A_0 (27-18+3-2) = 4 \), so that \( y_p(x) = \frac{1}{2} e^{3x} \). Consequently, \( y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + \frac{1}{2} e^{3x} \).

31. We have \( P(r) = (r-2)(r+3)(r+2) \implies y_c(x) = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{-2x} \). An appropriate trial solution is \( y_p(x) = A_0 \cos x + B_0 \sin x \). Substitution into the given differential equation and simplification leads to \((5A_0 - 15B_0) \sin x - (15A_0 + 5B_0) \cos x = 4 \cos x \). Consequently, \( A_0 \) and \( B_0 \) must satisfy: \( A_0 - 3B_0 = 0, \quad 15A_0 + 5B_0 = -4 \). This system has a solution \( A_0 = -\frac{6}{25}, B_0 = -\frac{2}{25} \). Hence, the general solution to the differential equation is \( y(x) = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{-2x} - \frac{6}{25} \cos x - \frac{2}{25} \sin x \).
33. We have \( P(r) = (r+1)^3 \implies y_e(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} \). We must use the modified trial solution \( y_p(x) = A_0 x^3 e^{-x} \). Differentiating this trial solution gives

\[
\begin{align*}
y_p' &= A_0 e^{-x} (-x^3 + 3x^2), \\
y_p'' &= A_0 e^{-x} (x^3 - 6x^2 + 6x), \\
y_p''' &= A_0 e^{-x} (-x^3 + 9x^2 - 18x + 6).
\end{align*}
\]

The given differential equation is \( y''' + 3y'' + 3y' + y = 0 \). Substituting the trial solution into this differential equation and simplifying yields \( A_0 = \frac{5}{6} \). Hence, \( y_p(x) = \frac{5}{6} x^3 e^{-x} \), and therefore the general solution to the differential equation is \( y(x) = e^{-x}(c_1 + c_2 x + c_3 x^2 + \frac{5}{6} x^3) \).

35. We have \( P(r) = (r^2 + 4r + 13)^2 \implies y_e(x) = e^{-2x} \left[ c_1 \cos 3x + c_2 \sin 3x + x(c_3 \cos 3x + c_4 \sin 3x) \right] \implies y_p(x) = x^2 e^{-2x} (A_0 \cos 3x + B_0 \sin 3x) \).

37. We have \( P(r) = r^2(r - 1)(r^2 + 4)^2 \implies y_e(x) = c_1 + c_2 x + c_3 e^x + c_4 2x + c_5 \sin 2x + x(c_6 \cos 2x + c_7 \sin 2x) \implies y_p(x) = A_0 x e^x + x^2 (A_1 \cos 2x + A_2 \sin 2x) \).

39. We have \( P(r) = r(r^2 - 9)(r^2 - 4r + 5) \implies y_e(x) = c_1 + c_2 e^{3x} + c_3 e^{-3x} + e^{2x} (c_4 \cos x + c_5 \sin x) \implies y_p(x) = A_0 x e^{3x} + x e^{2x} (A_1 \cos x + A_2 \sin x) \).

**Solutions to Section 6.4**

**True-False Review:**

1. **TRUE.**

3. **FALSE.** An appropriate complex-valued trial solution is \( y_p(x) = A_0 x e^{ix} \).

5. **FALSE.** An appropriate complex-valued trial solution is \( y_p(x) = A_0 x^2 e^{(2+5i)x} \).

**Problems:**

1. Consider \( z'' + 2z' + z = 50 e^{3ix} \). Let \( z_p(x) = A e^{3ix} \), \( z_p' = 3 A i e^{3ix} \), and \( z_p'' = -9 A e^{3ix} \). Substituting, we obtain

\[
-9 A e^{3ix} + 2(3 A i e^{3ix}) + A e^{3ix} = 50 e^{3ix} \implies A = -4 - 3i.
\]

Hence,

\[
z_p(x) = (-4 - 3i) e^{3ix} = (-4 - 3i)(\cos 3x + i \sin 3x) = (6 \sin 3x - 8 \cos 3x) + i(-4 \sin 3x - 3 \cos 3x).
\]

Consequently,

\[
y_p(x) = \text{Im}(z_p) = -3 \cos 3x - 4 \sin 3x.
\]

3. Consider \( z'' + 4z' + 4z = 169 e^{3ix} \). Let \( z_p(x) = A_0 e^{3ix} \). Substituting we get

\[
(-9 + 12i + 4) A_0 = 169 \implies A_0 = -5 - 12i.
\]

Hence,

\[
z_p = (-5 - 12i)(\cos 3x + i \sin 3x)
\]

so that

\[
y_p(x) = \text{Im}(z_p) = -12 \cos 3x - 5 \sin 3x.
\]
5. Consider \( z'' + z = 3e^{(1+2i)x} \). Let \( z_p(x) = A_0e^{(1+2i)x} \). Substituting into the differential equation we get
\([-3 + 4i]A_0 = 3 \implies A_0 = -\frac{3}{10}(1 + 2i) \). Hence,
\[ z_p(x) = -\frac{3}{10}(1 + 2i)e^x(\cos 2x + i \sin 2x) \]
so that
\[ y_p(x) = \text{Re}(z_p) = \frac{3}{10}e^x(2 \sin 2x - \cos 2x) \).

7. Consider \( z'' - 4z = 100xe^{(1+i)x} \). Let \( z_p(x) = (A_0 + A_1x)e^{(1+i)x} \). Substituting and solving the differential equation we get \( A_0 = 2 - 14i \) and \( A_1 = -20 - 10i \). Hence,
\[ z_p(x) = e^x[(2 - 14i) + (-20 - 10i)x](\cos x + i \sin x) \]
so that
\[ y_p(x) = \text{Im}(z_p) = e^x[(-14 \cos x + 2 \sin x) - 10x(2 \sin x + \cos x)]. \]

9. Consider \( z'' - 2z' + 10z = 24e^{(1+3i)x} \). Let \( z_p(x) = A xe^{(1+3i)x} \). Substituting into the differential equation and solving we get \( A = -4i \). Hence,
\[ z_p(x) = -4ixe^{(1+3i)x} \]
so that
\[ y_p(x) = \text{Re}(z_p) = 4xe^x \sin 3x. \]

11. Consider \( z'' + \omega_0^2z = F_0e^{\omega t} \).
Case 1: If \( \omega \neq \omega_0 \) let \( z_p(t) = Ae^{\omega_0 t} \). Substituting into the differential equation and solving we obtain
\[ A = \frac{F_0}{\omega_0^2 - \omega^2} \]. Hence, \( z_p(t) = A(\cos \omega t + i \sin \omega t) = \frac{F_0}{\omega_0^2 - \omega^2} \cos \omega t + i \frac{F_0}{\omega_0^2 - \omega^2} \sin \omega t \). Now \( y_p(t) = \text{Re}(z_p) \)
so \( y_p(t) = \frac{F_0}{\omega_0^2 - \omega^2} \cos \omega t. \)

Case 2: If \( \omega = \omega_0 \) let \( z_p(t) = Ate^{\omega_0 t} \). Substituting into the differential equation and solving we get \( A = -\frac{F_0i}{2\omega_0} \)
so \( z_p(t) = -\frac{F_0i}{2\omega_0}te^{\omega_0 t} = \frac{F_0t}{2\omega_0} \sin \omega_0 t - i \left( \frac{F_0t}{2\omega_0} \cos \omega_0 t \right) \). Hence, \( y_p(t) = \text{Re}(z_p) \) so \( y_p(t) = \frac{F_0t}{2\omega_0} \sin \omega_0 t. \)

**Solutions to Section 6.5**

**True-False Review:**

1. **TRUE.** This is reflected in the negative sign appearing in Hooke’s Law. The spring force \( F_s \) is given by \( F_s = -kL_0 \), where \( k > 0 \) and \( L_0 \) is the displacement of the spring from its equilibrium position. The spring force acts in a direction opposite to that of the displacement of the mass.

3. **TRUE.** The frequency of oscillation, denoted \( f \) in the text, and the period of oscillation, denoted \( T \) in the text, are inverses of one another: \( fT = 1 \). For instance, if a system undergoes \( f = 3 \) oscillations in one second, then each oscillation takes one-third of a second, so \( T = \frac{1}{3} \).
5. **FALSE.** In the cases of critical damping and overdamping, the system cannot oscillate. In fact, the system passes through the equilibrium position at most once in these cases.

7. **TRUE.** Air resistance tends to dampen the motion of the spring-mass system, since it acts in a direction opposite to that of the motion of the spring. This is reflected in Equation (6.5.4) by the negative sign appearing in the formula for the damping force $F_d$.

9. **TRUE.** We see in the section on free oscillations of a mechanical system, we see that the resulting motion of the mass is given by (6.5.11), (6.5.12), or (6.5.13), and in all of these cases, the amplitude of this system is bounded. Only in the case of forced oscillations with resonance can the amplitude increase without bound.

**Problems:**

1. We have $\frac{d^2 y}{dt^2} + 4y = 0 \implies r^2 + 4 = 0 \implies r = \pm 2i \implies y(t) = A\cos 2t + B\sin 2t$ and $y'(t) = -2A\sin 2t + B\cos 2t$. Now $y(0) = 2 \implies A = 2$ and $y'(0) = 4 \implies B = 2$. Thus, $y(t) = 2\cos 2t + 2\sin 2t$.

   Amplitude: $A_0 = \sqrt{2^2 + 2^2} = 2\sqrt{2}$,

   Natural frequency: $\omega_0 = 2$,

   Phase of the motion: $\phi = \tan^{-1}(1) = \frac{\pi}{4}$,

   Period of oscillation: $T = \frac{2\pi}{\omega_0} = \pi$.

3. (a) $3 N = k (1m) \implies k = 3 N/m$.

   (b) $\omega_0 = \sqrt{k/m} = \sqrt{3/4} = \sqrt{3}/2$. $\frac{d^2 y}{dt^2} + \frac{3}{4}y = 0 \implies y = A\cos (\sqrt{3}/2)t + B\sin (\sqrt{3}/2)t$. Since $y(0) = -1$, we conclude that $A = -1$. Now $y'(t) = -A\sqrt{3}/2 \sin (\sqrt{3}/2)t + B\sqrt{3}/2 \cos (\sqrt{3}/2)t$ and $y'(0) = -\frac{1}{2}$, so that $B = -\frac{\sqrt{3}}{3}$. Thus, $y(t) = -\cos (\sqrt{3}/2)t - \frac{\sqrt{3}}{3} \sin (\sqrt{3}/2)t$.

   Amplitude: $A_0 = \sqrt{(-1)^2 + (-\sqrt{3}/3)^2} = 2\sqrt{3}/3$,

   Phase of the motion: $\cos \phi = -\frac{\sqrt{3}}{2}, \ \sin \phi = \frac{1}{2} \implies \phi = \frac{5\pi}{6}$,

   Period of oscillation: $T = \frac{2\pi}{\omega_0} = \frac{4\sqrt{3}\pi}{3}$ sec.

5. We have $P(r) = r^2 + 3r + 2 = 0 \implies r = -1$ or $r = -2 \implies$ $y(t) = Ae^{-t} + Be^{-2t}$ and $y'(t) = -Ae^{-t} - 2Be^{-2t}$.

   Thus $y(0) = 1$ and $y'(0) = 0 \implies A + B = 1$ and $-A - 2B = 0 \implies A = 2$ and $B = -1$. Therefore, $y(t) = 2e^{-t} - e^{-2t}$. The motion is over-damped.

7. We have $P(r) = r^2 + 2r + 1 = 0 \implies r = -1$ (with multiplicity 2) $\implies$ $y(t) = Ae^{-t} + Bte^{-t}$ and $y'(t) = -Ae^{-t} + (1 - t)Bte^{-t}$.

   Since $y(0) = -1$ and $y'(0) = 2$, we find that $A + 0 = -1$ and $-A + B = 2 \implies A = -1$ and $B = 1$. Therefore, $y(t) = -e^{-t} + te^{-t}$. The motion is critically-damped.
Figure 0.0.43: Figure for Exercise 5

Figure 0.0.44: Figure for Exercise 7

9. We have $P(r) = r^2 + 4r + 7 = 0 \implies -2 \pm \sqrt{3}i \implies$

$$y(t) = e^{-2t}(A \cos \sqrt{3}t + B \sin \sqrt{3}t) \quad \text{and} \quad y'(t) = e^{-2t}[-(\sqrt{3}A - 2B) \sin \sqrt{3}t + (-2A + \sqrt{3}B) \cos 3t].$$

Since $y(0) = 2$ and $y'(0) = -6$, we find that $A + 0 = 2$ and $-2A + \sqrt{3}B = 6 \implies A = 2$ and $B = \frac{10\sqrt{3}}{3}$ so

$y(t) = e^{-2t}(2 \cos \sqrt{3}t + \frac{10\sqrt{3}}{3} \sin \sqrt{3}t)$. The motion is under-damped.

Figure 0.0.45: Figure for Exercise 9

11. At equilibrium, $e^{-2t} - 2e^{-3t} = 0 \implies t = \ln 2$. For maximum displacement, $y'(t) = 0 \implies -2e^{-2t} + 6e^{-3t} = 0 \implies t = \ln 3$ so the maximum displacement is given by $y(\ln 3) = \frac{1}{27}$.
13.
(a) We have \( P(r) = r^2 + 3r + 2 = 0 \iff r = -1 \) or \( r = -2 \iff \)
\[
y(t) = Ae^{-t} + Be^{-2t} \quad \text{and} \quad y'(t) = -Ae^{-t} - 2Be^{-2t}.
\]
Since \( y(0) = 1 \), we have \( A + B = 1 \), and since \( y'(0) = -3 \), we have \( -A - 2B = -3 \). Therefore, \( A = -1 \) and \( B = 2 \), so that \( y(t) = -e^{-t} + 2e^{-2t} \).

(b) \( y(t) = 0 \iff -e^{-t} + 2e^{-2t} = 0 \iff t = \ln 2 \).

(c) \[\begin{array}{c}
\text{Figure 0.0.46: Figure for Exercise 13}
\end{array}\]

15. We have \( P(r) = r^2 + \frac{c}{m}r + \frac{c^2}{4m^2} = 0 \iff r = -\frac{c}{2m} \) (with multiplicity 2). Thus,
\[
y(t) = Ae^{rt} + Be^{rt} \quad \text{and} \quad y'(t) = Are^{rt} + Brte^{rt} + Be^{rt}.
\]
Since \( y(0) = A \) and \( y'(0) = Ar + B \), we find that \( A = y_0 \) and \( Ar + B = v_0 \). Therefore, \( B = v_0 + \frac{vy_0}{2m} \). Thus \( y(t) = y_0e^{rt} + \left(v_0 + \frac{vy_0}{2m}\right)te^{rt} \), or \( y(t) = e^{rt}[y_0 + t(v_0 + \frac{c}{2m}y_0)] \). Since \( e^{rt} \) is never zero and \( y_0 + t(v_0 + \frac{c}{2m}y_0) \) is linear in \( t \), it follows that the product can have at most one zero.

17. Let \( \theta \) represent the angular displacement of the pendulum. From Equation (6.5.28), we have \( \frac{d^2\theta}{dt^2} + \frac{g}{L} \theta = 0 \). Since \( L = 0.5 \), this becomes \( \frac{d^2\theta}{dt^2} + 2g\theta = 0 \). Therefore,
\[
\theta(t) = c_1 \cos(\sqrt{2gt}) + c_2 \sin(\sqrt{2gt}).
\]
Since \( \theta(0) = \frac{1}{10} \), we have \( c_1 = \frac{1}{10} \), and since \( \frac{d\theta}{dt}(0) = 0 \), we have \( c_2 = 0 \). Thus \( \theta(t) = \frac{1}{10} \cos(\sqrt{2gt}) \).

19. In Problem 18, it was shown that the period of the simple pendulum is \( T = 2\pi\sqrt{\frac{L}{g}} \). The time required for the pendulum to swing from its extreme position on one side to its extreme position on the other side is half the period, \( T/2 \). If \( T/2 = 1 \), then \( \pi \sqrt{\frac{L}{g}} = 1 \) and \( L = \frac{g}{\pi^2} \), so that \( L \approx 0.993 \) meters.
21. Since each side is stretched $2a$, it follows that the vertical component of each side is given by $2a \cos (\theta_0) = \frac{4}{5} \frac{8a}{5}$ where $\theta_0$ is the angle the side of length $5a$ makes with the vertical. Consequently, in equilibrium, $2k \frac{8a}{5} = mg \implies k = \frac{5mg}{16a}$. Now when the mass is pulled down a small vertical distance, $y(t)$, from equilibrium the length $4a$ changes to $4a + y(t)$. Using the diagram below we see that the length of the hypotenuse is $\sqrt{y^2 + 8ay + 25a^2} = 5a \sqrt{1 + \frac{8}{25}y + \frac{y^2}{25a^2}} \approx 5a(1 + \frac{4}{25a}y + ...)$. Also, from the figure, $\cos \theta = \frac{y + 4a}{5a(1 + \frac{4}{25a}y + ...)} \approx \frac{y + 4a}{5a(1 - \frac{4}{25a}y + ...)} \approx \frac{4}{5} + y\left(\frac{1}{5a} - \frac{16}{125a}\right) + ... \approx \frac{4}{5} + \frac{9}{125a}y$.

Thus,

$$F = -2k \left[5a(1 + \frac{4}{25a}y + ...) - 3a\right] \cos \theta + mg$$

$$= -2k(2a + \frac{4}{5}y + ...)(\frac{4}{5} + \frac{9}{125a}y + ...) + mg$$

$$= -2k \left[\frac{8}{5a} + \left(\frac{16}{25} + \frac{18}{125}\right)y + ...\right] + \frac{16ak}{5}$$

$$= -\frac{98}{125}y + ...$$

$$= -2\left(\frac{5mg}{16a}\right)\frac{98}{125}y + ...$$

$$= -\frac{49mg}{100a}y + ... \approx -\frac{49mg}{100a}y.$$  

Thus, $\frac{d^2y}{dt^2} + \frac{49g}{100a}y = 0$, and hence, the period is $T = 2\pi \sqrt{\frac{100a}{49g}} = \frac{20\pi}{7} \sqrt{\frac{a}{g}}$.

23.

(a) The motion is under-damped.

(b) We have $P(r) = r^2 + 2r + 5 = 0 \implies r = -1 \pm 2i$ so $y_h(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$. Letting $y_p(t) = C \cos 2t + D \sin 2t$, we obtain $y_p'(t) = -2C \sin 2t + 2D \cos 2t$ and $y_p''(t) = -4C \cos 2t - 4D \sin 2t$. Substituting these results into the original equation results in the system: $-4C + 4D + 5C = 0$ and $-4D - 4C + 5D = 17$. The solution is $C = -4$ and $D = 1$; consequently, $y_p(t) = -4 \cos 2t + \sin 2t$. The general solution is $y(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) - 4 \cos 2t + \sin 2t$. Since $y(0) = -2$, we have $c_1 = 2$, and since $y'(0) = 0$ we
have \(2c_2 - c_1 + 2 = 0 \implies c_2 = 0\). Thus \(y(t) = 2e^{-t} \cos 2t - 4 \cos 2t + 2 \sin 2t\).

Transient part: \(2e^{-t} \cos 2t\).

Steady state part: \(-4 \cos 2t + 2 \sin 2t\).

25. Let \(y_p(t) = A \sin t + B \cos t \implies y_p'(t) = A \cos t - B \sin t \implies y_p''(t) = -A \sin t - B \cos t\). Substituting these results into \(y''(t) + 3y'(t) + 2y(t) = 10 \sin t\), we obtain the system \(A - 3B = 10\) and \(3A + B = 0\) so \(A = 1\) and \(B = -3\). Hence, \(y_p(t) = \sin t - 3 \cos t = \sqrt{A^2 + B^2} \sin [t + \tan^{-1} (B/A)] = \sqrt{10} \sin (t - \tan^{-1} 3)\).

27. If \(\omega = \frac{m}{n}\), where \(m\) and \(n\) are integers, then from the given equation

\[
y(t + \frac{2\pi n}{\omega_0}) = A_0 \cos \left[\omega_0 \left(t + \frac{2\pi n}{\omega_0}\right) - \phi\right] + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \left[\omega \left(t + \frac{2\pi n}{\omega_0}\right)\right]
\]

\[
 = A_0 \cos (\omega_0 t + 2\pi n - \phi) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos (\omega t + 2\pi m)
\]

\[
 = A_0 \cos (\omega_0 t - \phi) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos (\omega t + 2\pi m)
\]

Thus the motion is periodic with period \(T = \frac{2\pi n}{\omega_0}\).

29. Let \(y_p = A \cos \omega t + B \sin \omega t\) so that \(y'_p = -\omega A \sin \omega t + B \omega \cos \omega t\) and \(y''_p = -\omega^2 A \cos \omega t - B \omega^2 \sin \omega t\). Substituting these results into the original equation yields

\[-\omega^2 A \cos \omega t - B \omega^2 \sin \omega t + \frac{c}{m} (-\omega A \sin \omega t + B \omega \cos \omega t) + \frac{k}{m} (A \cos \omega t + B \sin \omega t) = \frac{F_0}{m} \cos \omega t,
\]

which implies that

\[
\cos \omega t (-\omega^2 A + \frac{c \omega}{m} B + \frac{k}{m} A) + \sin \omega t (-\omega^2 B - \frac{c \omega}{m} A + \frac{k}{m} B) = \frac{F_0}{m} \cos \omega t.
\]

Therefore,

\[-\omega^2 B - \frac{c \omega}{m} A + \frac{k}{m} B = 0 \quad \text{and} \quad -\omega^2 A + \frac{c \omega}{m} B + \frac{k}{m} A = \frac{F_0}{m}.
\]

Thus, \(\frac{k}{m} - \omega^2) A + \frac{m \omega}{c} B = \frac{F_0}{m}\). Solving this system for \(A\) and \(B\) yields

\[
B = \frac{F_0 c \omega}{(k - m \omega^2)^2 + c^2 \omega^2} \quad \text{and} \quad A = \frac{(k - m \omega^2) F_0}{(k - m \omega^2)^2 + c^2 \omega^2}.
\]

Thus, the particular solution is \(y_p(t) = \frac{F_0}{(k - m \omega^2)^2 + c^2 \omega^2} (k - m \omega^2) \cos \omega t + c \omega \sin \omega t\).

31.

(a) From \(\frac{d^2 y}{dt^2} + \frac{c}{m} \frac{dy}{dt} + \frac{k}{m} y = \frac{F(t)}{m}\), we substitute the given information: \(y''(t) + 2y'(t) + 5y(t) = 8 \cos \omega t\).

Then \(r^2 + 2r + 5 = 0 \implies r = -1 \pm 2i\) so let \(y_c(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)\) and \(y_p(t) = A \cos \omega t + B \sin \omega t\).
Thus we obtain the system \(-\omega^2 B - 2\omega A + 5B = 0\) and \(-\omega^2 A + 2\omega B + 5A = 8\). Solving this system yields:

\[
A = \frac{40 - 8\omega^2}{\omega^4 - 6\omega^2 + 25} \quad \text{and} \quad B = \frac{16\omega}{\omega^4 - 6\omega^2 + 25}.
\]

**Transient solution:** \(y_T(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)\).

**Steady state solution:** \(y_S(t) = \frac{40 - 8\omega^2}{\omega^4 - 6\omega^2 + 25} \cos \omega t + \frac{16\omega}{\omega^4 - 6\omega^2 + 25} \sin \omega t\).

(b) Since \(m = 1\), \(k = 5\), and \(c = 2\), we have \(\omega = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}} = \sqrt{5 - 2} = \sqrt{3}\) maximizes the amplitude of the steady-state solution (see Problem 30). Therefore, using (a), we have \(A = 1\) and \(B = \sqrt{3}\), so that

\[
y_p(t) = \cos (\sqrt{3}t) + \sqrt{3} \sin (\sqrt{3}t)
= 2 \left[ \frac{1}{2} \cos (\sqrt{3}t) + \frac{\sqrt{3}}{2} \sin (\sqrt{3}t) \right]
= 2 \cos (\sqrt{3}t - \pi/3).
\]

33. If \(\frac{d^2y}{dt^2} + 16y = 0\) then \(r^2 + 16 = 0 \Rightarrow r = \pm 4i \Rightarrow y_c(t) = A_0 \cos (4t - \phi)\). Now let a particular solution, \(y_p(t)\), take the form \(y_p(t) = e^{-t}(A \sin t + B \cos t)\). Substituting this expression into the differential equation, we obtain

\[
2e^{-t}(B \sin t - A \cos t) + 16e^{-t}(A \sin t + B \cos t) = 130e^{-t} \cos t,
\]
which implies that

\[
(2B + 16A) \sin t + (16B - 2A) \cos t = 130 \cos t.
\]
Thus, \(B + 8A = 0\) and \(8B - A = 65\). Hence, \(A = -1\) and \(B = 8\). Consequently, \(y_p(t) = e^{-t}(8 \cos t - \sin t)\).

**Transient part:** \(e^{-t}(8 \cos t - \sin t)\).

**Steady-state part:** \(A_0 \cos (4t - \phi)\).

---

### Solutions to Section 6.6

**True-False Review:**

1. **TRUE.** The differential equation governing the situation in which no driving electromotive force is present is (6.6.3), and its solutions are given in (6.6.4). In all cases, \(q(t) \to 0\) as \(t \to \infty\). Since the charge decays to zero, the rate of change of charge, which is the current in the circuit, eventually decreases to zero as well.

3. **TRUE.** The amplitude of the steady-state current is given by Equation (6.6.6), which is maximum when \(\omega^2 = \omega\). Substituting \(\omega^2 = \frac{1}{LC}\), it follows that the amplitude of the steady-state current will be a maximum when

\[
\omega = \omega_{\text{max}} = \frac{1}{\sqrt{LC}}.
\]

5. **FALSE.** The current \(i(t)\) in the circuit is the derivative of the charge \(q(t)\) on the capacitor, given in the solution to Example 6.6.1:

\[
q(t) = A_0 e^{-Rt/2L} \cos(\mu t - \phi) + \frac{E_0}{H} \cos(\omega t - \eta).
\]
The derivative of this is not inversely proportional to \( R \).

**Problems:**

1. We are given that \( R = \frac{3}{2} \), \( L = \frac{1}{2} \), \( C = \frac{2}{3} \), \( E_0 = 13 \), and \( \omega = 3 \). So \( \omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(1/2)(2/3)}} = \sqrt{3} \),

\[
H = \sqrt{L^2(\omega_0^2 - \omega^2) + R^2\omega^2} = \sqrt{\frac{1}{4}(3 - 9)^2 + \frac{9}{4}(9)} = \frac{3\sqrt{13}}{2}, \quad A = \frac{\omega E_0}{H} = \frac{3 \cdot 13}{3\sqrt{13}/2} = 2\sqrt{13}, \quad \cos \eta = \frac{R \omega}{H} = \frac{3 \sqrt{13}}{2}, \quad \sin \eta = \frac{\omega}{H} = \frac{3}{\sqrt{13}}. \]

Thus

\[
i_S(t) = -A \sin(\omega t - \eta) = -2\sqrt{13}\sin(3t - \eta) = -2\sqrt{13}\sin(3t \cos \eta - \cos 3t \sin \eta) = -2\sqrt{13} \left[ \sin 3t \left( -\frac{2}{\sqrt{13}} \right) - \cos 3t \left( \frac{3}{\sqrt{13}} \right) \right] = 2(2 \sin 3t + 3 \cos 3t).
\]

3. When \( R = 0 \), the differential equation is \( \frac{d^2q}{dt^2} + \frac{1}{LC}q = E_0 \cos \omega t \), and so \( q_p(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t \), where \( \omega_0 = \sqrt{\frac{1}{LC}} \).

**Case 1:** If \( \omega = \sqrt{\frac{1}{LC}} \), then let \( q_p(t) = t(A \cos \omega t + B \sin \omega t) \). Hence,

\[
q_p'' + \omega^2 q_p = E_0 \cos \omega t,
\]

and so

\[
2\omega B \cos \omega t - 2\omega A \sin \omega t = E_0 \cos \omega t.
\]

Therefore, \( A = 0 \) and \( B = \frac{E_0}{2\omega} \), and therefore \( q_p(t) = \frac{tE_0}{2\omega} \sin \omega t \) so that

\[
q(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{tE_0}{2\omega} \sin \omega t.
\]

Thus, as \( t \to \infty \), \( q(t) \to \infty \) when \( \omega = \sqrt{\frac{1}{LC}} \).

**Case 2:** If \( \omega \neq \sqrt{\frac{1}{LC}} \), then we take \( q_p(t) = A \cos \omega t \) so that

\[
q_p'' + \frac{1}{LC} q_p = E_0 \cos \omega t,
\]

which implies that

\[
-\omega^2 A \cos \omega t + \frac{A}{LC} \cos \omega t = E_0 \cos \omega t \implies \left( \frac{A}{LC} - \omega^2 A \right) \cos \omega t = E_0 \cos \omega t.
\]
Therefore, \( A = \frac{E_0 LC}{1 - LC \omega^2} \), and so \( q_p(t) = \frac{E_0 LC}{1 - LC \omega^2} \cos \omega t \). Hence,

\[
q(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{E_0 LC}{1 - LC \omega^2} \cos \omega t,
\]

which is a bounded expression. If \( \omega = \sqrt{\frac{1}{LC}} \), then \( i(t) = \frac{dq}{dt} = (\omega c_2 + \frac{t E_0}{2}) \cos \omega t + (\frac{E_0}{2\omega} - \omega c_1) \sin \omega t \) which implies that \( i(t) \rightarrow \infty \) as \( t \rightarrow \infty \). If \( \omega \neq \sqrt{\frac{1}{LC}} \), then

\[
i(t) = \frac{dq}{dt} = \frac{CL \omega E_0}{CL \omega^2 - L} \sin \omega t + \omega_0 c_2 \cos \omega t - \omega_0 c_1 \sin \omega t,
\]

which is bounded.

5. Given the RLC circuit with \( R = 3 \), \( L = \frac{1}{2} \), \( C = \frac{1}{5} \), \( E(t) = 2 \cos \omega t \), and \( E_0 = 2 \). So \( \mu = \sqrt{\frac{4L}{C} - R^2} = 1 \), \( \omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{10} \), and \( H = \sqrt{L^2 (\omega_0^2 - \omega^2) + R^2 \omega^2} = \sqrt{(1/2)^2 (10 - \omega^2)^2 + 3^2 \omega^2} = \sqrt{10 - \omega^2)^2 + 36 \omega^2}. \) Then \( \cos \eta = \frac{L(\omega_0^2 - \omega^2)}{H} = \frac{10 - \omega^2}{2H} \) and \( \sin \eta = \frac{R \omega}{H} = \frac{3 \omega}{H} \) so that

\[
q(t) = A_0 e^{-\frac{\mu t}{2}} \cos (\mu t - \phi) + \frac{E_0}{H} \cos (\omega t - \eta),
\]

where \( c_1 = A_0 \cos \phi \) and \( c_2 = A_0 \sin \phi \). Therefore,

\[
q(t) = A_0 e^{-3t} \cos (t - \phi) + \frac{2}{H} \cos (\omega t - \eta)
\]

and

\[
i(t) = \frac{dq}{dt} = -A_0 e^{-3t} [3 \cos (t - \phi) + \sin (t - \phi)] - \frac{2 \omega}{H} \sin (\omega t - \eta).
\]

Thus \( \omega_{\text{max}} = \sqrt{\frac{1}{LC}} = \sqrt{10} \).

7. Since \( E(t) = E_0 e^{-at} \), \( \frac{dE}{dt} = -a E_0 e^{-at} \). The equation for current is identical to the equation governing charge, except it has a different nonhomogeneous term.

\[
i_e(t) = A_0 e^{-\frac{\mu t}{2}} \cos (\mu t - \phi),
\]

and substituting \( i_p(t) = A e^{-at} \) into \( \frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} i = \frac{1}{L} \frac{dE}{dt} \), we have

\[
a^2 A e^{-at} + \frac{R}{L} (-a A e^{-at}) + \frac{1}{LC} (A e^{-at}) = \frac{1}{L} (-a E_0 e^{-at}).
\]

Therefore,

\[
A e^{-at} (a^2 - \frac{aR}{L} + \frac{1}{LC}) = -\frac{a E_0}{L} e^{-at},
\]
and so \( A = -\frac{aCE_0}{a^2LC - aCR + 1} \). Hence, \( i_p(t) = -\frac{aCE_0}{a^2LC - aCR + 1}e^{-at} \) and \( i(t) = A_0e^{-\frac{at}{2}}\cos(\mu t - \phi) - \frac{aCE_0}{a^2LC - aCR + 1}e^{-at} \), where \( \mu = \sqrt{\frac{4L}{C} - R^2}{2L} \).

**Solutions to Section 6.7**

**True-False Review:**

1. **TRUE.** This is essentially the statement of the variation-of-parameters method (Theorem 6.7.1).

2. **FALSE.** The requirement on the functions \( u_1, u_2, \ldots, u_n \), according to Theorem 6.7.6, is that they satisfy Equations (6.7.22). Because these equations involve the derivatives of \( u_1, u_2, \ldots, u_n \) only, constants of integration can be arbitrarily chosen in solving (6.7.22), and therefore, addition of constants to the functions \( u_1, u_2, \ldots, u_n \) again yields valid solutions to (6.7.22).

**Problems:**

1. Setting \( y'' + 6y' + 9y = 0 \implies r^2 + 6r + 9 = 0 \implies r \in \{-3, -3\} \implies y_c(x) = e^{-3x}(c_1 + c_2x) \). Let \( y_1(x) = e^{-3x} \) and \( y_2(x) = xe^{-3x} \). We have \( W[y_1, y_2](x) = e^{-6x} \). Then a particular solution to the given differential equation is \( y_p(x) = u_1y_1 + u_2y_2 \), where \( e^{-3x}u'_1 + xe^{-3x}u'_2 = 0 \) and \( -3e^{-3x}u'_1 + e^{-3x}(1 - 3x)u'_2 = \frac{2e^{-3x}}{x^2 + 1} \).

That is, \( u'_1 + xu'_2 = 0 \), \(-3u'_1 + (1 - 3x)u'_2 = \frac{2}{x^2 + 1} \). Solving for \( u_1 \) and \( u_2 \) (and setting integration constants to zero), we have

\[
u_1 = -\int^x \frac{2te^{-3t}(e^{-3t})}{e^{-6t}(t^2 + 1)}dt = -\int^x \frac{2t}{t^2 + 1}dt = -\ln(x^2 + 1)
\]

and

\[
u_2 = \int^x \frac{2e^{-3t}(e^{-3t})}{e^{-6t}(t^2 + 1)}dt = \int^x \frac{2}{t^2 + 1}dt = 2\tan^{-1}(x).
\]

Thus \( y_p(x) = -e^{-3x}\ln(x^2 + 1) + 2xe^{-3x}\tan^{-1}(x) \). Therefore, \( y(x) = e^{-3x}[c_1 + c_2x + 2x\tan^{-1}(x) - \ln(x^2 + 1)] \).

3. Setting \( y'' - 4y' + 5y = 0 \implies r^2 - 4r + 5 = 0 \implies r \in \{2 + i, 2 - i\} \implies y_c(x) = e^{2x}(c_1\cos x + c_2\sin x) \). Let \( y_1(x) = e^{2x}\cos x \) and \( y_2(x) = e^{2x}\sin x \). Then \( W[y_1, y_2](x) = e^{4x} \). A particular solution to the given differential equation is therefore \( y_p = u_1y_1 + u_2y_2 \), where

\[
e^{2x}\cos xu'_1 + e^{2x}\sin xu'_2 = 0
\]

and

\[
e^{2x}(2\cos x - \sin x)u'_1 + e^{2x}(2\sin x + \cos x)u'_2 = e^{2x}\tan x.
\]

Solving for \( u_1 \) and \( u_2 \) (and setting integration constants to zero), we have

\[
u_1 = -\int^x \frac{e^{2t}\sin(t(e^{2t}\tan t))}{e^{4t}}dt = \int^x \sin t\tan t\ dt = \sin x - \ln|\sec x + \tan x|
\]

and

\[
u_2 = \int^x \frac{e^{2t}\cos(t(e^{2t}\tan t))}{e^{4t}}dt = \int^x \sin t\ dt = -\cos x.
\]
Consequently, \( y_p(x) = e^{2x} \cos x (\sin x - \ln |\sec x + \tan x|) - e^{2x} \sin x \cos x = -e^{2x} \cos x \ln |\sec x + \tan x| \). Thus, \( y(x) = e^{2x} \cos x (c_1 - \ln |\sec x + \tan x|) + c_2 \sin x \).

5. Setting \( y'' + 4y' + 4y = 0 \implies r^2 + 4r + 4 = (r + 2)^2 = 0 \implies r \in \{-2, -2\} \implies y_c(x) = c_1 e^{-2x} + c_2 xe^{-2x}. \) Let \( y_1(x) = e^{-2x} \) and \( y_2(x) = xe^{-2x}. \) We have \( W[y_1, y_2](x) = e^{-4x}. \) Then a particular solution to the given differential equation is \( y_p(x) = y_1 u_1 + y_2 u_2, \) where

\[
 u' + xu_2' = 0, \quad -2u_1' + (1 - 2x)u_2' = \frac{1}{x^2}.
\]

Solving for \( u_1 \) and \( u_2 \) (and setting integration constants to zero), we have

\[
u_1 = -\int x \frac{(te^{-2t})e^{-2t}}{t^2e^{-4t}} dt = -\int x \frac{1}{t} dt = -\ln x
\]

and

\[
u_2 = \int x \frac{(e^{-2t})e^{-2t}}{t^2e^{-4t}} dt = -\frac{1}{x}.
\]

Consequently,

\[
y_p(x) = -e^{-2x} \ln x - e^{-2x}.
\]

Hence,

\[
y(x) = e^{-2x} (c_1 + c_2 x - \ln x - 1).
\]

7. Setting \( y'' - y = 0 \implies r^2 - 1 = 0 \implies r \in \{-1, 1\} \implies y_c(x) = c_1 e^x + c_2 e^{-x}. \) Let \( y_1(x) = e^x \) and \( y_2(x) = e^{-x}. \) Then,

\[
W[y_1, y_2](x) = (e^x)(e^{-x}) - (e^x)(e^{-x}) = -2 \neq 0
\]

so that

\[
y_p(x) = -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx = e^x \left[ \frac{2 \tan^{-1}(e^x) + e^{-x}}{e^x + e^{-x}} \right] dx = 2(e^x + e^{-x}) \tan^{-1}(e^x).
\]

Hence, \( y(x) = c_1 e^x + c_2 e^{-x} + 4 \cosh x \tan^{-1}(e^x). \)

9. Setting \( y'' - 2y' + y = 0 \implies r^2 - 2r + 1 = 0 \implies r \in \{1, 1\} \implies y_c(x) = c_1 e^x + c_2 xe^x. \) Let \( y_1(x) = e^x \) and \( y_2(x) = xe^x. \) Then,

\[
W[y_1, y_2](x) = (e^x)(e^x[x + 1]) - (e^x)(xe^x) = e^{2x} \neq 0
\]

so that

\[
y_p(x) = -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx = 4e^x \left[ \frac{x^2}{x^2 + 4e^x} \right] dx = \frac{e^x}{x} (2 \ln x + 3).
\]

Consequently, \( y(x) = e^x [c_1 + c_2 x + x^{-1}(2 \ln x + 3)]. \)
11. Setting \(y'' + 2y' + 17 = 0 \Rightarrow r^2 + 2r + 17 = 0 \Rightarrow r \in \{1 + 4i, 1 - 4i\} \Rightarrow y_c(x) = e^{-x}(c_1 \cos 4x + c_2 \sin 4x).\) Let \(y_1(x) = e^{-x} \cos 4x\) and \(y_2(x) = e^{-x} \sin 4x.\) Then

\[
W[y_1, y_2](x) = (e^{-x} \cos 4x)(e^{-x}[4 \cos 4x - \sin 4x]) - (e^{-x} \sin 4x)[-e^{-x}(4 \sin 4x + \cos 4x)] = 4e^{-2x}
\]

so that

\[
y_p(x) = -y_1\int \frac{y_2}{W} dx + y_2\int \frac{y_1}{W} dx
\]

\[
= -16e^{-x} \cos 4x \int \frac{\sin 4x}{3 + \sin^2 4x} dx + 16e^{-x} \sin 4x \int \frac{\cos 4x}{3 + \sin^2 4x} dx
\]

\[
= -16e^{-x} \cos 4x \left[-\frac{1}{8} \tanh^{-1}(\cos (4x/2))\right] + 16e^{-x} \sin 4x \left[\frac{\sqrt{3}}{12} \tan^{-1}(\sin (4x/\sqrt{3}))\right]
\]

\[
= 2e^{-x} \cos 4x \tan^{-1}(\cos (4x/2)) + \frac{4\sqrt{3}}{3} e^{-x} \sin 4x \tan^{-1}(\sin (4x/\sqrt{3}))
\]

Therefore, \(y(x) = e^{-x}\left[c_1 \cos 4x + c_2 \sin 4x + \cos 4x \ln \left(\frac{\cos 4x + 2}{\cos 4x - 2}\right) + \frac{4\sqrt{3}}{3} \sin 4x \tan^{-1}(\sin (4x/\sqrt{3}))\right].\)

13. Setting \(y'' - 10y' + 25y = 0 \Rightarrow r^2 - 10r + 25 = 0 \Rightarrow r \in \{5, 5\} \Rightarrow y_c(x) = e^{5x}(c_1 + c_2 x).\) Let \(y_1(x) = e^{5x}\) and \(y_2(x) = xe^{5x}.\) Then,

\[
W[y_1, y_2](x) = (e^{5x})[\alpha(5x + 1)] - (xe^{5x})(\alpha e^{5x}) = e^{10x} \neq 0
\]

so that

\[
y_p(x) = -y_1\int \frac{y_2}{W} dx + y_2\int \frac{y_1}{W} dx
\]

\[
= -2e^{5x} \int \frac{x}{4 + x^2} dx + 2xe^{5x} \int \frac{1}{4 + x^2} dx
\]

\[
= -e^{5x} \ln (4 + x^2) + xe^{5x} \tan^{-1}(x/2)
\]

Hence, \(y(x) = e^{5x}[c_1 + c_2 x - \ln (4 + x^2) + x \tan^{-1}(x/2)].\)

15. The complementary function for the differential equation is \(y_c(x) = c_1 \cos x + c_2 \sin x.\) To determine a particular solution to the given differential equation we first find particular solutions to each of the differential equations \(y'' + y = \sec x,\) and \(y'' + y = 4e^x,\) and then add the two solutions together. Using variation-of-parameters, a particular solution to \(y'' + y = \sec x\) is \(y_{p_1}(x) = u_1 \cos x + u_2 \sin x,\) where

\[
\cos xu_1' + \sin xu_2' = 0, \quad -\sin xu_1' + \cos xu_2' = \sec x.
\]

This pair of equations has a solution \(u_1' = -\tan x\) and \(u_2' = 1.\) Consequently, we can choose

\[
u_1(x) = \ln (\cos x) \quad \text{and} \quad u_2(x) = x,
\]

so that

\[
y_{p_1}(x) = \cos x \ln (\cos x) + x \sin x.
\]

From the method of undetermined coefficients, the differential equation \(y'' + y = 4e^x\) has a particular solution of the form \(y_{p_2}(x) = 2e^x.\) A particular solution to the given differential equation is therefore

\[
y_p(x) = y_{p_1}(x) + y_{p_2}(x) = \cos x \ln (\cos x) + x \sin x + 2e^x.
\]
and the general solution is

\[ y(x) = c_1 \cos x + c_2 \sin x + \cos x \ln (\cos x) + x \sin x + 2e^x. \]

17. The complementary function for the differential equation is \( y_c(x) = c_1 e^{-2x} + c_2 xe^{-2x} \). To determine a particular solution to the given differential equation, we first find particular solutions to each of the differential equations \( y'' + 4y' + 4y = 15e^{-2x} \ln x \) and \( y'' + 4y' + 4y = 25 \cos x \) separately, and then we add the two solutions together. Using variation-of-parameters, a particular solution to the differential equation is \( y_p(x) = u_1 e^{-2x} + u_2 xe^{-2x} \), where

\[
e^{-2x}u_1' + xe^{-2x}u_2' = 0, \quad -2e^{-2x}u_1' + e^{-2x}(1 - 2x)u_2' = 15e^{-2x} \ln x.
\]

This pair of equations has a solution

\[ u_1' = -15x \ln x, \quad u_2' = 15 \ln x. \]

Consequently, we can choose \( u_1(x) = -\frac{15}{4} x^2(2 \ln x - 1) \) and \( u_2(x) = 15x(\ln x - 1) \) so that

\[ y_p(x) = -\frac{15}{4} x^2 e^{-2x}(2 \ln x - 1) + 15x^2 e^{-2x}(\ln x - 1) = \frac{15}{4} x^2 e^{-2x}(2 \ln x - 3). \]

According to the method of undetermined coefficients, the differential equation \( y'' + 4y' + 4y = 25 \cos x \) has a particular solution of the form \( y_{p_2}(x) = A_0 \cos x + A_1 \sin x \). Substituting into the preceding differential equation and equating the coefficients yields \( 3A_0 + 4A_1 = 25 \) and \(-4A_0 + 3A_1 = 0\), with a solution \( A_0 = 3, \quad A_1 = 4 \). Consequently,

\[ y_{p_2}(x) = 3 \cos x + 4 \sin x. \]

A particular solution to the given differential equation is therefore

\[ y_p(x) = y_{p_1}(x) + y_{p_2}(x) = \frac{15}{4} x^2 e^{-2x}(2 \ln x - 3) + 3 \cos x + 4 \sin x, \]

and the general solution is

\[ y(x) = c_1 e^{-2x} + c_2 xe^{-2x} + \frac{15}{4} x^2 e^{-2x}(2 \ln x - 3) + 3 \cos x + 4 \sin x. \]

19. The complementary function for the differential equation is \( y_c(x) = c_1 e^{2x} + c_2 xe^{2x} + c_3 x^2 e^{2x} \). Let \( y_1(x) = e^{2x}, \ y_2(x) = xe^{2x} \) and \( y_3(x) = x^2 e^{2x} \). The system (6.7.22) from the text assumes the form

\[
u_1' + u_2 + x^2 u_3' = 0, \quad 2u_1' + (1 + 2x)u_2' + (2x + 2x^2)u_3' = 0, \quad 4u_1' + (4 + 4x)u_2' + (2 + 8x + 4x^2)u_3' = 36 \ln x,
\]

where we have divided each equation by \( e^{2x} \). Solving the system we obtain \( u_1' = 18x^2 \ln x, \quad u_2' = -36x \ln x, \) and \( u_3' = 18 \ln x \) from which it follows that \( u_1 = 2x^3(3 \ln x - 1), \quad u_2 = 9x^2(1 - 2 \ln x) \) and \( u_3 = 18x(\ln x - 1) \). Hence,

\[ y_p(x) = u_1y_1 + u_2y_2 + u_3y_3 = [2x^3(3 \ln x - 1)] e^{2x} + [9x^2(1 - 2 \ln x)] xe^{2x} + [18x(\ln x - 1)] x^2 e^{2x} = x^3 e^{2x}(6 \ln x - 11). \]
Therefore, the general solution is
\[ y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} + x^3 e^{3x} (6 \ln x - 11). \]

21. The complementary function for the differential equation is \( y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}. \) The system (6.7.22) from the text assumes the form

\[
\begin{align*}
  u_1' + xu_2' + x^2 u_3' &= 0, \\
  -u_1' + (1 - x)u_2' + (2x - x^2)u_3' &= 0, \\
  u_1' + (x - 2)u_2' + (2 - 4x + x^2)u_3' &= \frac{2}{1 + x^2},
\end{align*}
\]

where we have divided each equation by \( e^{-x}. \) Solving the system we obtain \( u_1' = \frac{x^2}{1 + x^2}, \quad u_2' = \frac{-2x}{1 + x^2}, \) and \( u_3' = \frac{1}{1 + x^2} \) from which it follows that \( u_1 = x - \tan^{-1} x, \quad u_2 = -\ln (x^2 + 1), \) and \( u_3 = \tan^{-1} x. \) Thus,

\[
y_p(x) = u_1 y_1 + u_2 y_2 + u_3 y_3 = (x - \tan^{-1} x)e^{-x} - x e^{-x} \ln (x^2 + 1) + x^2 e^{-x} \tan^{-1} x.
\]

Therefore, the general solution to the differential equation is

\[ y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + (x - \tan^{-1} x)e^{-x} - x e^{-x} \ln (x^2 + 1) + x^2 e^{-x} \tan^{-1} x. \]

23. Given \( y'' - y = F(x) \implies y_c(x) = c_1 e^x + c_2 e^{-x}. \) Choose \( y_1(x) = e^x \) and \( y_2(x) = e^{-x}. \) Then \( W[y_1, y_2](x) = (e^x)(-e^{-x}) - (e^{-x})(e^x) = -2. \) Hence, \( K(x, t) = \frac{e^t e^{-x} - e^{-t} e^x}{e^t - e^{-t}} = \frac{1}{2} (e^{x-t} - e^{-x-t}) = \sinh (x - t), \) so that

\[ y_p(x) = \int_{x_0}^{x} \sinh (x - t) F(t) dt. \]

25. Given \( y'' + 5y' + 4y = F(x) \implies y_c(x) = c_1 e^{-4x} + c_2 e^{-x}. \) Choose \( y_1(x) = e^{-4x} \) and \( y_2(x) = e^{-x}. \) Therefore, the general solution is \( W[y_1, y_2](x) = (e^{-4x})(-e^{-x}) - (e^{-x})(-4e^{-4x}) = 3e^{-5x}. \) Hence, \( K(x, t) = \frac{e^{-4t} e^{-x} - e^{-t} e^{-4x}}{3e^{-5t}} = \frac{1}{3} [e^{-x} - e^{4(t-x)}], \) so that

\[ y_p(x) = \frac{1}{3} \int_{x_0}^{x} [e^{t-x} - e^{4(t-x)}] F(t) dt. \]

27. Given \( y'' + y = \sec x \implies y_c(x) = c_1 \cos x + c_2 \sin x. \) Choose \( y_1(x) = \cos x \) and \( y_2(x) = \sin x. \) Then \( W[y_1, y_2](x) = (\cos x) (\cos x) - (\sin x) (-\sin x) = 1. \) Hence, \( K(x, t) = \cos t \sin x - \sin t \cos x, \) so that

\[ y_p(x) = \int_{0}^{x} \cos t \sin x - \sin t \cos x (\sec t) dt = \int_{0}^{x} (\sin x - \tan t \cos x) dt. \]

Thus \( y_p(x) = x \sin x + \ln (\cos x) \cos x. \) Consequently, \( y(x) = c_1 \cos x + c_2 \sin x + x \sin x + \ln (\cos x) \cos x. \) Then \( y(0) = 0 \) yields \( c_1 = 0 \) and \( y'(0) = 1 \) yields \( c_2 = 1. \) Therefore,

\[ y(x) = \sin x + x \sin x + \ln (\cos x) \cos x. \]
29. Given \( y'' - 2ay' + a^2y = F(x) \implies y_e(x) = c_1e^{ax} + c_2xe^{ax} \). Choose \( y_1(x) = e^{ax} \) and \( y_2(x) = xe^{ax} \). Then \( W[y_1, y_2](x) = e^{2ax} \). Hence, \( K(x, t) = \frac{e^{atxe^{ax}} - te^{atxe^{ax}}}{e^{2at}} = e^{a(x-t)}(x-t). \)

(a) 
\[ y_p(x) = \alpha \int_0^x e^{a(x-t)}(x-t) \frac{e^{at}}{t^2 + \beta^2} dt = \alpha e^{ax} \int_0^x \left( \frac{x}{t^2 + \beta^2} - \frac{t}{t^2 + \beta^2} \right) dt. \]

Thus \( y_p(x) = \alpha e^{ax} \left[ \frac{x}{\beta} \tan^{-1} \left( \frac{x}{\beta} \right) - \frac{1}{2} \ln \left( \frac{x^2 + \beta^2}{\beta^2} \right) \right]. \)

(b) 
\[ y_p(x) = \alpha \int_0^x e^{a(x-t)}(x-t) \frac{e^{at}}{(\beta^2 - t^2)^{1/2}} dt = \alpha e^{ax} \int_0^x \left[ \frac{x}{(\beta^2 - t^2)^{1/2}} - \frac{t}{(\beta^2 - t^2)^{1/2}} \right] dt. \]

Thus \( y_p(x) = \alpha e^{ax} \left[ x \sin^{-1} \left( \frac{x}{\beta} \right) + (\beta^2 - x^2)^{1/2} - \beta^2 \right]. \)

(c) 
\[ y_p(x) = \alpha \int_0^x e^{a(x-t)}(x-t) e^{at} \ln t dt = -\alpha e^{ax} \int_0^x \ln t + x e^{ax} \int_0^x e^{at} \ln t dt. \]

If \( \alpha = -1 \) then \( y_p(x) = \frac{x e^{ax}}{2} \left[ (\ln x)^2 - 2 \ln x - 2 \right]. \)

If \( \alpha = -2 \) then \( y_p(x) = -\alpha e^{ax} \left[ \frac{(\ln x)^2}{2} - \ln x + 1 \right]. \)

If \( \alpha \neq -1 \) and \( \alpha \neq -2 \) then \( y_p(x) = \frac{e^{ax}x^{\alpha+2}}{(\alpha + 1)(\alpha + 2)} \left[ \ln x - \frac{2\alpha + 3}{(\alpha + 1)(\alpha + 2)} \right]. \)

31. The complementary equation for the differential equation is \( y_e(x) = c_1e^{rx} + c_2xe^{rx} + c_3x^2e^{rx} \). Let \( y_1(x) = e^{rx}, \quad y_2(x) = xe^{rx}, \) and \( y_3(x) = x^2e^{rx} \). The system (6.7.22) in the text therefore assumes the form

\[
\begin{align*}
0 &= u_1' + tu_2' + t^2u_3' \\
0 &= ru_1' + (rt + 1)u_2' + (r^2t + 2t)u_3' \\
r^2e^{rt}u_1' + (r^2 + 2r)e^{rt}u_2' + (r^2 + 4rt + 2)e^{rt}u_3' &= F(x)
\end{align*}
\]

Row reducing the augmented matrix,
\[
\begin{bmatrix}
1 & t & t^2 & 0 \\
t & rt + 1 & rt^2 + 2t & 0 \\
r^2e^{rt} & (r^2t + 2r)e^{rt} & (r^2t^2 + 4rt + 2)e^{rt} & F(x)
\end{bmatrix},
\]

we obtain
\[
\begin{bmatrix}
1 & t & t^2 & 0 \\
0 & 1 & 2t & 0 \\
0 & 0 & 1 & F(x) / 2e^{rt}
\end{bmatrix}.
\]

By back-substitution, we find that \( u_3' = \frac{F(x)}{2e^{rt}}, \quad u_2' = \frac{tF(x)}{e^{rt}} \) and \( u_1' = \frac{t^2F(x)}{2e^{rt}} \). Thus,
\[
\begin{align*}
u_1(x) &= \frac{1}{2} \int_a^x \frac{t^2F(x)}{2e^{rt}} dt, \\
u_2(x) &= -\frac{1}{2} \int_a^x \frac{2tF(x)}{e^{rt}} dt, \\
u_3(x) &= \frac{1}{2} \int_a^x \frac{F(x)}{2e^{rt}} dt.
\end{align*}
\]
Hence,

\[ y_p(x) = e^{\gamma x}u_1 + xe^{\gamma x}u_2 + x^2 e^{\gamma x}u_3 \]

\[ = \frac{e^{\gamma x}}{2} \int_a^x t^2 F(t) \, dt - \frac{x e^{\gamma x}}{2} \int_a^x 2t F(t) \, dt + \frac{x^2 e^{\gamma x}}{2} \int_a^x F(t) \, dt \]

\[ = \frac{1}{2} \int_a^x F(t)(t^2 - 2tx + x^2)e^{(\gamma - t)x} \, dt \]

\[ = \frac{1}{2} \int_a^x F(t)(x-t)^2 e^{(\gamma - t)x} \, dt. \]

33. Three linearly independent solutions to the associated homogeneous problem are \( y_1(x) = e^{-3x}, y_2(x) = e^{3x}, \) and \( y_3(x) = e^{-5x}. \) Then

\[ W[y_1, y_2, y_3](t) = \begin{vmatrix} e^{-3t} & e^{3t} & e^{-5t} \\ -3e^{-3t} & 3e^{3t} & -5e^{-5t} \\ e^{-3t} & 3e^{3t} & 25e^{-5t} \end{vmatrix} = 96e^{-5t}, \]

\[ \tilde{W}_1 = \begin{vmatrix} 0 & e^{3t} & e^{-5t} \\ 0 & 3e^{3t} & -5e^{-5t} \\ 1 & 9e^{3t} & 25e^{-5t} \end{vmatrix} = -8e^{-2t}, \]

\[ \tilde{W}_2 = \begin{vmatrix} e^{-3t} & 0 & e^{-5t} \\ -3e^{-3t} & 0 & -5e^{-5t} \\ e^{-3t} & 0 & 25e^{-5t} \end{vmatrix} = 2e^{-8t}, \]

\[ \tilde{W}_3 = \begin{vmatrix} e^{-3t} & e^{3t} & 0 \\ -3e^{-3t} & 3e^{3t} & 0 \\ 9e^{-3t} & 9e^{3t} & 1 \end{vmatrix} = 6. \]

So, \( K(x, t) = \frac{e^{-3x}(-8e^{-2t}) + e^{3x}(2e^{-8t}) + e^{-5x}(6)}{96e^{-5t}} = \frac{1}{48}[e^{3x}(x-t) + 3e^{-5(x-t)} - 4e^{-3(x-t)}]. \) Hence,

\[ y_p(x) = \frac{1}{48} \int_{x_0}^x [e^{3(x-t)} + 3e^{-5(x-t)} - 4e^{-3(x-t)}]F(t) \, dt. \]

35. Three linearly independent solutions to the associated homogeneous problem are \( y_1(x) = e^{-4x}, y_2(x) = xe^{-4x}, \) and \( y_3(x) = e^{2x}. \) Then

\[ W[y_1, y_2, y_3](t) = \begin{vmatrix} e^{-4t} & te^{-4t} & e^{2t} \\ -4e^{-4t} & (1 - 4t)e^{-4t} & 2e^{2t} \\ 16e^{-4t} & (-8 + 16t)e^{-4t} & 4e^{2t} \end{vmatrix} = 36e^{-6t}, \]

\[ \tilde{W}_1 = \begin{vmatrix} 0 & te^{-4t} & e^{2t} \\ 0 & (1 - 4t)e^{-4t} & 2e^{2t} \\ 1 & (-8 + 16t)e^{-4t} & 4e^{2t} \end{vmatrix} = e^{-2t}(6t - 1), \]

\[ \tilde{W}_2 = \begin{vmatrix} e^{-4t} & 0 & e^{2t} \\ -4e^{-4t} & 0 & 2e^{2t} \\ 16e^{-4t} & 1 & 4e^{2t} \end{vmatrix} = -6e^{-2t}, \]

\[ \tilde{W}_3 = \begin{vmatrix} e^{-4t} & te^{-4t} & 0 \\ -4e^{-4t} & (1 - 4t)e^{-4t} & 0 \\ 16e^{-4t} & (-8 + 16t)e^{-4t} & 1 \end{vmatrix} = e^{-8t}. \]

Hence, \( K(x, t) = \frac{e^{-2t}(6t - 1)e^{-4x} - 6e^{-2t}(xe^{-4x}) + e^{-8t}(e^{2x})}{36e^{-6t}} = \frac{1}{36}[e^{4(t-x)}(6(t-x) - 1) + e^{2(x-t)}]. \) Consequently,

\[ y_p(x) = \frac{1}{36} \int_{x_0}^x [e^{4(t-x)}(6(t-x) - 1) + e^{2(x-t)}]F(t) \, dt. \]
37. Three linearly independent solutions to the associated homogeneous problem are \(y_1(x) = e^x\), \(y_2(x) = e^{2x}\), and \(y_3(x) = e^{-4x}\). Then

\[
W[y_1, y_2, y_3](t) = \begin{vmatrix}
e^t & e^{2t} & e^{-4t} \\
e^t & 2e^{2t} & -4e^{-4t} \\
e^t & 4e^{2t} & 16e^{-4t}
\end{vmatrix}
\]

Furthermore,

\[
\tilde{W}_1(t) = \begin{vmatrix}
0 & e^{2t} & e^{-4t} \\
0 & 2e^{2t} & -4e^{-4t} \\
1 & 4e^{2t} & 16e^{-4t}
\end{vmatrix} = -6e^{-2t}, \quad \tilde{W}_2(t) = \begin{vmatrix}
e^t & 0 & e^{-4t} \\
e^t & 0 & -4e^{-4t} \\
e^t & 1 & 16e^{-4t}
\end{vmatrix} = 5e^{-3t}, \quad \tilde{W}_3(t) = \begin{vmatrix}
e^t & e^{2t} & 0 \\
e^t & 2e^{2t} & 0 \\
e^t & 4e^{2t} & 1
\end{vmatrix} = e^{3t}.
\]

Consequently,

\[
K(x, t) = \frac{1}{30e^{-t}} \left[ e^t(-6e^{-2t}) + e^{2x}(5e^{-3t}) + e^{-4x}(e^{3t}) \right] = \frac{1}{30e^{-t}} \left[ e^{-4(x-t)} + 5e^{2(x-t)} - 6e^{x-t} \right],
\]

and a particular solution to the given differential equation is

\[
y_p(x) = \frac{1}{30} \int_0^x \left[ e^{-4(x-t)} + 5e^{2(x-t)} - 6e^{x-t} \right] F(t)dt.
\]

**Solutions to Section 6.8**

**True-False Review:**

1. **False.** First of all, the given equation only addresses the case of a second-order Cauchy-Euler equation. Secondly, the term containing \(y'\) must contain a factor of \(x\) (see Equation 6.8.1):

\[
x^2 y'' + a_1 xy' + a_2 y = 0.
\]

2. **False.** The indicial equation (6.8.2) in this case is

\[
r(r-1) + 9r + 16 = r^2 + 8r + 16 = (r+4)^2 = 0,
\]

and so we have only one real root, \(r = 4\). Therefore, the only solutions to this Cauchy-Euler equation of the form \(y = x^r\) take the form \(y = cx^{-4}\). Therefore, only one linearly independent solution of the form \(y = x^r\) to the Cauchy-Euler equation has been obtained.

3. **True.** A solution obtained by the method in this section containing the function \(\ln x\) implies a repeated root to the indicial equation. In fact, such a solution takes the form \(y_2(x) = x^{r'\ln x}\). Therefore, if \(y(x) = \ln x\) is a solution, we conclude that \(r_1 = r_2 = -1\) is the only root of the indicial equation \(r(r-1)-a_1r+a_2 = 0\). From Case 2 in this section, we see that \(-1 = -\frac{a_1-1}{2}\), and so \(a_1 = 3\). Moreover, the discriminant of Equation (6.8.3) must be zero:

\[
(a_1 - 1)^2 - 4a_2 = 0.
\]

That is, \(2^2 - 4a_2 = 0\). Therefore, \(a_2 = 1\). Therefore, the Cauchy-Euler equation in question, with \(a_1 = 3\) and \(a_2 = 1\), must be as given.

**Problems:**

1. We are given that \(x^2 y'' - xy' + 5y = 0\). If \(y(x) = x^r\) then the indicial equation is \(r^2 - 2r + 5 = 0 \implies r = 1 \pm 2i \implies y_1(x) = x \sin(2 \ln x), \quad y_2(x) = x \cos(2 \ln x) \implies y(x) = x_1 \sin(2 \ln x) + c_2 \cos(2 \ln x)].\)
3. We are given that \(x^2y'' - 3xy' + 4y = 0\). If \(y(x) = x^r\) then the indicial equation is \(r^2 - 4r + 4 = 0 \Rightarrow r \in \{2, 2\} \Rightarrow y_1(x) = x^2, y_2(x) = x^2 \ln x \Rightarrow y(x) = x^2(c_1 + c_2 \ln x)\).

5. We are given that \(x^2y'' + 3xy' + y = 0\). If \(y(x) = x^r\) then the indicial equation is \(r^2 + 2r + 1 = 0 \Rightarrow r \in \{-1, -1\} \Rightarrow y_1(x) = x^{-1}, y_2(x) = x^{-1} \ln x \Rightarrow y(x) = x^{-1}(c_1 + c_2 \ln x)\).

7. We are given that \(x^2y'' - xy' - 35y = 0\). If \(y(x) = x^r\) then the indicial equation is \(r^2 - 2r - 35 = 0 \Rightarrow r \in \{-5, 7\} \Rightarrow y_1(x) = x^7, y_2(x) = x^{-5} \Rightarrow y(x) = c_1x^7 + c_2x^{-5}\).

9. If \(y(x) = x^r\) then the indicial equation is \(r^2 - m^2 = 0 \Rightarrow r \in \{-m, m\} \Rightarrow y(x) = c_1x^m + c_2x^{-m}\).

11. If \(y(x) = x^r\) then the indicial equation is \(r^2 - 2mr + (m^2 + k^2) = 0 \Rightarrow r \in \{m + ki, m - ki\} \Rightarrow y(x) = x^m(c_1 \cos(k \ln x) + c_2 \sin(k \ln x))\).

13. If \(y(x) = (-x)^r, y' = r(-x)^{-1}, y'' = r(r - 1)(-x)^{-2}\). Substituting these results into the original equation and simplifying yields \(x^2 r(r - 1)(-x)^{-2} + ax(r)(-x)^{-1} + b(-x)^r = 0 \Rightarrow r^2 - r + ar + b = 0 \Rightarrow r^2 + r(a - 1) + b = 0\).

15. Consider \(x^2y'' - 4xy' + 6y = 0\). For \(y(x) = x^r\), the indicial equation is given by \(r^2 - 5r + 6 = 0 \Rightarrow r \in \{2, 3\} \Rightarrow y_c(x) = c_1x^2 + c_2x^3\). Let \(y_1(x) = x^2\) and \(y_2(x) = x^3\). Then \(W[y_1, y_2](x) = (x^2)(3x^2) - (x^3)(2x) = x^4 \neq 0\) for \(x > 0\), so that

\[
y_p(x) = -y_1 \int \frac{y_2F}{W} dx + y_2 \int \frac{y_1F}{W} dx = -x^2 \int \frac{x^3(x^4 \sin x)}{x^2 x^4} dx + x^3 \int \frac{x^2(x^4 \sin x)}{x^2 x^4} dx
\]

\[
= -x^2 \int x \sin x dx + x^3 \int \sin x dx = -x^2 \sin x.
\]

Thus, \(y(x) = c_1x^2 + c_2x^3 - x^2 \sin x\).

17. Consider \(x^2y'' - 3xy' + 4y = 0\). For \(y(x) = x^r\), the indicial equation is given by \(r^2 - 4r + 4 = 0 \Rightarrow r \in \{2, 2\} \Rightarrow y_c(x) = c_1x^2 + c_2x^2 \ln x\). Let \(y_1(x) = x^2\) and \(y_2(x) = x^2 \ln x\). Then \(W[y_1, y_2](x) = (x^2)(2x \ln x) - (x^2 \ln x)(2x) = x^3 \neq 0\) for \(x > 0\), so that

\[
y_p(x) = -y_1 \int \frac{y_2F}{W} dx + y_2 \int \frac{y_1F}{W} dx = -x^2 \int \frac{x^2 \ln x(x^2/ \ln x)}{x^2 x^3} dx + x^2 \ln x \int \frac{x^2(x^2/ \ln x)}{x^2 x^3} dx
\]

\[
= -x^2 \ln x + x^2 \ln x \ln |x|
\]

for \(x > 0\). Hence, \(y(x) = x^2[c_1 + c_2 \ln x + \ln x (| \ln x |)]\).

19. Consider \(x^2y'' + xy' + 9y = 0\). For \(y(x) = x^r\), the indicial equation is given by \(r^2 + 9 = 0 \Rightarrow r \in \{-3i, 3i\} \Rightarrow y_c(x) = c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)\). Let \(y_1(x) = \cos(3 \ln x)\) and \(y_2(x) = \sin(3 \ln x)\). Then

\[
W[y_1, y_2](x) = [\cos(3 \ln x)] \left[3 \cos(3 \ln x) \right] - [\sin(3 \ln x)] \left[-3 \sin(3 \ln x) \right] = \frac{3}{x} \neq 0
\]

for \(x > 0\), so that

\[
y_p(x) = -y_1 \int \frac{y_2F}{W} dx + y_2 \int \frac{y_1F}{W} dx = -\cos(3 \ln x) \int \frac{\sin(3 \ln x)9 \ln x}{x^2(3/x)} dx + \sin(3 \ln x) \int \frac{\cos(3 \ln x)(9 \ln x)}{x^2(3/x)} dx.
\]
Making the change of variables $u = 3 \ln x$ in the preceding integrals yields

$$y_p(x) = -\frac{1}{3} \cos(3 \ln x) \int u \sin u du + \frac{1}{3} \sin(3 \ln x) \int u \cos u du$$

$$= -\frac{1}{3} \cos(3 \ln x)(-u \cos u + \sin u) + \frac{1}{3} \sin(3 \ln x)(u \sin u + \cos u) = \frac{1}{3} \ln x.$$

Consequently, $y(x) = c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x) + \frac{1}{3} \ln x.$

21. Consider $x^2 y'' - (2m - 1)xy' + m^2 y = 0$. For $y(x) = x^r$, the indicial equation is given by $r^2 - 2mr + m^2 = 0$ $\implies r \in \{m, -m\} \implies y_c(x) = c_1 x^m + c_2 x^{-m} \ln x$. Let $y_1(x) = x^m$ and $y_2(x) = x^m \ln x$. Then

$$W[y_1, y_2](x) = (x^m)[x^{m-1}(1 + m \ln x)] - (x^m \ln x)(mx^{m-1}) = x^{2m-1} \neq 0$$

for $x > 0$, so that

$$y_p(x) = -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx = -x^m \int \frac{x^m \ln x[x^m(\ln x)^k]}{x^2(x^2)^{m-1}} dx + x^m \ln x \int \frac{x^m[x^m(\ln x)^k]}{x^2(x^2)^{m-1}} dx$$

$$= -x^m \int (\ln x)^{k+1} \frac{dx}{x} + x^m \ln x \int (\ln x)^k \frac{dx}{x}.$$ (21.1)

Case 1: If $k \neq -1$ and $k \neq -2$, then Equation (21.1) becomes

$$y_p(x) = -\frac{x^m(\ln x)^{k+2}}{k+2} + x^m \ln x \left(\frac{(\ln x)^{k+1}}{k+1}\right)^{(k+1)(k+2)}.$$

Case 2: If $k = -1$, then Equation (21.1) becomes

$$y_p(x) = -x^m \int \frac{1}{x} dx + x^m \ln x \int \frac{1}{x \ln x} dx = (\ln |\ln x| - 1)x^m \ln x.$$

Case 3: If $k = -2$ then Equation (21.1) becomes

$$y_p(x) = -x^m \int \frac{1}{x \ln x} dx + x^m \ln x \int \frac{1}{x(\ln x)^2} dx = -x^m(1 + \ln |\ln x|).$$

23.

(a) The indicial equation is $r(r - 1) + r + 25 = 0$ $\implies r \in \{5i, -5i\}$. The general solution to the differential equation is therefore $y(t) = c_1 \cos(5 \ln t) + c_2 \sin(5 \ln t)$. Given $y(1) = \frac{3\sqrt{3}}{2}$ then $c_1 = \frac{3\sqrt{3}}{2}$ and $y'(1) = \frac{15}{2}$ then $c_2 = \frac{3}{2}$. Hence, the solution to the initial-value problem is

$$y(t) = 3 \left[\frac{\sqrt{3}}{2} \cos(5 \ln t) + \frac{1}{2} \sin(5 \ln t)\right] = 3 \cos(5 \ln t - \pi/6).$$

(b) The zeros of $y(t)$ occur when $5 \ln t - \pi/6 = (2n + 1)\pi/2$, $n = 0, \pm 1, \pm 2, ...$

Solving for $t$ yields

$$t = e^{(6n+5)/30}.$$
The motion is oscillatory but not periodic. Consequently the system is not performing simple harmonic motion.

Solutions to Section 6.9

Problems:

1. We are given that \( y_1(x) = x^2 \). Let \( y_2(x) = x^2 u \) so \( y_2' = 2xu + x^2 u' \), and \( y_2'' = x^2 u'' + 4xu' + 2u \). Substituting these results into the given differential equation and simplifying we obtain \( xu'' + u' = 0 \implies \frac{u''}{u'} = -\frac{1}{x} \implies u(x) = c_1 \ln x + c_2 \). Letting \( c_1 = 1 \) and \( c_2 = 0 \), we obtain a second linearly independent solution, \( y_2(x) = x^2 \ln x \).

3. We are given \( y_1(x) = e^x \). Let \( y_2(x) = ue^x \) so that \( y_2' = e^x u + e^x u' \), and \( y_2'' = e^x u + 2e^x u' + e^x u'' \). Substituting these results into the given differential equation and simplifying we obtain \( xu'' + u' = 0 \implies \frac{u''}{u'} = -\frac{1}{x} \implies u' = \frac{c_1}{x} \implies u(x) = c_1 \ln x + c_2 \). Letting \( c_1 = 1 \) and \( c_2 = 0 \), we obtain a second linearly independent solution, \( y_2(x) = e^x \ln x \).

5. We are given \( y_1(x) = x \) and \(-1 < x < 1 \). Let \( y_2(x) = ux \) so that \( y_2' = u'x + u \) and \( y_2'' = u''x + 2u' \). Substituting these results into the original equation and simplifying we obtain \( u''(x - x^3) + u'(2 - 4x^2) = 0 \implies \frac{u''}{u'} = \frac{4x^2 - 2}{x - x^3} \implies \frac{u''}{u'} = -\frac{2}{x} - \frac{1}{1 + x} + \frac{1}{1 - x} \implies \ln |u'| = -\ln |x| - \ln |1 + x| + \ln |1 - x| + c_1 \implies u'(x) = \frac{c_1}{(1 - x^2)x^2}, -1 < x < 1 \implies u(x) = c_2 \left[ \frac{1}{2} \ln \left( \frac{x + 1}{1 - x} \right) - \frac{1}{x} \right] + c_3 \). Letting \( c_2 = 1 \) and \( c_3 = 0 \) we obtain a second linearly independent solution, \( y_2(x) = \frac{1}{2} x \ln \left( \frac{x + 1}{1 - x} \right) - 1 \).
7.
(a) If \( y_1(x) = x^\lambda \), then the corresponding indicial equation is given by \( \lambda^2 - 2m\lambda + m^2 = 0 \implies \lambda \in \{m, m\} \) so one particular solution of the given differential equation is \( y_1(x) = x^m \).

(b) Let \( y_2(x) = x^m u \) so \( y'_2 = x^m u' + m x^{m-1} u \) and \( y''_2 = x^m u'' + m x^{m-2} u + 2 m x^{m-1} u' \). Substituting these results into the original equation and simplifying yields \( x^m u' + x^{m+1} u'' = 0 \implies \frac{u''}{u'} = -\frac{1}{x} \implies u'(x) = \frac{c}{x} \implies u(x) = c \ln x + c_1 \). Letting \( c = 1 \) and \( c_1 = 0 \) we obtain a second linearly independent solution, \( y_2(x) = x^m \ln x \).

9.
(a) If \( y_1(x) = e^{\alpha x} \) then \( y'_1(x) = \alpha e^{\alpha x} \) and \( y''_1(x) = \alpha^2 e^{\alpha x} \) so from the original equation we have \( x y''_1 + (\alpha + \beta) y'_1 + \alpha \beta y_1 = x (\alpha^2 e^{\alpha x}) - (\alpha x + \beta) (\alpha e^{\alpha x}) + \alpha e^{\alpha x} = \alpha^2 x - (\alpha x + \beta) \alpha + \alpha \beta = 0 \). Hence, \( y_1(x) = e^{\alpha x} \) is a solution to the differential equation.

(b) If \( y(x) = e^{\alpha x} u \), then \( y' = (u' + \alpha u) e^{\alpha x} \) and \( y'' = [u'' + \alpha (2u' + \alpha u)] e^{\alpha x} \). Substituting these results into the original equation and simplifying yields \( x u'' + (\alpha x - \beta) u' = 0 \implies \frac{u''}{u'} = -\alpha + \beta x^{-1} \implies u' = c_1 x^\beta e^{-\alpha x} \implies u(x) = c_1 \int x^\beta e^{-\alpha x} dx + c_2 \). Letting \( c_1 = 1 \) and \( c_2 = 0 \) we obtain \( y_2(x) = \int x^\beta e^{-\alpha x} dx \) as a second linearly independent solution.

(c) If \( \alpha = 1 \) and \( \beta \) is a non-negative integer then \( y_2(x) = e^x \int x^\beta e^{-x} dx \). Use repeated integration by parts:

\[
y_2(x) = e^x \left( -x^\beta e^{-x} + \beta \int x^{\beta-1} e^{-x} dx \right) = -\beta! \left[ \frac{x^\beta}{\beta!} - \frac{e^x}{(\beta-1)!} \int x^{\beta-1} e^{-x} dx \right] = -\beta! \left[ \frac{x^\beta}{\beta!} - \frac{e^x}{(\beta-1)!} \left( -x^{\beta-1} e^{-x} + (\beta - 1) \int x^{\beta-2} e^{-x} dx \right) \right] = -\beta! \left[ \frac{x^\beta}{\beta!} + \frac{e^x}{(\beta-1)!} \right] + \beta! \frac{e^x}{(\beta-2)!} \int x^{\beta-2} e^{-x} dx \right] = -\beta! \left[ \frac{x^\beta}{\beta!} + \frac{e^x}{(\beta-1)!} + \cdots + \frac{e^x}{(\beta+1)!} \right].
\]

11. We are given that \( y_1(x) = e^{2x} \). If \( y_2(x) = u e^{2x} \), then \( y'_2 = (u' + 2u) e^{2x} \) and \( y''_2 = (u'' + 4u' + 4u) e^{2x} \). Substituting these results into the original equation and simplifying yields \( u'' \sin x + 2u' \cos x = \csc x \implies u'' \sin^2 x + 2u' \sin x \cos x = 1 \implies u' = \frac{u}{\sin^2 x} + \frac{c_3}{\sin^2 x} \implies u(x) = -x \cot x + \ln (\sin x) - c_3 \cot x + c_2 \). Hence, the general solution is \( y(x) = c_1 \cos x + c_2 \sin x [\ln (\sin x) - x \cot x] \).

13. We are given that \( y_1(x) = \sin x \) where \( 0 < x < \pi \). If \( y_2(x) = u \sin x \), then \( y'_2 = u' \sin x + u \cos x \) and \( y''_2 = u'' \sin x + 2u' \cos x - u \sin x \). Substituting these results into the original equation and simplifying yields \( u'' \sin x + 2u' \cos x = \csc x \implies u'' \sin^2 x + 2u' \sin x \cos x = 1 \implies u' = u \sin^{-2} x + \frac{c_3}{\sin^2 x} \implies u(x) = -x \cot x + \ln (\sin x) - c_3 \cot x + c_2 \). Hence, the general solution is \( y(x) = c_1 \cos x + c_2 \sin x [\ln (\sin x) - x \cot x] \).
15. We are given that \( y_1(x) = x^2 \), where \( x > 0 \). If \( y_2(x) = ux^2 \) then \( y_2' = u'x^2 + 2ux \) and \( y_2'' = u''x^2 + 4u'x + 2u \). Substituting these results into the original differential equation and simplifying we obtain
\[
 u''x + u' = 8x \implies xu' = 4x^2 + c_2 \implies u(x) = 2x^2 + c_2 \ln x + c_1.
\]
Hence, the general solution is
\[
y(x) = x^2(c_1 + c_2 \ln x + 2x^2).
\]

**Solutions to Section 6.10**

**Problems:**

1. \( Ly = (D^2 + 3)(e^{x^3}) = D^2(e^{x^3}) + 3e^{x^3} = D(3x^2e^{x^3}) + 3e^{x^3} = e^{x^3}(9x^4 + 6x) + 3e^{x^3} = 3e^{x^3}(3x^4 + 2x + 1) \).

3. \( Ly = \left( \frac{1}{x}D^2 + xD - 2 \right)(4 \sin x) = \frac{1}{x}D^2(4 \sin x) + xD(4 \sin x) - 2(4 \sin x) = -\frac{4}{x} \sin x + 4x \cos x - 8 \sin x. \)

5. \( Ly = [(x^2 + 1)D^3 - (\cos x)D + 5x^2](\ln x + 8x^5) = (x^2 + 1)D^3(\ln x + 8x^5) - (\cos x)D(\ln x + 8x^5) + 5x^2(\ln x + 8x^5) = (x^2 + 1) \left( \frac{D}{x} + 40x^4 \right) - (\cos x) \left( \frac{D}{x} + 40x^4 \right) + 5x^2(\ln x + 8x^5). \)

7. \( P(r) = r^3 + 3r^2 - 4 = (r - 1)(r + 2)^2 = 0 \implies r = 1, r = -2 \) (multiplicity 2).

General solution:
\[
y(x) = c_1e^r + c_2e^{-2x} + c_3xe^{-2x}.
\]

9. \( P(r) = r^4 + 13r^2 + 36 = (r^2 + 4)(r^2 + 9) \implies r = \pm 2i, r = \pm 3i. \)

General solution:
\[
y(x) = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 3x + c_4 \sin 3x.
\]

11. \( P(r) = (r + 3)^2(r^2 - 4r + 13) \implies r = -3 \) (multiplicity 2), \( r = 2 \pm 3i \).

General solution:
\[
y(x) = c_1e^{-3x} + c_2xe^{-3x} + e^{2x}(c_3 \cos 3x + c_4 \sin 3x).
\]

13. \( P(r) = (r^2 + 4r + 4)(r - 3) = (r + 2)^2(r - 3) \implies r = -2 \) (multiplicity 2), \( r = 3 \).

General solution:
\[
y(x) = c_1e^{-2x} + c_2xe^{-2x} + c_3e^{3x}.
\]

15. \( A(D) = D^2 - 6D + 10 \).

17. \( A(D) = (D^2 + 1)^2(D + 2) \).

19. In operator form the given differential equation is
\[
(D^2 + 6D + 9)y = 4e^{-2x},
\]
that is
\[
(D + 3)^2y = 4e^{-2x}.
\] (0.0.22)

Therefore the complementary function is
\[
y_c(x) = c_1e^{-3x} + c_2xe^{-3x}.
\]
The annihilator of $F(x) = e^{-2x}$ is $A(D) = D + 2$. Operating on (0.0.22) with $D + 2$ yields

$$(D + 2)(D + 3)^2 y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 e^{-2x}.$$  

Consequently, an appropriate trial solution for (0.0.22) is

$$y_p(x) = A_0 e^{-2x}.$$

21. In operator form the given differential equation is

$$(D^3 - 6D^2 + 25D)y = \sin 4x,$$

that is

$$D(D^2 - 6D + 25)y = \sin 4x. \quad (0.0.23)$$

Therefore the complementary function is

$$y_c(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + c_3.$$  

The annihilator of $F(x) = \sin 4x$ is $A(D) = D^2 + 16$. Operating on (0.0.23) with $D^2 + 16$ yields

$$(D^2 + 16)D(D^2 - 6D + 25)y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 \cos 4x + A_1 \sin 4x.$$  

Consequently, an appropriate trial solution for (0.0.23) is

$$y_p(x) = A_0 \cos 4x + A_1 \sin 4x.$$  

23. In operator form the given differential equation is

$$(D^6 + 3D^4 + 3D^2 + 1)y = 2 \sin x,$$

that is

$$(D^2 + 1)^3 y = 2 \sin x. \quad (0.0.24)$$

Therefore the complementary function is

$$y_c(x) = c_1 \cos x + c_2 \sin x + x(c_3 \cos x + c_4 \sin x) + x^2(c_5 \cos x + c_6 \sin x).$$  

The annihilator of $F(x) = 2 \sin x$ is $A(D) = D^2 + 1$. Operating on (0.0.24) with $D^2 + 1$ yields

$$(D^2 + 1)^4 y = 0$$

which has general solution

$$y(x) = y_c(x) + x^3(A_0 \cos x + A_1 \sin x).$$  

Consequently, an appropriate trial solution for (0.0.24) is

$$y_p(x) = x^3(A_0 \cos x + A_1 \sin x).$$
25.

(a) From Problem 20 the complementary function for the given differential equation is

\[ y_c(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + c_3, \]

and an appropriate trial solution is

\[ y_p(x) = A_0 x + A_1 x^2 + A_2 x^3. \]

Inserting this expression for \( y_p(x) \) into the given differential equation yields

\[ 6A_2 - 6(2A_1 + 6A_2 x) + 25(A_0 + 2A_1 x + 3A_2 x^2) = x^2, \]

so that \( A_0, A_1, A_2 \) must satisfy

\[ 25A_0 - 12A_1 + 6A_2 = 0, \quad 50A_1 - 36A_2 = 0, \quad 75A_2 = 1. \]

Hence,

\[ A_0 = \frac{22}{15625}, \quad A_1 = \frac{6}{625}, \quad A_2 = \frac{1}{75}, \]

so that

\[ y_p(x) = \frac{22}{15625} x + \frac{6}{625} x^2 + \frac{1}{75} x^2. \]

Consequently, the general solution to the given differential equation is

\[ y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + c_3 + \frac{22}{15625} x + \frac{6}{625} x^2 + \frac{1}{75} x^2. \]

(b) Choosing

\[ y_1(x) = 1, \quad y_2(x) = e^{3x} \cos 4x, \quad y_3(x) = e^{3x} \sin 4x, \]

we have

\[ W[y_1, y_2, y_3](x) = \begin{vmatrix} 1 & e^{3x} \cos 4x & e^{3x} \sin 4x \\ 0 & e^{3x}(3 \cos 4x - 4 \sin 4x) & e^{3x}(3 \sin 4x + 4 \cos 4x) \\ 0 & e^{3x}(-7 \cos 4x - 24 \sin 4x) & e^{3x}(-7 \sin 4x + 24 \cos 4x) \end{vmatrix} = 100e^{6x}. \]

Then a particular solution to the given differential equation is

\[ y_p(x) = u_1 + u_2 e^{3x} \cos 4x + u_3 e^{3x} \sin 4x \]

where

\( u_1' + e^{3x} \cos 4x u_2' + e^{3x} \sin 4x u_3' = 0, \)
\( e^{3x}(3 \cos 4x - 4 \sin 4x)u_2' + e^{3x}(3 \sin 4x + 4 \cos 4x)u_3' = 0, \)
\( e^{3x}(-7 \cos 4x - 24 \sin 4x)u_2' + e^{3x}(-7 \sin 4x + 24 \cos 4x)u_3' = x^2. \)

Solving this system yields:

\[ u_1' = \frac{1}{25} x^2, \quad u_2' = -\frac{1}{100} e^{-3x} x^2(3 \sin 4x + 4 \cos 4x), \quad u_3' = \frac{1}{100} e^{-3x} x^2(3 \cos 4x - 4 \sin 4x). \]
Hence,
\[
y_p(x) = \frac{1}{75} x^3 - \frac{1}{100} e^{3x} \cos 4x \int e^{-3x} x^2 (3 \sin 4x + 4 \cos 4x) \, dx \\
+ \frac{1}{100} e^{3x} \sin 4x \int e^{-3x} x^2 (3 \cos 4x - 4 \sin 4x) \, dx
\]
\[
= \frac{1}{75} x^3 + \frac{6}{625} x^2 + \frac{22}{15625} x - \frac{168}{390625},
\]
and the general solution to the given differential equation is
\[
y(x) = e^{3x} (c_1 \cos 4x + c_2 \sin 4x) + c_3 + \frac{1}{75} x^3 + \frac{6}{625} x^2 + \frac{22}{15625} x.
\]

27.

(a) In operator form the given differential equation is
\[
(D^2 - 4)y = 5e^x
\]
that is
\[
(D - 2)(D + 2)y = 5e^x. \tag{0.0.25}
\]
Therefore the complementary function is
\[
y_c(x) = c_1 e^{2x} + c_2 e^{-2x}.
\]
The annihilator of \( F(x) = 5e^x \) is \( A(D) = D - 1 \). Operating on (0.0.25) with \( D - 1 \) yields
\[
(D - 1)(D - 2)(D + 2)y = 0
\]
which has general solution
\[
y(x) = y_c(x) + A_0 e^x.
\]
Consequently, an appropriate trial solution for (0.0.25) is
\[
y_p(x) = A_0 e^x.
\]
Inserting this expression for \( y_p(x) \) into the given differential equation yields
\[
A_0 e^x (1 - 4) = 5e^x
\]
so that \( A_0 = - \frac{5}{3} \). Hence,
\[
y_p(x) = -\frac{5}{3} e^x,
\]
and the general solution to the given differential equation is
\[
y(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{5}{3} e^x.
\]

(b) Choosing \( y_1(x) = e^{2x}, \ y_2(x) = e^{-2x} \), we have \( W[y_1, y_2](x) = -4 \). Hence a particular solution to the given differential equation is
\[
y_p(x) = -e^{2x} \int \frac{e^{-2x} \cdot 5e^x}{-4} \, dx + e^{-2x} \int \frac{e^{2x} \cdot 5e^x}{-4} \, dx = -\frac{5}{3} e^x.
\]
Therefore the differential equation has general solution

\[ y(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{5}{3} e^x. \]

29.

(a) In operator form the given differential equation is

\[ (D^2 - 1)y = 4e^x \]

that is

\[ (D - 1)(D + 1)y = 4e^x. \]  \hspace{1cm} (0.0.26)

Therefore the complementary function is

\[ y_c(x) = c_1 e^x + c_2 e^{-x}. \]

The annihilator of \( F(x) = 4e^x \) is \( A(D) = D - 1 \). Operating on (0.0.26) with \( D - 1 \) yields

\[ (D - 1)^2(D + 1)y = 0 \]

which has general solution

\[ y(x) = y_c(x) + A_0 xe^x. \]

Consequently, an appropriate trial solution for (0.0.26) is

\[ y_p(x) = A_0 xe^x. \]

Differentiating this trial solution with respect to \( x \) yields

\[ y_p'(x) = A_0 e^x(x + 1), \quad y_p''(x) = A_0 e^x(x + 2). \]

Inserting these expressions into the given differential equation yields

\[ A_0 e^x(x + 2) - A_0 xe^x = 4e^x, \]

so that \( A_0 = 2 \). Hence,

\[ y_p(x) = 2xe^x, \]

and the general solution to the given differential equation is

\[ y(x) = c_1 e^x + c_2 e^{-x} + 2xe^x. \]

(b) Choosing \( y_1(x) = e^x \), \( y_2(x) = e^{-x} \), we have \( W[y_1, y_2](x) = -2 \). Hence a particular solution to the given differential equation is

\[ y_p(x) = -e^x \int \frac{e^{-x} \cdot 4e^x}{-2} \, dx + e^{-x} \int \frac{e^x \cdot 4e^x}{-2} \, dx = 2xe^x - e^x. \]

The last term in this expression for \( y_p(x) \) can be omitted since it is part of the complementary function. Consequently, the given differential equation has general solution

\[ y(x) = c_1 e^x + c_2 e^{-x} + 2xe^x. \]
31. The nonhomogeneous term \( F(x) = \ln x \) cannot be annihilated, and therefore the annihilator method cannot be applied.

33. The nonhomogeneous term \( F(x) = \tan x \) cannot be annihilated, and therefore the annihilator method cannot be applied.

35. In operator form the given differential equation is

\[
(D^2 - 8D + 16)y = 7e^{4x}.
\]

that is

\[
(D - 4)^2y = 7e^{4x}. \tag{0.0.27}
\]

Therefore the complementary function is

\[
y_c(x) = c_1 e^{4x} + c_2 xe^{4x}.
\]

The annihilator of \( F(x) = 7e^{4x} \) is \( A(D) = D - 4 \). Operating on (0.0.27) with \( D - 4 \) yields

\[
(D - 4)^3y = 0
\]

which has general solution

\[
y(x) = y_c(x) + A_0 x^2 e^{4x}.
\]

Consequently, an appropriate trial solution for (0.0.27) is

\[
y_p(x) = A_0 x^2 e^{4x}.
\]

37. In operator form the given differential equation is

\[
(D^2 - 2D + 5)y = 7e^{at}(4t + \cos bt). \tag{0.0.29}
\]

Therefore the complementary function is

\[
y_c(x) = c_1 e^{x^2}(c_1 \cos 2x + c_2 \sin 2x).
\]

The annihilator of \( F(x) = 7e^{at} \cos x + \sin x \) is \( A(D) = (D^2 - 2D + 2)(D^2 + 1) \). Operating on (0.0.28) with \( (D^2 - 2D + 2)(D^2 + 1) \) yields

\[
(D^2 - 2D + 5)(D^2 + 1)(D^2 - 2D + 5)y = 0
\]

which has general solution

\[
y(x) = y_c(x) + e^x(A_0 \cos x + A_1 \sin x) + A_2 \cos x + A_3 \sin x.
\]

Consequently, an appropriate trial solution for (0.0.28) is

\[
y_p(x) = e^x(A_0 \cos x + A_1 \sin x) + A_2 \cos x + A_3 \sin x.
\]
Therefore the complementary function is
\[ y_c(t) = e^{at}(c_1 \cos bt + c_2 \sin bt). \]
The annihilator of \( F(t) = e^{at}(4t + \cos bt) \) is \( A(D) = (D - a)^2(D^2 - 2aD + a^2 + b^2) \). Operating on (0.0.29) with \((D - a)^2(D^2 - 2aD + a^2 + b^2)y = 0\) yields
\[ (D - a)^2(D^2 - 2aD + a^2 + b^2)^2y = 0 \]
which has general solution
\[ y(t) = y_c(t) + e^{at}(A_0 + A_1 t) + te^{at}(A_2 \cos 2t + A_3 \sin bt). \]
Consequently, an appropriate trial solution for (0.0.29) is
\[ y_p(t) = e^{at}(A_0 + A_1 t) + te^{at}(A_2 \cos 2t + A_3 \sin bt). \]

41. In operator form the given differential equation is
\[ (D^2 + 2D - 3)y = 2xe^{-3x} \]
that is
\[ (D + 3)(D - 1)y = 2xe^{-3x}. \] (0.0.30)
Therefore the complementary function is
\[ y_c(x) = c_1 e^{-3x} + c_2 e^x. \]
The annihilator of \( F(x) = 2xe^{-3x} \) is \( A(D) = (D + 3)^2 \). Operating on (0.0.30) with \((D + 3)^2y = 0\) yields
\[ (D + 3)^3(D - 1)y = 0 \]
which has general solution
\[ y(x) = y_c(x) + e^{-3x}(A_0 x + A_1 x^2). \]
Consequently, an appropriate trial solution for (0.0.30) is
\[ y_p(x) = e^{-3x}(A_0 x + A_1 x^2). \]
Differentiating this trial solution with respect to \( x \) yields
\[ y_p'(x) = e^{-3x}(-3A_0 x - 3A_1 x^2 + A_0 + 2A_1), \]
\[ y_p''(x) = e^{-3x}(9A_0 x + 9A_1 x^2 - 3A_0 - 6A_1 x - 3A_0 - 6A_1 x + 2A_1). \]
Inserting these expressions into the given differential equation yields
\[ e^{-3x}(9A_0 x + 9A_1 x^2 - 6A_0 - 12A_1 x + 2A_1) + 2e^{-3x}(-3A_0 x - 3A_1 x^2 + A_0 + 2A_1 x) \]
\[ - 3e^{-3x}(A_0 x + A_1 x^2) = 2xe^{-3x}, \]
which simplifies to
\[ -4A_0 + 2A_1 - 8A_1 x = 2x. \]
Therefore $A_0$ and $A_1$ must satisfy

\[-4A_0 + 2A_1 = 0, \quad -8A_1 = 2,\]

so that $A_0 = -\frac{1}{8}$ and $A_1 = -\frac{1}{4}$. Hence,

\[y_p(x) = e^{-3x} \left(\frac{1}{8} - \frac{1}{4}x^2\right) = -\frac{1}{8} xe^{-3x}(2x + 1),\]

and the general solution to the given differential equation is

\[y(x) = c_1 e^{-3x} + c_2 e^x - \frac{1}{8} xe^{-3x}(2x + 1).\]

43. In operator form the given differential equation is

\[(D^2 + 4)y = 8 \cos 2x. \tag{0.0.31}\]

Therefore the complementary function is

\[y_c(x) = c_1 \cos 2x + c_2 \sin 2x.\]

The annihilator of $F(x) = 8 \cos 2x$ is $A(D) = D^2 + 4$. Operating on (0.0.31) with $D^2 + 4$ yields

\[(D^2 + 4)^2 y = 0\]

which has general solution

\[y(x) = y_c(x) + x(A_0 \cos 2x + A_1 \sin 2x).\]

Consequently, an appropriate trial solution for (0.0.31) is

\[y_p(x) = x(A_0 \cos 2x + A_1 \sin 2x).\]

Differentiating this trial solution with respect to $x$ yields

\[y'_p(x) = A_0 \cos 2x + A_1 \sin 2x + x(-2A_0 \sin 2x + A_1 \cos 2x),\]
\[y''_p(x) = -4A_0 \sin 2x + 4A_1 \cos 2x + x(-4A_0 \cos 2x - A_1 \sin 2x).\]

Inserting these expressions into the given differential equation yields

\[-4A_0 \sin 2x + 4A_1 \cos 2x + x(-4A_0 \cos 2x - A_1 \sin 2x) + 4x(A_0 \cos 2x + A_1 \sin 2x) = 8 \cos 2x,\]

that is,

\[-4A_0 \sin 2x + 4A_1 \cos 2x = 8 \cos 2x.\]

Consequently, $A_0 = 0$, and $A_1 = 2$. Therefore,

\[y_p(x) = 2x \sin 2x,\]

and the general solution to the given differential equation is

\[y(x) = c_1 \cos 2x + c_2 \sin 2x + 2x \sin 2x.\]
45. In operator form the given differential equation is

\[(D^2 - 1)y = 3e^{2x} + \sin x,\]

that is

\[(D - 1)(D + 1)y = 3e^{2x} + \sin x.\] (0.0.32)

Therefore the complementary function is

\[y_c(x) = c_1 e^x + c_2 e^{-x}.\]

The annihilator of \(F(x) = 3e^{2x} + \sin x\) is \(A(D) = (D^2 + 1)(D - 2).\) Operating on (0.0.32) with \((D^2 + 1)(D - 2)\) yields

\[(D^2 + 1)(D - 2)(D - 1)(D + 1)y = 0\]

which has general solution

\[y(x) = y_c(x) + A_0 e^{2x} + A_1 \cos x + A_2 \sin x.\]

Consequently, an appropriate trial solution for (0.0.32) is

\[y_p(x) = A_0 e^{2x} + A_1 \cos x + A_2 \sin x.\]

Differentiating this trial solution with respect to \(x\) yields

\[y'_p(x) = 2A_0 e^{2x} - A_1 \sin x + A_2 \cos x, \quad y''_p(x) = 4A_0 e^{2x} - A_1 \cos x - A_2 \sin x.\]

Inserting these expressions into the given differential equation yields

\[4A_0 e^{2x} - A_1 \cos x - A_2 \sin x - (A_0 e^{2x} + A_1 \cos x + A_2 \sin x) = 3e^{2x} + \sin x,\]

that is,

\[3A_0 e^{2x} - 2A_1 \cos x - 2A_2 \sin x = 3e^{2x} + \sin x.\]

Therefore, \(A_0 = 1, A_1 = 0, A_2 = -\frac{1}{2},\) so that

\[y_p(x) = e^{2x} - \frac{1}{2} \sin x,\]

and the general solution to the given differential equation is

\[y(x) = c_1 e^x + c_2 e^{-x} + e^{2x} - \frac{1}{2} \sin x.\]

47. The complementary function is

\[y_c(x) = c_1 \cos x + c_2 \sin x.\]

Choosing

\[y_1(x) = \cos x, \quad y_2(x) = \sin x\]

we have

\[W[y_1, y_2](x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1\]

so that

\[y_p(x) = -\cos x \int \sin x \cdot \frac{1}{\sin x} dx + \sin x \int \cos x \cdot \frac{1}{\sin x} dx = -x \cos x + \sin x \ln |\sin x|.\]
Hence,

\[ y(x) = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \ln |\sin x|. \]

49. The complementary function is

\[ y_c(x) = c_1 e^{mx} + c_2 xe^{mx}. \]

Choosing

\[ y_1(x) = e^{mx}, \quad y_2(x) = xe^{mx} \]

we have

\[ W[y_1, y_2](x) = \left| \begin{array}{cc} e^{mx} & xe^{mx} \\ me^{mx} & e^{mx}(mx + 1) \end{array} \right| = e^{2mx} \]

so that

\[ y_p(x) = -e^{mx} \int \frac{xe^{mx} \cdot e^{mx} \ln x}{e^{2mx}} \, dx + xe^{mx} \int \frac{e^{mx} \cdot e^{mx} \ln x}{e^{2mx}} \, dx = \frac{1}{4} x^2 e^{mx}(2 \ln x - 3). \]

Hence,

\[ y(x) = c_1 e^{mx} + c_2 xe^{mx} + \frac{1}{4} x^2 e^{mx}(2 \ln x - 3). \]

51. The complementary function is

\[ y_c(x) = c_1 e^x + c_2 xe^x. \]

Choosing

\[ y_1(x) = e^x, \quad y_2(x) = xe^x \]

we have

\[ W[y_1, y_2](x) = \left| \begin{array}{cc} e^x & xe^x \\ e^x & e^x(x + 1) \end{array} \right| = e^{2x} \]

so that

\[ y_p(x) = -e^x \int \frac{xe^x \cdot e^x \ln x}{e^{2x}} \, dx + xe^x \int \frac{e^x \cdot e^x \ln x}{e^{2x}} \, dx = \frac{1}{4} x^2 e^x(2 \ln x - 3). \]

Hence,

\[ y(x) = c_1 e^x + c_2 xe^x + \frac{1}{4} x^2 e^x(2 \ln x - 3) \]

53. The indicial equation is

\[ r(r - 1) + 9r + 6 = 0 \implies r^2 + 8r + 16 = 0 \implies (r + 4)^2 = 0, \]

with root \( r = -4 \) (multiplicity 2). Consequently, the given differential equation has general solution

\[ y(x) = c_1 x^{-4} + c_2 x^{-4} \ln x. \]

55. The indicial equation is

\[ r(r - 1) - 11r + 37 = 0 \implies r^2 - 12r + 37 = 0, \]

with roots \( r = 6 \pm i \). Consequently, the given differential equation has general solution

\[ y(x) = c_1 x^6 \cos(\ln x) + c_2 x^6 \sin(\ln x). \]
The indicial equation is
\[ r(r - 1) - 2r - 18 = 0 \implies r^2 - 3r - 18 = 0 \implies (r + 3)(r - 6) = 0, \]
with roots \( r = -3, r = 6 \). Consequently, the given differential equation has general solution
\[ y(x) = c_1 x^{-3} + c_2 x^6. \]

The indicial equation is
\[ r(r - 1) + 9r + 6 = 0 \implies r^2 + 8r + 16 = 0 \implies (r + 4)^2 = 0, \]
with root \( r = -4 \) (multiplicity 2). Consequently, the complementary function is
\[ y_c(x) = c_1 x^{-4} + c_2 x^{-4} \ln x. \]
Choosing
\[ y_1(x) = x^{-4}, \quad y_2(x) = x^{-4} \ln x \]
we have
\[ W[y_1, y_2](x) = \begin{vmatrix} x^{-4} & x^{-4} \ln x \\ -4x^{-5} & x^{-5}(1 - 4 \ln x) \end{vmatrix} = x^{-9} \]
so that
\[ y_p(x) = -x^{-4} \int \frac{x^{-4} \ln x \cdot x^{-5}}{x^{-9}} dx + x^{-4} \ln x \int \frac{x^{-4} \cdot x^{-5}}{x^{-9}} dx = x^{-3}. \]
Hence,
\[ y(x) = c_1 x^{-4} + c_2 x^{-4} \ln x + x^{-3}. \]

The indicial equation is
\[ r(r - 1) - 5r + 10 = 0 \implies r^2 - 6r + 10 = 0, \]
with roots \( r = 3 \pm i \). Consequently, the complementary function is
\[ y_c(x) = c_1 x^3 \cos(\ln x) + c_2 x^3 \sin(\ln x). \]
Choosing
\[ y_1(x) = x^3 \cos(\ln x), \quad y_2(x) = x^3 \sin(\ln x) \]
we have
\[ W[y_1, y_2](x) = \begin{vmatrix} x^3 \cos(\ln x) & x^3 \sin(\ln x) \\ x^2[3 \cos(\ln x) - \sin(\ln x)] & x^2[3 \sin(\ln x) + \cos(\ln x)] \end{vmatrix} = x^5 \]
so that
\[ y_p(x) = -x^3 \cos(\ln x) \int \frac{x^3 \sin(\ln x) \cdot x}{x^5} dx + x^3 \sin(\ln x) \int \frac{x^3 \cos(\ln x) \cdot x}{x^5} dx = x^3. \]
Hence,
\[ y(x) = c_1 x^3 \cos(\ln x) + c_2 x^3 \sin(\ln x) + x^3. \]

The complementary function is
\[ y_c(x) = c_1 e^{-2x} + c_2 x e^{-2x}. \]
Choosing 
\[ y_1(x) = e^{-2x}, \quad y_2(x) = xe^{-2x} \]
we have 
\[ W[y_1, y_2](x) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x}(1 - 2x) \end{vmatrix} = e^{-4x}. \]
so that 
\[ y_p(x) = -e^{-2x} \int \frac{xe^{-2x}}{e^{-4x}} \ln x \cdot e^{-2x} d\!x + xe^{-2x} \int \frac{e^{-2x}}{e^{-4x}} \cdot \ln x \cdot e^{-2x} d\!x = \frac{1}{2} xe^{-2x}[(\ln x)^2 - 2 \ln x + 2]. \]
Hence, 
\[ y(x) = c_1 e^{-2x} + c_2 xe^{-2x} + \frac{1}{2} xe^{-2x}[(\ln x)^2 - 2 \ln x]. \]

65. In operator form the given differential equation is 
\[ (D^2 - 9D + 20)y = x^3 e^{5x}, \]
that is, 
\[ (D - 4)(D - 5)y = x^3 e^{5x}. \] (0.0.33)
Therefore the complementary function is 
\[ y_c(x) = c_1 e^{4x} + c_2 e^{5x}. \]
The annihilator of \( F(x) = x^3 e^{5x} \) is \( A(D) = (D - 5)^4 \). Operating on (0.0.33) with \( (D - 5)^4 \) yields 
\[ (D - 5)^5(D - 4)y = 0 \]
which has general solution 
\[ y(x) = y_c(x) + e^{5x}(A_0x + A_1x^2 + A_2x^3 + A_3x^4). \]
Consequently, an appropriate trial solution for (0.0.33) is 
\[ y_p(x) = e^{5x}(A_0x + A_1x^2 + A_2x^3 + A_3x^4). \]
Differentiating this trial solution with respect to \( x \) yields 
\[
\begin{align*}
y''_p(x) & = e^{5x}(5A_0x + 5A_1x^2 + 5A_2x^3 + 5A_3x^4 + A_0 + 2A_1x + 3A_2x^2 + 4A_3x^3), \\
y'''_p(x) & = e^{5x}(25A_0x + 25A_1x^2 + 25A_2x^3 + 10A_0 + 20A_1x + 30A_2x^2 + 40A_3x^3 \\
& \quad + 25A_3x^4 + 12A_3x^2 + 2A_1 + 6A_2x) \\
& + 2A_1 + 6A_2x) - 9(5A_0x + 5A_1x^2 + 5A_2x^3 + 5A_3x^4 + A_0 + 2A_1x + 3A_2x^2 + 4A_3x^3) \\
& + 20(A_0x + A_1x^2 + A_2x^3 + A_3x^4) = x^3.
\end{align*}
\]
Inserting these expressions into the given differential equation yields 
\[
(25A_0x + 25A_1x^2 + 25A_2x^3 + 10A_0 + 20A_1x + 30A_2x^2 + 40A_3x^3 + 25A_3x^4 + 12A_3x^2 \\
+ 2A_1 + 6A_2x) - 9(5A_0x + 5A_1x^2 + 5A_2x^3 + 5A_3x^4 + A_0 + 2A_1x + 3A_2x^2 + 4A_3x^3) \\
+ 20(A_0x + A_1x^2 + A_2x^3 + A_3x^4) = x^3.
\]
which simplifies to 
\[ 4A_3x^3 + 3(2A_2 + 4A_3)x^2 + 2(A_1 + 3A_2)x + A_0 + 2A_1 = x^3. \]
Therefore \( A_0, A_1, A_2, A_3 \) must satisfy

\[
4A_3 = 1, \quad A_2 + 4A_3 = 0, \quad A_1 + 3A_2 = 0, \quad A_0 + 2A_1 = 0,
\]

so that \( A_0 = -6, A_1 = 3, A_2 = -1, A_3 = \frac{1}{4} \). Consequently,

\[
y_p(x) = \frac{1}{4} xe^{5x} (x^3 - 4x^2 + 12x - 24),
\]

and the general solution to the given differential equation is

\[
y(x) = c_1 e^{4x} + c_2 e^{5x} + \frac{1}{4} xe^{5x} (x^3 - 4x^2 + 12x - 24).
\]

**Solutions to Section 7.1**

**True-False Review:**

1. **TRUE.** If we make the substitution \( x \to x_1 \) and \( y \to x_2 \), then in terms of Definition 7.1.1, we have two equations with \( a_{11}(t) = t^2, a_{12}(t) = -t, a_{21}(t) = \sin t, a_{22}(t) = 5 \), and \( b_1(t) = b_2(t) = 0 \).

3. **FALSE.** The term \( tx_2y \) on the right-hand side of the formula for \( x' \) is non-linear because the unknown functions \( x \) and \( y \) are multiplied together, and this is prohibited in the form of a first-order linear system of differential equations.

5. **TRUE.** If we consider Equation (7.1.16) with \( n = 3 \), then the text describes how the substitution \( x_1 = x, x_2 = x', x_3 = x'' \) enables us to replace (7.1.16) with the equivalent first-order linear system

\[
\begin{align*}
x_1' &= x_2, \\
x_2' &= x_3, \\
x_3' &= -a_3(t)x_1 - a_2(t)x_2 - a_1(t)x_1 + F(t).
\end{align*}
\]

7. **FALSE.** As indicated in Definition 7.1.6, an initial-value problem consists of auxiliary conditions all of which are applied at the same time \( t_0 \). In this system, the value of \( x \) is specified at \( t = 0 \) while the value of \( y \) is specified at \( t = 1 \).

9. **FALSE.** There is no initial-value specified for the function \( y(t) \). The value \( y(2) \) would be required in order for the to be an initial-value problem.

**Problems:**

1. Writing the given system of differential equations in operator form gives us the equations

\[
(D - 2)x_1 + 3x_2 = 0 \quad \text{and} \quad -x_1 + (D + 2)x_2 = 0.
\]

Operating on the first equation with \( D + 2 \), we obtain \((D + 2)(D - 2)x_1 + 3(D + 2)x_2 = 0\). Combining this with the second equation above yields \((D + 2)(D - 2)x_1 + 3x_1 = 0\), or \((D^2 - 1)x_1 = 0\). Therefore, since the characteristic equation for this differential equation has roots \( r = \pm 1 \), we obtain

\[
x_1(t) = c_1 e^t + c_2 e^{-t}.
\]

Hence,

\[
x_2(t) = -\frac{1}{3}(D - 2)x_1
\]

\[
= -\frac{1}{3} \left( c_1 e^t - c_2 e^{-t} - 2c_1 e^t - 2c_2 e^{-t} \right)
\]

\[
= \frac{1}{3} c_1 e^t + c_2 e^{-t}.
\]
To summarize, the solution to the system of differential equations is

\[ x_1(t) = c_1 e^t + c_2 e^{-t} \quad \text{and} \quad x_2(t) = \frac{1}{3} c_1 e^t + c_2 e^{-t}. \]

3. Writing the given system of differential equations in operator form gives us the equations

\[(D - 2)x_1 - 4x_2 = 0 \quad \text{and} \quad 4x_1 + (D + 6)x_2 = 0.\]

Operating on the first equation with \(D + 6\), we obtain \((D + 6)(D - 2)x_1 - 4(D + 6)x_2 = 0\). Combining this with the second equation above yields \((D + 6)(D - 2)x_1 + 16x_1 = 0\), or \((D^2 + 4D + 4)x_1 = 0\). Since the characteristic equation for this differential equation has root \(r = -2\) with multiplicity 2, we obtain

\[ x_1(t) = c_1 e^{-2t} + c_2 te^{-2t}. \]

Hence,

\[
x_2(t) = \frac{1}{4}(D - 2)x_1
\]
\[
= \frac{1}{4} (-2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 e^{-2t} - 2c_1 e^{-2t} - 2c_2 te^{-2t})
\]
\[
= \frac{1}{4} (-4c_1 e^{-2t} + c_2 (1 - 4t) e^{-2t})
\]
\[
= -c_1 e^{-2t} + \frac{1}{4} c_2 (1 - 4t) e^{-2t}.
\]

To summarize, the solution to the given system of differential equations is

\[ x_1(t) = c_1 e^{-2t} + c_2 te^{-2t} \quad \text{and} \quad x_2(t) = -c_1 e^{-2t} + \frac{1}{4} c_2 (1 - 4t) e^{-2t}. \]

5. Writing the given system of differential equations in operator form gives us the equations

\[(D - 1)x_1 + 3x_2 = 0 \quad \text{and} \quad -3x_1 + (D - 1)x_2 = 0.\]

Operating on the first equation with \(D - 1\), we obtain \((D - 1)^2x_1 + 3(D - 1)x_2 = 0\). Combining this with the second equation above yields \((D - 1)^2x_1 + 9x_1 = 0\), or \((D^2 - 2D + 10)x_1 = 0\). Since the characteristic equation for this differential equation has roots \(r = 1 \pm 3i\), we obtain

\[ x_1(t) = c_1 e^t \cos 3t + c_2 e^t \sin 3t. \]

Hence,

\[
x_2(t) = -\frac{1}{3} (D - 1)x_1
\]
\[
= -\frac{1}{3} (c_1 e^t \cos 3t - 3c_1 e^t \sin 3t + 3c_2 e^t \cos 3t + c_2 e^t \sin 3t - c_1 e^t \cos 3t - c_2 e^t \sin 3t)
\]
\[
= -\frac{1}{3} (-3c_1 e^t \sin 3t + 3c_2 e^t \cos 3t)
\]
\[
= c_1 e^t \sin 3t - c_2 e^t \cos 3t.
\]

To summarize, the solution to the given system of differential equations is

\[ x_1(t) = c_1 e^t \cos 3t + c_2 e^t \sin 3t \quad \text{and} \quad x_2(t) = c_1 e^t \sin 3t - c_2 e^t \cos 3t. \]
7. Writing the given system of differential equations in operator form gives us the equations

\[(D + 2)x_1 - x_2 - x_3 = 0, \quad -x_1 + (D + 1)x_2 - 3x_3 = 0, \quad x_2 + (D + 3)x_3 = 0.\]

Operating on the middle equation with \(D + 2\), we obtain \(- (D + 2)x_1 + (D + 2)(D + 1)x_2 - 3(D + 2)x_3 = 0\). Combining this with the first equation to eliminate \(x_1\) yields \((D^2 + 3D + 1)x_2 - (3D + 7)x_3 = 0\). Operating on the third equation with \(D^2 + 3D + 1\) yields \((D^2 + 3D + 1)x_2 + (D^2 + 3D + 1)(D + 3)x_3 = 0\). Combining this with previous result, we obtain \([((D^2 + 3D + 1)(D + 3) + (3D + 7))x_3 = 0\). That is, \((D^3 + 6D^2 + 13D + 10)x_3 = 0\), or \((D + 2)(D^2 + 4D + 5)x_3 = 0\). Since the characteristic equation for this differential equation has roots \(r = -2\) and \(r = -2 \pm i\), we obtain

\[x_3(t) = c_1 e^{-2t} + c_2 e^{-2t} \cos t + c_3 e^{-2t} \sin t.\]

From the third equation in the original system, we have

\[x_2(t) = -(D + 3)x_3 = -e^{-2t} [c_1 + c_2 (\cos t - \sin t) + c_3 (\cos t + \sin t)],\]

and from the second equation in the original system, we have

\[x_1(t) = (D + 1)x_2 - 3x_3 = -e^{-2t} [2c_1 + c_2 \cos t + c_3 \sin t].\]

To summarize, the solution to the given system of differential equations is

\[x_1(t) = -e^{-2t} [2c_1 + c_2 \cos t + c_3 \sin t],\]

\[x_2(t) = -e^{-2t} [c_1 + c_2 (\cos t - \sin t) + c_3 (\cos t + \sin t)],\]

\[x_3(t) = c_1 e^{-2t} + c_2 e^{-2t} \cos t + c_3 e^{-2t} \sin t.\]

9. Writing the given system of differential equations in operator form gives us the equations

\[(D - 2)x_1 - 5x_2 = 0 \quad \text{and} \quad x_1 + (D + 2)x_2 = 0.\]

Operating on the first equation with \(D + 2\), we obtain \((D + 2)(D - 2)x_1 - 5(D + 2)x_2 = 0\). Combining this with the second equation above, we obtain \((D + 2)(D - 2)x_1 + 5x_1 = 0\), or \((D^2 + 1)x_1 = 0\). Since the characteristic equation for this differential equation has roots \(r = \pm i\), we obtain

\[x_1(t) = c_1 \cos t + c_2 \sin t.\]

Hence,

\[x_2(t) = \frac{1}{5} (D - 2)x_1 = \frac{1}{5} (-c_1 \sin t + c_2 \cos t - 2c_1 \cos t - 2c_2 \sin t) = \frac{1}{5} (-c_1 (\sin t + 2 \cos t) + c_2 (\cos t - 2 \sin t)).\]

Imposing the initial conditions \(x_1(0) = 0\) and \(x_2(0) = 1\), we find that \(c_1 = 0\) and \(c_2 = 5\). Therefore,

\[x_1(t) = 5 \sin t \quad \text{and} \quad x_2(t) = \cos t - 2 \sin t.\]

11. Writing the given system of differential equations in operator form gives us the equations

\[(D - 1)x_1 - 2x_2 = 5e^{4t} \quad \text{and} \quad -2x_1 + (D - 1)x_2 = 0.\]
Operating of the first equation with $D - 1$ yields

$$(D - 1)^2 x_1 - 2(D - 1)x_2 = 5(D - 1)e^{4t},$$

or

$$(D - 1)^2 x_1 - 2(D - 1)x_2 = 15e^{4t}.$$  
Combining this with the second equation above, we obtain $(D - 1)^2 x_1 - 4x_1 = 15e^{4t}$, or $(D^2 - 2D - 3)x_1 = 15e^{4t}$. The complementary function for $x_1$ is

$$x_{1c}(t) = c_1 e^{-t} + c_2 e^{3t}.$$  
Using the trial solution $x_{1p}(t) = Ae^{4t}$, we find that $A = 3$, so that

$$x_1(t) = c_1 e^{-t} + c_2 e^{3t} + 3e^{4t}.$$  
Hence,

$$x_2(t) = \frac{1}{2} [(D - 1)x_1 - 5e^{4t}] = \frac{1}{2} [-c_1 e^{-t} + 3c_2 e^{3t} + 12e^{4t} - c_1 e^{-t} - c_2 e^{3t} - 3e^{4t} - 5e^{4t}]$$

$$= \frac{1}{2} [-2c_1 e^{-t} + 2c_2 e^{3t} + 4e^{4t}]$$

$$= -c_1 e^{-t} + c_2 e^{3t} + 2e^{4t}.$$  
Therefore,

$$x_1(t) = c_1 e^{-t} + c_2 e^{3t} + 3e^{4t} \quad \text{and} \quad x_2(t) = -c_1 e^{-t} + c_2 e^{3t} + 2e^{4t}.$$  

13. Writing the given system of differential equations in operator form gives us the equations

$$(D - 1)x_1 - x_2 = e^{2t} \quad \text{and} \quad -3x_1 + (D + 1)x_2 = 5e^{2t}.$$  
Operating on the first equation with $D + 1$ yields $(D + 1)(D - 1)x_1 - (D + 1)x_2 = (D + 1)e^{2t}$. Combining this with the second equation above yields

$$[(D + 1)(D - 1) - 3]x_1 = 5e^{2t} + (D + 1)e^{2t} = 8e^{2t},$$

or

$$(D^2 - 4)x_1 = 8e^{2t}.$$  
The complementary function for $x_1$ is

$$x_{1c}(t) = c_1 e^{2t} + c_2 e^{-2t}.$$  
Using the trial solution $x_{1p}(t) = Ate^{2t}$, we find that $A = 2$. Therefore,

$$x_1(t) = c_1 e^{2t} + c_2 e^{-2t} + 2te^{2t}.$$  
Hence,

$$x_2(t) = x_1'(t) - x_1(t) - e^{2t}$$

$$= 2c_1 e^{2t} - 2c_2 e^{-2t} + 2e^{2t} + 4te^{2t} - c_1 e^{2t} - c_2 e^{-2t} - 2te^{2t} - e^{2t}$$

$$= (c_1 + 2t + 1)e^{2t} - 3c_2 e^{-2t}.$$
Therefore,
\[ x_1(t) = c_1 e^{2t} + c_2 e^{-2t} + 2te^{2t} \quad \text{and} \quad x_2(t) = (c_1 + 2t + 1)e^{2t} - 3c_2 e^{-2t}. \]

15. Letting \( x_1 = x, x_2 = \frac{dx}{dt}, x_3 = y, \) and \( x_4 = \frac{dy}{dt} \) yields the first-order system
\[
\begin{align*}
\frac{dx_1}{dt} &= x_2, \\
\frac{dx_2}{dt} &= -x_1 + 3x_4 + \sin t, \\
\frac{dx_3}{dt} &= x_4, \\
\frac{dx_4}{dt} &= tx_2 + e^t x_3 + t^2.
\end{align*}
\]

17. Letting \( y_1 = y \) and \( y_2 = \frac{dy}{dt} \), we obtain the system
\[
\begin{align*}
\frac{dy_1}{dt} &= y_2 \\
\frac{dy_2}{dt} &= -by_1 - ay_2 + F(t).
\end{align*}
\]

19. Letting \( u_1 = x, u_2 = \frac{dx}{dt}, u_3 = y, \) and \( u_4 = \frac{dy}{dt} \), we obtain the system
\[
\begin{align*}
\frac{du_1}{dt} &= u_2, \\
\frac{du_2}{dt} &= \frac{1}{m_1} [-k_1 u_1 + k_2 (u_3 - u_1)], \\
\frac{du_3}{dt} &= u_4, \\
\frac{du_4}{dt} &= -\frac{k_2}{m_2} (u_3 - u_1),
\end{align*}
\]
with initial conditions
\[ u_1(0) = \alpha_1, \quad u_2(0) = \alpha_2, \quad u_3(0) = \alpha_3, \quad u_4(0) = \alpha_4. \]

**Solutions to Section 7.2**

1. **TRUE.** If the unknown function \( x(t) \) is a column vector function, then \( A(t) \) must contain \( n \) columns in order to be able to compute \( A(t)x(t) \). And since \( x'(t) \) is also a column vector function, then \( A(t)x(t) \) must have \( n \) rows, and therefore, \( A(t) \) must also have \( n \) rows. Therefore, \( A(t) \) contains the same number of rows and columns.

3. **FALSE.** Many counterexamples are possible. For instance, if
\[
\begin{align*}
x_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
x_2 &= \begin{bmatrix} 1 \\ t \end{bmatrix}, \\
x_3 &= \begin{bmatrix} 1 \\ t^2 \end{bmatrix},
\end{align*}
\]
then if we set
\[
c_1 x_1(t) + c_2 x_2(t) + c_2 x_3(t) = 0,
\]
then we arrive at the equations
\[
c_1 + c_2 + c_3 = 0 \quad \text{and} \quad c_1 + tc_2 + t^2 c_3 = 0.
\]
Setting \( t = 0 \), we conclude that \( c_1 = 0 \). Next, setting \( t = -1 \), we have \(-c_2 + c_3 = 0 = c_2 + c_3 \), and this requires \( c_2 = c_3 = 0 \). Therefore \( c_1 = c_2 = c_3 = 0 \). Hence, \( \{x_1, x_2, x_3\} \) is linearly independent.

5. **FALSE.** To the contrary, an \( n \times n \) linear system \( x'(t) = Ax(t) \) always has \( n \) linearly independent solutions, regardless of any condition on the determinant of the matrix \( A \). This is explained in the paragraph preceding Example 7.2.7 and is proved in Theorem 7.3.2 in the next section.

7. **FALSE.** We have
\[
(x_0(t) + b(t))' = x'_0(t) + b'(t) = A(t)x_0(t) + b'(t),
\]
and this in general is not the same as
\[ A(t)(x_0(t) + b(t)) + b(t). \]

**Problems:**

1. We compute the Wronskian:

\[
W[x_1, x_2](t) = \begin{vmatrix}
    e^t & e^t \\
    -e^t & e^t
\end{vmatrix} = 2e^{2t} \neq 0,
\]

so that by Theorem 7.2.4 the vector functions are linearly independent on \((-\infty, \infty)\).

3. We compute the Wronskian:

\[
W[x_1, x_2, x_3](t) = \begin{vmatrix}
    t + 1 & e^t & 1 \\
    t - 1 & e^{2t} & \sin t \\
    2t & e^{3t} & \cos t
\end{vmatrix}
= (t + 1)e^{2t}\cos t + 2te^t\sin t + (t - 1)e^{3t} - 2te^{2t} - (t + 1)e^{3t}\sin t - (t - 1)e^t\cos t.
\]

Substituting \( t = 0, \) we find that \( W[x_1, x_2, x_3](0) = 1, \) so that the Wronskian is not identically zero on \((-\infty, \infty)\). Therefore, by Theorem 7.2.4, the vector functions are linearly independent on \((-\infty, \infty)\).

5. We compute the Wronskian:

\[
W[x_1, x_2, x_3](t) = \begin{vmatrix}
    \sin t & t & \sinh t \\
    \cos t & 1 - t & \cosh t \\
    1 & 1 & 1
\end{vmatrix}
= (1 - t)\sin t + t\cosh t + \cos t\sinh t - (1 - t)\sin t - \sin t\cosh t - t\cos t.
\]

A quick computation with, say \( t = 1, \) shows that this Wronskian is not identically zero on \((-\infty, \infty)\). Therefore, by Theorem 7.2.4, the given vector functions are linearly independent on \((-\infty, \infty)\).

7. By inspection we see that \( 4x_1 - x_2 = 0 \) so that \( \{x_1, x_2\} \) is linearly dependent.

9. We see that \( 3x_1 - x_2 + x_3 = 0 \) so that \( \{x_1, x_2, x_3\} \) is linearly dependent.

11. Let

\[ S = \{x \in V_n(I) : x' = Ax\}. \]

Note that \( x = 0 \) is a solution to \( x' = Ax, \) so that \( 0 \) belongs to \( S. \) Therefore, \( S \) is nonempty. Next, we must verify that \( S \) is closed under addition and scalar multiplication:

*Closure under addition:* Suppose that \( u \) and \( v \) belong to \( S. \) This means that \( u' = Au \) and \( v' = Av. \) Therefore \( (u + v)' = u' + v' = Au + Av = A(u + v), \) which shows that \( u + v \) belongs to \( S. \) Hence, \( S \) is closed under addition.

*Closure under scalar multiplication:* Suppose that \( u \) belongs to \( S \) and \( c \) is a scalar. From the fact that \( u' = Au, \) we deduce that \( (cu)' = cu' = c(Au) = A(cu), \) which shows that \( cu \) belongs to \( S. \) Hence, \( S \) is closed under scalar multiplication.

13.
(a) Letting \( x_1 = y \) and \( x_2 = y' \), the given differential equation can be replaced by the first-order system
\[
\begin{align*}
x'_1 &= x_2 \\
and \\
x'_2 &= -bx_1 - ax_2.
\end{align*}
\]
That is \( \mathbf{x}' = A\mathbf{x} \), where \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) and \( A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \).

(b) Supposing that \( y_1 = f_1(t) \) and \( y_2 = f_2(t) \) are solutions to (7.2.4), we have that
\[
\begin{align*}
\frac{d^2 y_1}{dt^2} + a \frac{dy_1}{dt} + by_1 &= 0 \\
\text{and} \\
\frac{d^2 y_2}{dt^2} + a \frac{dy_2}{dt} + by_2 &= 0,
\end{align*}
\]
or
\[
\begin{align*}
f''_1(t) + af'_1(t) + bf_1(t) &= 0 \\
\text{and} \\
f''_2(t) + af'_2(t) + bf_2(t) &= 0.
\end{align*}
\]
Substituting \( \mathbf{x}_1(t) = \begin{bmatrix} f_1(t) \\ f'_1(t) \end{bmatrix} \) into \( \mathbf{x}' = A\mathbf{x} \), we obtain
\[
\begin{bmatrix}
f'_1(t) \\
af'_1(t)
\end{bmatrix} = \begin{bmatrix}
-bf_1(t) - af_1(t)
\end{bmatrix},
\]
which corresponds precisely to the differential equation for \( f_1 \) above. Likewise, substituting \( \mathbf{x}_2(t) = \begin{bmatrix} f_2(t) \\ f'_2(t) \end{bmatrix} \) into \( \mathbf{x}' = A\mathbf{x} \), we obtain
\[
\begin{bmatrix}
f'_2(t) \\
af'_2(t)
\end{bmatrix} = \begin{bmatrix}
-bf_2(t) - af_2(t)
\end{bmatrix},
\]
which corresponds precisely to the differential equation for \( f_2 \) above.

(c) Using the Wronskian in \( V_2(I) \) we have \( W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} f_1 & f_2 \\ f'_1 & f'_2 \end{vmatrix} \) Similarly, using the Wronskian in \( C^1(I) \),
\[
W[y_1, y_2](t) = \begin{vmatrix} f_1 & f_2 \\ f'_1 & f'_2 \end{vmatrix}
\]

### Solutions to Section 7.3

**True-False Review:**

1. **FALSE.** With \( b(t) \neq 0 \), note that \( \mathbf{x}(t) = \mathbf{0} \) is not a solution to the system \( \mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t) \), and lacking the zero vector, the solution set to the differential equation cannot form a vector space.

3. **TRUE.** More than this, a fundamental solution set for \( \mathbf{x}'(t) = A(t)\mathbf{x}(t) \) is in fact a *basis* for the space of solutions to the linear system. In particular, it spans the space of all solutions to \( \mathbf{x}' = A\mathbf{x} \).

**Problems:**

1. We first show that \( \mathbf{x}_1 \) is a solution to \( \mathbf{x}' = A\mathbf{x} \):
\[
A\mathbf{x}_1 = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ 8e^{4t} \end{bmatrix} = \mathbf{x}'_1.
\]
Next we show that \( \mathbf{x}_2 \) is a solution to \( \mathbf{x}' = A\mathbf{x} \):
\[
A\mathbf{x}_2 = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} -3e^{-t} \\ -e^{-t} \end{bmatrix} = \mathbf{x}'_2.
\]
To check \( \{x_1, x_2\} \) for linear independence, we compute the Wronskian:

\[
W[x_1, x_2](t) = \begin{vmatrix}
  e^{4t} & 3e^{-t} \\
  2e^{4t} & e^{-t}
\end{vmatrix} = -5e^{3t} \neq 0,
\]

so \( \{x_1, x_2\} \) is linearly independent. Therefore, the general solution to the system is

\[
x(t) = c_1 \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix}.
\]

Finally, we use the initial condition \( x(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \) to determine \( c_1 \) and \( c_2 \): We have

\[
\begin{bmatrix} -2 \\ 1 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix},
\]

which implies that

\[
c_1 + 3c_2 = -2 \quad \text{and} \quad 2c_1 + c_2 = 1.
\]

Solving this system, we obtain \( c_1 = 1 \) and \( c_2 = -1 \). Thus, the particular solution to this initial value problem is

\[
x(t) = \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} - \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix}.
\]

3. We first show that \( x_1 \) is a solution to \( x' = Ax \):

\[
Ax_1 = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = x_1.
\]

Next we show that \( x_2 \) is a solution to \( x' = Ax \):

\[
Ax_2 = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} e^{2t} \\ 3e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 6e^{2t} \\ 2e^{2t} \end{bmatrix} = x'_2.
\]

Next we show that \( x_3 \) is a solution to \( x' = Ax \):

\[
Ax_3 = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ 4e^{4t} \\ 4e^{4t} \end{bmatrix} = x'_3.
\]

To check \( \{x_1, x_2, x_3\} \) for linear independence, we use the Wronskian:

\[
W[x_1, x_2, x_3](t) = \begin{vmatrix}
  -3 & e^{2t} & e^{4t} \\
  9 & 3e^{2t} & e^{4t} \\
  5 & e^{2t} & e^{4t}
\end{vmatrix} = -16e^{6t} \neq 0,
\]

so \( \{x_1, x_2, x_3\} \) is linearly independent. Therefore, the general solution to the system is

\[
x(t) = c_1 \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ 3e^{2t} \\ e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix}.
\]
5. We first show that \( x_1 \) is a solution to \( x' = Ax \):

\[
Ax_1 = \begin{bmatrix}
\frac{1}{t} & t \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
t \sin t \\
\cos t
\end{bmatrix} = \begin{bmatrix}
\sin t + t \cos t \\
-\sin t
\end{bmatrix} = x_1'.
\]

Next we show that \( x_2 \) is a solution to \( x' = Ax \):

\[
Ax_2 = \begin{bmatrix}
\frac{1}{t} & t \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
-t \cos t \\
\sin t
\end{bmatrix} = \begin{bmatrix}
-\cos t + t \sin t \\
\cos t
\end{bmatrix} = x_2'.
\]

To check \( \{x_1, x_2\} \) for linear independence, we use the Wronskian:

\[
W[x_1, x_2](t) = \begin{vmatrix}
t \sin t & -t \cos t \\
\cos t & \sin t
\end{vmatrix} = t,
\]

which is nonzero for all choices of \( t \neq 0 \). Therefore, \( \{x_1, x_2\} \) is linearly independent. Therefore, the general solution to the system is

\[
x(t) = c_1 \begin{bmatrix}
t \sin t \\
\cos t
\end{bmatrix} + c_2 \begin{bmatrix}
-t \cos t \\
\sin t
\end{bmatrix}.
\]

7. (a) Write the fundamental matrix as \( X(t) = [x_1, x_2, \ldots, x_n] \), where the set of vector functions \( \{x_1, x_2, \ldots, x_n\} \) is linearly independent. Therefore, by Theorem 7.3.2, the general solution to the system \( x' = A(t)x \) is

\[
x(t) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = [x_1, x_2, \ldots, x_n] \begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix} = X(t)c,
\]

where \( c \) is a vector of constants.

(b) Note that \( x = X(t)X^{-1}(t_0)x_0 \) is a solution to \( x' = Ax \) by part (a). Moreover, if we evaluate \( x(t_0) \) we get \( x(t_0) = X(t_0)X^{-1}(t_0)x_0 = x_0 \), so that \( x \) also satisfies the initial condition in this problem. Since solutions to such an initial-value problem are unique (Theorem 7.3.1), \( x \) is the solution to this initial-value problem.

**Solutions to Section 7.4**

**True-False Review:**

1. **TRUE.** If \( x(t) = e^{\lambda t}v \), then

\[
x'(t) = \lambda e^{\lambda t}v = e^{\lambda t}(\lambda v) = e^{\lambda t}(Av) = A(e^{\lambda t}v) = Ax(t).
\]

This calculation appears prior to Theorem 7.4.1.

3. **FALSE.** Many counterexamples can be given. For instance, if \( A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), then the matrices \( A \) and \( B \) have the same characteristic equation: \( \lambda^2 = 0 \). However, whereas any constant vector function \( x(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \) is a solution to \( x' = Ax \), we have \( Bx = \begin{bmatrix} c_2 \\ 0 \end{bmatrix} \) and \( x' = 0 \). Therefore, unless \( c_2 = 0 \),
\( x \) is not a solution to the system \( x' = Bx \). Hence, the systems \( x' = Ax \) and \( x' = Bx \) have differing solution sets.

5. TRUE. The two real-valued solutions forming a basis for the set of solutions to \( x' = Ax \) in this case are

\[
x_1(t) = e^{at}(\cos bt \mathbf{r} - \sin bt \mathbf{s}) \quad \text{and} \quad x_2(t) = e^{at}(\sin bt \mathbf{r} + \cos bt \mathbf{s}).
\]

The general solution is a linear combination of these, and since \( a > 0 \), \( e^{at} \to \infty \) as \( t \to \infty \). Since \( x_1 \) and \( x_2 \) cannot cancel out in a linear combination (since they are linearly independent), \( e^{at} \) remains as a factor in any particular solution to the vector differential equation, and as this factor tends to \( \infty \), \( \|x(t)\| \to \infty \) as \( t \to \infty \).

Problems:

1. Note that

\[
\det(A - \lambda I) = 0 \iff \begin{vmatrix} -2 - \lambda & -7 \\ -1 & 4 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 2\lambda - 15 = 0 \iff \lambda = -3 \text{ or } \lambda = 5.
\]

Eigenvectors for \( \lambda = -3 \): Here we have

\[
A + 3I = \begin{bmatrix} 1 & -7 \\ -1 & 7 \end{bmatrix},
\]

and so we have an eigenvector corresponding to \( \text{nullspace}(A + 3I) \) as follows: \( \mathbf{v}_1 = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \).

Hence, we obtain the first solution to the linear system: \( x_1(t) = e^{-3t} \begin{bmatrix} 7 \\ 1 \end{bmatrix} \).

Eigenvectors for \( \lambda = 5 \): Here we have

\[
A - 5I = \begin{bmatrix} -7 & -7 \\ -1 & -1 \end{bmatrix},
\]

and so we have an eigenvector corresponding to \( \text{nullspace}(A - 5i) \) as follows: \( \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \).

Hence, we obtain the second solution to the linear system: \( x_2(t) = e^{5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \).

Putting the two solutions \( x_1(t) \) and \( x_2(t) \) together, we obtain the general solution to this linear system:

\[
x(t) = c_1 e^{-3t} \begin{bmatrix} 7 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.
\]

3. Note that

\[
\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & -2 \\ 5 & -5 - \lambda \end{vmatrix} \iff \lambda^2 + 4\lambda + 5 = 0 \iff \lambda = -2 \pm i.
\]

Eigenvectors for \( \lambda = -2 + i \): Here we have

\[
A - (-2 + i)I = \begin{bmatrix} 3 - i & -2 \\ 5 & 3 - i \end{bmatrix}.
\]
The two rows of this matrix must be proportional, since we must be able to find a nonzero vector in its nullspace corresponding to \( \lambda = -2 + i \), so concentrating on the first row, we can determine an eigenvector corresponding to \( \lambda = -2 + i \) as follows: \( v = \begin{bmatrix} \frac{2}{3} \\ -i \end{bmatrix} \). Note that other choices of \( v \) are possible here, such as \( \begin{bmatrix} \frac{3}{5} + i \\ 5 \end{bmatrix} \), obtained by concentrating on the second row instead of the first row. Proceeding with our choice of \( v \), we obtain the solution

\[
x(t) = e^{-2+it} \begin{bmatrix} \frac{2}{3} \\ -i \end{bmatrix}.
\]

Therefore, we obtain the two real-valued solutions

\[
x_1(t) = e^{-2t} \left( \cos t \begin{bmatrix} \frac{2}{3} \\ -1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) \quad \text{and} \quad x_2(t) = e^{-2t} \left( \cos t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin t \begin{bmatrix} \frac{2}{3} \end{bmatrix} \right).
\]

Putting the two solutions \( x_1(t) \) and \( x_2(t) \) together, we obtain the general solution to this linear system:

\[
x(t) = c_1 e^{-2t} \begin{bmatrix} \frac{2}{3} \\ -1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + c_2 e^{-2t} \left( \cos t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin t \begin{bmatrix} \frac{2}{3} \end{bmatrix} \right).
\]

Equivalently, this can be expressed as

\[
x(t) = c_1 e^{-2t} \begin{bmatrix} 2\cos t \\ 3\cos t + \sin t \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2\sin t \\ -\cos t + 3\sin t \end{bmatrix}.
\]

Note that one can also obtain the solution by working with the eigenvalue \( \lambda = -2 - i \). The calculations are similar.

5. Note that

\[
\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 0 & 0 \\ 0 & 5 - \lambda & -7 \\ 0 & 2 & -4 - \lambda \end{vmatrix} = 0 \iff (2 - \lambda)(\lambda^2 - \lambda - 6) = 0 \iff (2 - \lambda)(\lambda - 3)(\lambda + 2) = 0.
\]

Thus, \( \lambda = 2, 3, \) or \( -2 \).

Eigenvectors for \( \lambda = -2 \): Here we have

\[
A + 2I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & -7 \\ 0 & 2 & -2 \end{bmatrix},
\]

and so we have an eigenvector corresponding to \( \text{nullspace}(A + 2I) \) as follows: \( v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \). Hence, we obtain

the first solution to the linear system:

\[
x_1(t) = e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.
\]
Eigenvectors for $\lambda = 2$: Here we have

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -7 \\ 0 & 2 & -6 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - 2I)$ as follows: $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Hence, we obtain the second solution to the linear system: $x_2(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = 3$: Here we have

$$A - 3I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & -7 \\ 0 & 2 & -7 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - 3I)$ as follows: $v_3 = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$. Hence, we obtain the third solution to the linear system: $x_3(t) = e^{3t} \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$.

Putting the three solutions $x_1(t)$, $x_2(t)$, and $x_3(t)$ together, we obtain the general solution to this linear system:

$$x(t) = c_1 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}.$$

7. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 5 - \lambda \end{vmatrix} = 0 \iff (5 - \lambda)(\lambda^2 + 1) = 0 \iff \lambda = 5, i, \text{ or } -i.$$

Eigenvectors for $\lambda = 5$: Here we have

$$A - 5I = \begin{bmatrix} -5 & 1 & 0 \\ -1 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - 5I)$ as follows: $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence, we obtain the first solution to the linear system: $x_1(t) = e^{5t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
Eigenvectors for $\lambda = i$: Here we have

$$A - iI = \begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & 5 - i \end{bmatrix},$$

and we obtain an eigenvector corresponding to $\lambda = i$ by computing $\text{nullspace}(A - iI)$. We obtain $v = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$.

We therefore obtain the corresponding solution

$$x(t) = e^{it} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = (\cos t + i \sin t) \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \left( \cos t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + i \left( \cos t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

Therefore, we obtain two real-valued solutions corresponding to $\lambda = i$:

$$x_2(t) = \cos t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x_3(t) = \cos t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Putting the solutions $x_1(t)$, $x_2(t)$, and $x_3(t)$ together, we obtain the general solution to the linear system:

$$x(t) = c_1 e^{5t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \left( \cos t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + c_3 \left( \cos t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

Equivalently, we can express this solution as

$$x(t) = c_1 e^{5t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ -\sin t \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \sin t \\ \cos t \\ 0 \end{bmatrix}. $$

9. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3 - \lambda & 2 & 6 \\ -2 & 1 - \lambda & -2 \\ -1 & -2 & -4 - \lambda \end{vmatrix} = 0 \iff \lambda = 1, 2, \text{ or } -3.$$

Eigenvectors for $\lambda = 1$: Here we have

$$A - I = \begin{bmatrix} 2 & 2 & 6 \\ -2 & 0 & -2 \\ -1 & -2 & -5 \end{bmatrix},$$
and so we have an eigenvector corresponding to nullspace($A - I$) as follows: $v_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$. Hence, we obtain the first solution to the linear system: $x_1(t) = e^t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

**Eigenvectors for $\lambda = 2$:** Here we have

$$A - 2I = \begin{bmatrix} 1 & 2 & 6 \\ -2 & -1 & -2 \\ -1 & -2 & -6 \end{bmatrix},$$

and so we have an eigenvector corresponding to nullspace($A - 2I$) as follows: $v_2 = \begin{bmatrix} 2 \\ -10 \\ 3 \end{bmatrix}$. Hence, we obtain the second solution to the linear system: $x_2(t) = e^{2t} \begin{bmatrix} 2 \\ -10 \\ 3 \end{bmatrix}$.

**Eigenvectors for $\lambda = -3$:** Here we have

$$A + 3I = \begin{bmatrix} 6 & 2 & 6 \\ -2 & 4 & -2 \\ -1 & -2 & -1 \end{bmatrix},$$

and so we have an eigenvector corresponding to nullspace($A + 3I$) as follows: $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Hence, we obtain the third solution to the linear system: $x_3(t) = e^{-3t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Putting the three solutions $x_1(t)$, $x_2(t)$, and $x_3(t)$ together, we obtain the general solution to this linear system:

$$x(t) = c_1 e^t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ -10 \\ 3 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

**11.** Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3 - \lambda & 0 & -1 \\ 0 & -3 - \lambda & -1 \\ 0 & 2 & -1 - \lambda \end{vmatrix} = 0 \iff (3 - \lambda)(\lambda^2 + 4\lambda + 5) = 0 \iff \lambda^3, -2 + i, \text{ or } -2 - i.$$

**Eigenvectors for $\lambda = 3$:** Here we have

$$A - 3I = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -6 & -1 \\ 0 & 2 & -4 \end{bmatrix},$$
and so we have an eigenvector corresponding to \( \text{nullspace}(A - 3I) \) as follows: \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \). Hence, we obtain the first solution to the linear system: \( \mathbf{x}_1(t) = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \).

**Eigenvectors for \( \lambda = -2 + i \):** Here we have

\[
A - (-2 + i)I = \begin{bmatrix} 5 - i & 0 & -1 \\ 0 & -1 - i & -1 \\ 0 & 2 & 1 - i \end{bmatrix}
\]

13. Note that

\[
\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & -1 & 3 \\ 2 & -1 - \lambda & 3 \\ 2 & -1 & 3 - \lambda \end{vmatrix} = 0 \iff \lambda = 0 \text{ or } \lambda = 4.
\]

**Eigenvectors for \( \lambda = 0 \):** Here we have

\[
A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & -1 & 3 \\ 2 & -1 & 3 \end{bmatrix}.
\]

The corresponding equation here is \( 2x - y + 3z = 0 \). Setting \( x = t \) and \( z = s \), we obtain \( y = 2t + 3s \). Therefore, a typical vector in \( \text{nullspace}(A) \) takes the form \( \begin{bmatrix} t \\ 2t + 3s \\ s \end{bmatrix} \), and thus we obtain two linearly independent eigenvectors corresponding to \( \lambda = 0 \): \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \) and \( \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \). Hence, we obtain two linearly independent solutions to the linear system:

\[
\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.
\]

**Eigenvectors for \( \lambda = 4 \):** Here we have

\[
A - 4I = \begin{bmatrix} -2 & -1 & 3 \\ 2 & -5 & 3 \\ 2 & -1 & -1 \end{bmatrix},
\]

and so we have an eigenvector corresponding to \( \text{nullspace}(A - 4I) \) as follows: \( \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \). Therefore, we obtain the third linearly independent solution to the linear system: \( \mathbf{x}_3(t) = e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \).
Putting the three solutions $x_1(t)$, $x_2(t)$, and $x_3(t)$ together, we obtain the general solution to this linear system:

$$x(t) = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + c_3 e^{it} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$  

15. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ -1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = 0 \iff (\lambda^2 + 1)^2 = 0 \iff \lambda = \pm i.$$  

Eigenvectors for $\lambda = i$: Here we have

$$A = iI = \begin{bmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -i & -1 \\ 0 & 0 & 1 & -i \end{bmatrix},$$

and the system reduces to the equations $-x - iy = 0$ and $z - iw = 0$. Setting $y = t$ and $w = s$, we have $x = -it$ and $z = is$. Therefore, a typical vector in $\text{nullspace}(A - iI)$ takes the form $\begin{bmatrix} -it \\ t \\ is \\ s \end{bmatrix}$ and gives two complex-valued eigenvectors

$$v_1 = \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}.$$  

These eigenvectors correspond to two complex-valued solutions

$$u_1(t) = e^{it} \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad u_2(t) = e^{it} \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}.$$  

We next seek two real-valued solutions corresponding to each of these complex-valued solutions:

$$x(t) = e^{it} \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix} = (\cos t + i \sin t) \left( \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= \left( \cos t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + i \left( \cos t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right).$$
Thus, we obtain the real-valued solutions
\[
x_1(t) = \cos t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2(t) = \cos t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]
or
\[
x_1(t) = \begin{bmatrix} \sin t \\ \cos t \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2(t) = \begin{bmatrix} -\cos t \\ \sin t \\ 0 \\ 0 \end{bmatrix}.
\]

Finally, we use \( u_2(t) \) to generate two more real-valued solutions:
\[
x(t) = e^{it} \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}
\]
\[
= (\cos t + i \sin t) \left( \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)
\]
\[
= \left( \begin{bmatrix} \cos t \\ 0 \\ 0 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) + i \left( \begin{bmatrix} \cos t \\ 0 \\ 0 \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).
\]

Thus, we obtain the real-valued solutions
\[
x_3(t) = \cos t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad x_4(t) = \cos t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},
\]
or
\[
x_3(t) = \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad x_4(t) = \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix}.
\]

Putting together the four real-valued solutions \( x_1(t), x_2(t), x_3(t) \) and \( x_4(t) \) obtained above, we find the general solution
\[
x(t) = c_1 \begin{bmatrix} \sin t \\ \cos t \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -\cos t \\ \sin t \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix}.
\]

17. Note that
\[
\text{det}(A - \lambda I) = 0 \iff \begin{vmatrix} -1 - \lambda & -6 \\ 3 & 5 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 4\lambda + 13 = 0 \iff \lambda = 2 \pm 3i.
\]
Eigenvectors for $\lambda = 2 + 3i$: Here we have

$$A - (2 + 3i)I = \begin{bmatrix} -3 - 3i & -6 \\ 3 & 3 - 3i \end{bmatrix}.$$ 

The two rows of this matrix must be proportional, since we must be able to find a nonzero vector in its nullspace corresponding to $\lambda = 2 + 3i$, so concentrating on the first row, we can determine an eigenvector corresponding to $\lambda = 2 + 3i$ as follows: $v = \begin{bmatrix} -2 \\ 1 + i \end{bmatrix}$. Note that other choices of $v$ are possible here. Using our choice, we obtain

$$x(t) = e^{(2+3i)t}v = e^{2t}(\cos 3t + i \sin 3t) \begin{bmatrix} -2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$ 

Thus, we find two real-valued solutions to the linear system:

$$x_1(t) = e^{2t} \left( \cos 3t \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \sin 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad x_2(t) = e^{2t} \left( \cos 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 3t \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right).$$ 

Therefore, the general solution to the linear system is

$$x(t) = c_1 e^{2t} \left( \cos 3t \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \sin 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + c_2 e^{2t} \left( \cos 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 3t \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right).$$ 

Since

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we can quickly solve for the constants $c_1$ and $c_2$ to obtain $c_1 = -1$ and $c_2 = 3$. Thus, the particular solution to this initial-value problem is

$$x(t) = -e^{2t} \begin{bmatrix} -2 \cos 3t \\ \cos 3t - \sin 3t \end{bmatrix} + 3e^{2t} \begin{bmatrix} -2 \sin 3t \\ \cos 3t + \sin 3t \end{bmatrix}.$$ 

19. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 4 \\ -4 & -\lambda \end{vmatrix} = 0 \iff \lambda^2 + 16 = 0 \iff \lambda = \pm 4i.$$ 

Eigenvectors for $\lambda = 4i$: Here we have

$$A - 4i I = \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix}.$$
and focusing on the first row of this matrix (since the second row must be row-equivalent to the first row), we obtain \(-4ix + 4y = 0\). We may choose the eigenvector \(v = \begin{bmatrix} 1 \\ i \end{bmatrix}\). Thus, we obtain the solution
\[
x(t) = e^{4it}v = (\cos 4t + i \sin 4t) \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)
= \left( \cos 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 4t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + i \left( \cos 4t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).
\]
This generates two real-valued solutions:

\[
x_1(t) = \cos 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 4t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad x_2(t) = \cos 4t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]
or

\[
x_1(t) = \begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix} \quad \text{and} \quad x_2(t) = \sin 4t \begin{bmatrix} \cos 4t \\ \cos 4t \end{bmatrix}.
\]
Therefore, putting these solutions together, we obtain the general solution to the linear system:

\[
x(t) = c_1 \begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix} + c_2 \begin{bmatrix} \sin 4t \\ \cos 4t \end{bmatrix}.
\]
Since
\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]
we can quickly solve for the constants \(c_1\) and \(c_2\): \(c_1 = c_2 = 1\). Therefore, the particular solution to this initial-value problem is

\[
x(t) = \begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix} + \begin{bmatrix} \sin 4t \\ \cos 4t \end{bmatrix} = \begin{bmatrix} \cos 4t + \sin 4t \\ -\sin 4t + \cos 4t \end{bmatrix}.
\]
Since \(x^2_1 + x^2_2 = 2\) for all \(t \in \mathbb{R}\), the graph of the solution is a circle of radius \(\sqrt{2}\), centered at \((0, 0)\).

21. Let us assume that \(c_1x_1 + c_2x_2 = 0\). We must show that \(c_1 = c_2 = 0\). Substituting the expressions given in the problem for \(x_1\) and \(x_2\), we obtain

\[
e^{at} \left[ c_1 (r \cos bt - s \sin bt) + c_2 (r \sin bt + s \cos bt) \right] = 0,
\]
\[
(c_1 r + c_2 s) \cos bt + (c_2 r - c_1 s) \sin bt = 0,
\]
\[
c_1 r + c_2 s = 0 \quad \text{and} \quad c_2 r - c_1 s = 0.
\]
But \(r\) and \(s\) are linearly independent by assumption. Therefore, \(c_1 = c_2 = 0\).

23. Note that \(\lim_{t \to \infty} e^{\lambda t} = 0\) if and only if \(\text{Re}(\lambda) < 0\). All solutions to the system are of the form \(x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2\), where \(\lambda_1\) and \(\lambda_2\) are the eigenvalues of \(A\), and \(v_1\) and \(v_2\) are corresponding linearly independent eigenvectors. Thus we are given that for all values of \(c_1\) and \(c_2\) \(\lim_{t \to \infty} x(t) = 0\). In particular,

\[
\lim_{t \to \infty} e^{\lambda_1 t} v_1 = 0 \quad \text{and} \quad \lim_{t \to \infty} e^{\lambda_2 t} v_2 = 0.
\]
Since \( v_1 \) and \( v_2 \) are constant vectors, it must therefore be the case that \( \lim_{t \to \infty} e^{\lambda_1 t} = 0 \) and \( \lim_{t \to \infty} e^{\lambda_2 t} = 0 \). Consequently, \( \text{Re}(\lambda_1) < 0 \) and \( \text{Re}(\lambda_2) < 0 \).

25. We have \( \det(A - \lambda I) = 0 \) if and only if \( \begin{vmatrix} a - \lambda & b \\ -b & a - \lambda \end{vmatrix} = 0 \) if and only if \( \lambda^2 - 2a\lambda + a^2 + b^2 = 0 \). Therefore, \( \lambda = a \pm bi \). When \( \lambda = a - bi \), then \( A - \lambda I \) becomes \( \begin{bmatrix} bi & b \\ -bi & bi \end{bmatrix} \), and the linear system \((A - \lambda I)v = 0\) has solutions of the form \( v = r(i, 1) \), where \( r \in \mathbb{C} \). A complex-valued solution of the system is therefore

\[
e^{(a-b)it} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = e^{at}(\cos bt - i \sin bt) \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.
\]

This gives rise to two real-valued solutions

\[
x_1(t) = e^{at} \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix} \quad \text{and} \quad x_2(t) = e^{at} \begin{bmatrix} \cos bt \\ -\sin bt \end{bmatrix}.
\]

Therefore, the general solution is given by

\[
x(t) = e^{at} \left\{ c_1 \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix} + c_2 \begin{bmatrix} \cos bt \\ -\sin bt \end{bmatrix} \right\}.
\]

Since \( a < 0 \), as \( t \to \infty \) we have \( x(t) \to 0 \). More specifically, we have

\[
x_1(t) = e^{at}(c_1 \sin bt + c_2 \cos bt) \quad \text{and} \quad x_2(t) = e^{at}(c_1 \cos bt - c_2 \sin bt).
\]

Hence, \( x_1(t)^2 + x_2(t)^2 = e^{2at}(c_1^2 + c_2^2) \). Therefore, the solution curves lie on circles of ever decreasing radii as \( t \) increases. Thus, the solution curves in the \( x_1x_2 \)-plane spiral towards the origin as \( t \) increases.

27. The given linear system can be expressed in matrix form as \( x' = Ax \), where \( A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \). We have

\[
\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -b & -a - \lambda \end{vmatrix} = 0 \quad \text{if and only if} \quad \lambda^2 + a\lambda + b = 0 \quad \text{if and only if} \quad \lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.
\]

Thus, both eigenvalues have negative real part. From Problem 22, this implies that \( \lim_{t \to \infty} x(t) = 0 \).

### Solutions to Section 7.5

**True-False Review:**

1. **FALSE.** Every \( n \times n \) linear system of differential equations has \( n \) linearly independent solutions (Theorem 7.3.2), regardless of whether \( A \) is defective or nondefective.

3. **FALSE.** The number of linearly independent solutions to \( x' = Ax \) corresponding to \( \lambda \) is equal to the algebraic multiplicity of the eigenvalue \( \lambda \) as a root of the characteristic equation for \( A \), not the dimension of the eigenspace \( E_\lambda \).

**Problems:**

1. We have \( \det(A - \lambda I) = 0 \) if and only if \( \begin{vmatrix} -\lambda & -2 \\ 2 & 4 - \lambda \end{vmatrix} = 0 \) if and only if \( \lambda^2 - 4\lambda + 4 = 0 \). Therefore \( \lambda = 2 \) (with multiplicity two). When \( \lambda = 2 \), the corresponding eigenvectors take the form \( v = r(-1, 1) \), where \( r \in \mathbb{R} \). We will determine a second linearly independent solution of the form

\[
x_1(t) = e^{2t}(v_1 + tv_0)
\]
where \( v_1 \) and \( v_0 \) are determined from

\[
(A - 2I)^2 v_1 = 0, \quad (A - 2I) v_1 \neq 0, \quad v_0 = (A - 2I) v_1.
\]

In this case, we have \( A - 2I = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} \) and \( (A - 2I)^2 = 0_2 \). Therefore, we may choose \( v_1 \) to be any vector such that \( (A - 2I) v_1 \neq 0 \). For simplicity, we take \( v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Thus, \( v_0 = (A - 2I) v_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \).

From the expressions for \( v_0 \) and \( v_1 \), we can write down two linearly independent solutions to the vector differential equation:

\[
x_0(t) = e^{2t} \begin{bmatrix} -2 \\ 2 \end{bmatrix} \quad \text{and} \quad x_1(t) = e^{2t} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\} = e^{2t} \begin{bmatrix} 1 - 2t \\ 2t \end{bmatrix}.
\]

Consequently, the general solution to the vector differential equation is

\[
x(t) = c_1 e^{2t} \begin{bmatrix} -2 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 - 2t \\ 2t \end{bmatrix}.
\]

3. We have \( \det(A - \lambda I) = 0 \) if and only if

\[
\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & -1 - \lambda \end{vmatrix} = 0 \quad \text{if and only if} \quad (\lambda + 1)(1 - \lambda^2) = 0.
\]

Therefore, the eigenvalues of \( A \) are \( \lambda = 1 \) and \( \lambda = -1 \) (with multiplicity two).

Eigenvectors for \( \lambda = 1 \): The corresponding eigenvectors are \( v = r(1, 1, 1) \), where \( r \in \mathbb{R} \). Hence, one solution of the system is \( x(t) = e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \).

Eigenvectors for \( \lambda = -1 \): The corresponding eigenvectors are \( v = r(1, -1, 1) \), where \( r \in \mathbb{R} \). Thus, we obtain one solution corresponding to \( \lambda = -1 \) of \( x_0(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \). We will determine a second linearly independent solution corresponding to \( \lambda = -1 \) of the form

\[
x_1(t) = e^{-t}(v_1 + tv_0)
\]

where \( v_1 \) and \( v_0 \) are determined from

\[
(A + I)^2 v_1 = 0, \quad (A + I) v_1 \neq 0, \quad v_0 = (A + I) v_1.
\]

In this case, we have \( A + I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \) and \( (A + I)^2 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \). Many choices for \( v_1 \) are possible that meet the requirements above. Let us take, as one possibility, \( v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \). Then \( v_0 = (A + I) v_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \).

Hence, we have the solution

\[
x_1(t) = e^{-t} \left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right).
\]
We have therefore obtained two linearly independent solutions to the system of differential equations corresponding to $\lambda = -1$:

$$x_0(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad x_1(t) = e^{-t} \left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right).$$

Putting this together with the solution obtained above corresponding to $\lambda = 1$, we obtain the general solution to this system of differential equations:

$$x(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{-t} \left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right),$$

or equivalently,

$$x(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + e^{-t} \left( c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 + t \\ -t \\ -1 + t \end{bmatrix} \right).$$

5. We have $\det(A - \lambda I) = 0$ if and only if

$$\begin{vmatrix} -2 - \lambda & 0 & 0 \\ 1 & -3 - \lambda & -1 \\ -1 & 1 & -1 - \lambda \end{vmatrix} = 0 \quad \text{if and only if} \quad (-2 - \lambda)(\lambda^2 + 4\lambda + 4) = 0 \quad \text{if and only if} \quad (\lambda + 2)^3 = 0.$$

This means that the only eigenvalue of $A$ is $\lambda = -2$ (with multiplicity 3).

Eigenvectors for $\lambda = -2$: The matrix $A + 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. There are two free variables, and hence we obtain two linearly independent eigenvectors. In fact, the eigenvectors must take the form $(r + s, r, s) = r(1, 1, 0) + s(1, 0, 1)$, where $r, s \in \mathbb{R}$. Therefore, we can construct two linearly independent solutions to the system of differential equations as follows:

$$x_0^{(1)}(t) = e^{-2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x_0^{(2)}(t) = e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$  

Consequently, we seek a third linearly independent solution $x_1(t)$ to the vector differential equation of the form

$$x_1(t) = e^{-2t}(v_1 + tv_0)$$

where $v_1$ and $v_0$ are determined from

$$(A + 2I)^2 v_1 = 0, \quad (A + 2I) v_1 \neq 0, \quad v_0 = (A + 2I) v_1.$$

In this case, we have $A + 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ and $(A + 2I)^2 = 0$. Therefore, we can choose $v_1$ to be any vector such that $(A + 2I) v_1 \neq 0$. For instance, we can choose $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then $v_0 = (A + 2I) v_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. 


Hence, we have the solution $x_1(t) = e^{-2t} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$. Therefore, the general solution to this vector differential equation is
\[
x(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-2t} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right),
\]
or equivalently,
\[
x(t) = e^{-2t} \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ t \\ -t \end{bmatrix} \right\}.
\]

7. We have $\det(A - \lambda I) = 0$ if and only if
\[
\begin{vmatrix}
4 - \lambda & 0 & 0 \\
1 & 4 - \lambda & 0 \\
0 & 1 & 4 - \lambda
\end{vmatrix} = 0
\]
if and only if $(4 - \lambda)^3 = 0$. This means that the only eigenvalue of $A$ is $\lambda = 4$ (with multiplicity 3).

Eigenvectors for $\lambda = 4$: The matrix $A - 4I = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}$. Therefore, we obtain eigenvectors of the form $v_0 = r(0, 0, 1)$, where $r \in \mathbb{R}$, and we obtain a solution to $x' = Ax$ of the form
\[
x_0(t) = e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

According to Theorem 7.5.4, there exist two further linearly independent solutions to $x' = Ax$ of the form
\[
x_1(t) = e^{4t}(v_1 + tv_0),
\]
\[
x_2(t) = e^{4t}(v_2 + tv_1 + \frac{1}{2!}t^2v_0),
\]
where
\[
(A - 4I)^3v_2 = 0, \quad (A - 4I)^2v_2 \neq 0,
\]
and
\[
v_1 = (A - 4I)v_2, \quad v_0 = (A - 4I)^2v_2.
\]

A short calculation shows that
\[
A - 4I = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad (A - 4I)^2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad (A - 4I)^3 = 0_3.
\]

Therefore, we may take
\[
v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_1 = (A - 4I)v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_0 = (A - 4I)^2v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]
Hence, we obtain
\[ x_1(t) = e^{4t} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = e^{4t} \begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix} \]
and
\[ x_2(t) = e^{4t} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2!} t^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = e^{4t} \begin{bmatrix} 1 \\ t \\ t^2/2 \end{bmatrix}. \]
Therefore, the general solution to the vector differential equation is
\[ x(t) = c_1 e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \\ t \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 1 \\ t \\ t^2/2 \end{bmatrix} \]
\[ = e^{4t} \left\{ c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ t \\ t^2/2 \end{bmatrix} \right\}. \]

9. We have \( \det(A - \lambda I) = 0 \) if and only if \( \begin{vmatrix} 3 - \lambda & 1 & 0 \\ -1 & 5 - \lambda & 0 \\ 0 & 0 & 4 - \lambda \end{vmatrix} = 0 \) if and only if \( (4 - \lambda)(\lambda^2 - 8\lambda + 16) = 0 \) if and only if \( (\lambda - 4)^3 = 0 \). This means that the only eigenvalue of \( A \) is \( \lambda = 4 \) (with multiplicity 3).

**Eigenvectors for \( \lambda = 4 \):** The matrix \( A - 4I = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 4 - \lambda \end{bmatrix} \). There are two free variables, and hence we obtain two linearly independent eigenvectors. In fact, the eigenvectors must take the form \((r, r, s) = r(1, 1, 0) + s(0, 0, 1)\), where \( r, s \in \mathbb{R} \). Therefore, we can construct two linearly independent solutions to the system of differential equations as follows:
\[ x^{(1)}_0(t) = e^{4t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x^{(2)}_0(t) = e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \]
Consequently, we seek a third linearly independent solution \( x_1(t) \) to the vector differential equation of the form
\[ x_1(t) = e^{4t}(v_1 + tv_0) \]
where \( v_1 \) and \( v_0 \) are determined from
\[ (A - 4I)^2 v_1 = 0, \quad (A - 4I)v_1 \neq 0, \quad v_0 = (A - 4I)v_1. \]
In this case,
\[ A - 4I = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (A - 4I)^2 = 0. \]
Therefore, we may choose any \( v_1 \) such that \( (A - 4I)v_1 = 0 \). For instance, we may take \( v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \). Then
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\[ v_0 = (A - 4I)v_1 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}. \]  
Hence, we have the solution

\[ x_1(t) = e^{4t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}. \]

Therefore, the general solution to this vector differential equation is

\[ x(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 1-t \\ -t \\ 0 \end{bmatrix} \]

11. We have \( \det(A - \lambda I) = 0 \) if and only if

\[
\begin{array}{ccc}
-\lambda & -1 & 0 \\
1 & -\lambda & 0 \\
1 & 0 & 2 - \lambda \\
0 & 1 & 0 & 2 - \lambda
\end{array} = 0 \]

This means that the eigenvalues of \( A \) are \( \lambda = \pm i \) and \( \lambda = 2 \) (with multiplicity 2).

Eigenvectors for \( \lambda = i \): We have \( A - iI = \begin{bmatrix} -i & -1 & 0 & 0 \\
1 & -i & 0 & 0 \\
1 & 0 & 2 - i & 1 \\
0 & 1 & 0 & 2 - i
\end{bmatrix} \). A short calculation shows that a corresponding complex eigenvector for \( \lambda = i \) is \( v = (-5(1 + 2i), 5(-2 + i), -2 + 4i, 5) \). Therefore, we obtain the complex-valued solution

\[ u(t) = (\cos t + i \sin t) \begin{bmatrix} -5(1 + 2i) \\
5(-2 + i) \\
-2 + 4i \\
5 \end{bmatrix}. \]

We can then derive two real-valued linearly independent solutions:

\[ x_1(t) = \begin{bmatrix} 5(2 \sin t - \cos t) \\
-5(2 \cos t + \sin t) \\
-2(\cos t + 2 \sin t) \\
5 \cos t
\end{bmatrix} \quad \text{and} \quad x_2(t) = \begin{bmatrix} -5(2 \cos t + \sin t) \\
5(\cos t - 2 \sin t) \\
2(\cos t - \sin t) \\
5 \sin t
\end{bmatrix}. \]

Eigenvectors for \( \lambda = 2 \): We have \( A - 2I = \begin{bmatrix} -2 & -1 & 0 & 0 \\
1 & -2 & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix} \). A short calculation shows that we obtain only one linearly independent eigenvector: \( v = r(0, 0, 1, 0) \), where \( r \in \mathbb{C} \). Therefore, another real-valued solution to this system of differential equation is given by

\[ x_3(t) = e^{2t} \begin{bmatrix} 0 \\
0 \\
1 \\
0
\end{bmatrix}. \]
To find a fourth real-valued linearly independent solution, we seek a solution of the form $x_4(t) = e^{2t}(v_1 + tv_0)$, where $v_1$ and $v_0$ are determined from

$$(A - 2I)v_1 = 0, \quad (A - 2I)v_1 \neq 0, \quad v_0 = (A - 2I)v_1.$$ 

In this case, we have $A - 2I = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $(A - 2I)^2 = \begin{bmatrix} 3 & 4 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix}$. We may choose $v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Then $v_0 = (A - 2I)v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Therefore, we have a solution $x_4(t) = e^{2t}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = e^{2t}\begin{bmatrix} 0 \\ 0 \\ t \\ 1 \end{bmatrix}$.

Therefore, the general solution to this system of differential equations is

$$x(t) = c_1 \begin{bmatrix} 5(2\sin t - \cos t) \\ -5(2\cos t + \sin t) \\ -2(\cos t + 2\sin t) \\ 5\cos t \end{bmatrix} + c_2 \begin{bmatrix} -5(2\cos t + \sin t) \\ 5(\cos t - 2\sin t) \\ 2(\cos t - \sin t) \\ 5\sin t \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

13. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & -1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix} = 0$ if and only if $(\lambda^2 + 1)^2 = 0$. This means that the eigenvalues of $A$ are $\lambda = \pm i$ (with multiplicity 2).

Eigenvectors for $\lambda = i$: We have $A - iI = \begin{bmatrix} -i & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 0 & -i & -1 \\ 0 & 1 & 1 & -i \end{bmatrix}$, which yields corresponding eigenvectors of the form $v = r(0, 0, i, 1)$, where $r \in \mathbb{C}$. Therefore, we obtain a solution to the linear system $x' = Ax$ as follows:

$$u(t) = (\cos t + i\sin t) \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} = (\cos t + i\sin t) \left( \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix}.$$
This gives rise to the two real-valued solutions

\[ x_1(t) = \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad x_2(t) = \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix}. \]

We seek to find further real-valued and linearly independent solutions to \( x' = Ax \) of the form \( x(t) = e^{it}(v_1 + tv_0) \), where

\[ (A - iI)^2v_1 = 0, \quad (A - iI)v_1 \neq 0, \quad (A - iI)v_1 = v_0. \]

One possible choice for \( v_1 \) is \( v_1 = (i, 1, 1, -i) \). Then \( v_0 = (A - iI)v_1 = (0, 0, i, 1) \). Therefore, we obtain the solution

\[ u(t) = (\cos t + i \sin t) \begin{bmatrix} i \\ 1 \\ 1 \\ -i \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \]

This gives rise to two more real-valued solutions:

\[ x_3(t) = \begin{bmatrix} -\sin t \\ \cos t \\ \cos t - t \sin t \\ \sin t + t \cos t \end{bmatrix} \quad \text{and} \quad x_4(t) = \begin{bmatrix} \cos t \\ \sin t \\ \sin t + t \cos t \\ t \sin t - \cos t \end{bmatrix}. \]

Therefore, the general solution to this linear system of differential equations is

\[ x(t) = c_1 \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix} + c_3 \begin{bmatrix} -\sin t \\ \cos t \\ \cos t - t \sin t \\ \sin t + t \cos t \end{bmatrix} + c_4 \begin{bmatrix} \cos t \\ \sin t \\ \sin t + t \cos t \\ t \sin t - \cos t \end{bmatrix}. \]

15. We have \( \det(A - \lambda I) = 0 \) if and only if

\[ \begin{vmatrix} -2 - \lambda & -1 & 4 \\ 0 & -1 - \lambda & 0 \\ -1 & -3 & 2 - \lambda \end{vmatrix} = 0 \]

This means that the eigenvalues of \( A \) are \( \lambda = -1 \) and \( \lambda = 0 \) (with multiplicity 2).

**Eigenvectors for \( \lambda = -1 \):** We have \( A + I = \begin{bmatrix} -1 & -1 & 4 \\ 0 & 0 & 0 \\ -1 & -3 & 3 \end{bmatrix} \), which row reduces to \( \begin{bmatrix} 1 & 1 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \). This gives rise to corresponding eigenvectors of the form \( v = r(9, -1, 2) \), where \( r \in \mathbb{R} \). Taking \( r = 1 \), we obtain the following solution to the linear system: \( x_1(t) = e^{-t} \begin{bmatrix} 9 \\ -1 \\ 2 \end{bmatrix} \).

**Eigenvectors for \( \lambda = 0 \):** We have \( A = \begin{bmatrix} -2 & -1 & 4 \\ 0 & -1 & 0 \\ -1 & -3 & 2 \end{bmatrix} \), which row reduces to \( \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \). This gives rise to corresponding eigenvectors of the form \( v = r(2, 0, 1) \), where \( r \in \mathbb{R} \). Taking \( r = 1 \), we obtain the following...
solution to the linear system: $x_2(t) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. To find a second linearly independent solution corresponding to $\lambda = 0$, we seek a solution of the form $x_3(t) = v_1 + tv_0$, where

$$A^2v_1 = 0, \quad Av_1 \neq 0, \quad v_0 = Av_1.$$ 

In this case, we have $A^2 = \begin{bmatrix} 0 & -9 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$. We can choose $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then $v_0 = Av_1 = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$. Hence, we have another solution $x_3(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$. Therefore, the general solution to this linear system is

$$x(t) = c_1e^{-t} \begin{bmatrix} 9 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c_3 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \right).$$

We have been given that $x(0) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, so that

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 9 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

We quickly solve this system to find $c_1 = -1$, $c_2 = 3$, and $c_3 = 1$. Therefore, the solution to this initial-value problem is

$$x(t) = -e^{-t} \begin{bmatrix} 9 \\ -1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 7 - 2t - 9e^{-t} \\ -2e^{-t} \\ 3 - t - 2e^{-t} \end{bmatrix}.$$

17. All solutions to the system are linear combinations of solutions of the form $x(t) = e^{\lambda t}(v_1 + tv_0)$, where $\lambda$ is an eigenvalue of $A$ and $v_0$ may be the zero vector. Writing $\lambda = a + ib$, we have $x(t) = e^{at}(\cos bt + i \sin bt)(v_1 + tv_0)$. Now for $k = 0$ or $k = 1$, we have $\lim_{t \to \infty} t^k e^{at} = 0$ if and only if $a < 0$. Therefore, it follows that

$$\lim_{t \to \infty} e^{at}(\cos bt + i \sin bt)(v_1 + tv_0) = 0 \text{ if and only if } a < 0,$$

and so all solutions to the system satisfy

$$\lim_{t \to \infty} x(t) = 0$$

if and only if all eigenvalues of $A$ have negative real part.

19. (a) The number of cycles of generalized eigenvectors of $A$ corresponding to $\lambda$ is precisely the number of linearly independent eigenvectors of $A$ corresponding to $\lambda$. Since our assumption in the Theorem is that the eigenspace corresponding to $\lambda$ is $r$-dimensional, we conclude that we have $r$ cycles of generalized eigenvectors of $A$ corresponding to $\lambda$. 
(b) The length $l_i$ of a cycle of generalized eigenvectors is given as $k_i + 1$ in the formulation of Theorem 7.5.4. By the fact cited on p. 434 of the text, since the multiplicity of $\lambda$ is $m$, the total number of linearly independent generalized eigenvectors corresponding to $\lambda$ is $m$. However, the number of such generalized eigenvectors found in part (a) is

$$l_1 + l_2 + \cdots + l_r = (k_1 + 1) + (k_2 + 1) + \cdots + (k_r + 1) = (k_1 + k_2 + \cdots + k_r) + r.$$ 

Therefore,

$$k_1 + k_2 + \cdots + k_r = m - r.$$

(c) A cycle of generalized eigenvectors corresponding to $\lambda$ of length $l_i = k_i + 1$ must take the form $\{(A - \lambda I)^{k_i+1}v, (A - \lambda I)^{k_i}v, \ldots, (A - \lambda I)v, v\}$, where $v$ is a vector chosen such that $(A - \lambda I)^{k_i+1}v = 0$ and $(A - \lambda I)^{k_i}v \neq 0$. This vector $v$ is precisely the vector $v_{k_i}$ referred to in the statement of Theorem 7.5.4. The statements (7.5.16)-(7.5.19) describe the remaining vectors comprising the cycle.

(d) Assume that

$$c_0x_0^{(i)}(t) + c_1x_1^{(i)}(t) + \cdots + c_{k_i}x_{k_i}^{(i)}(t) = 0.$$ 

We must show that $c_0 = c_1 = \cdots = c_{k_i} = 0$. Substituting the expressions for $x_j^{(i)}(t)$ from (7.5.11)-(7.5.14), we obtain

$$c_0e^{\lambda t}v_0^{(i)} + c_1e^{\lambda t}(v_1^{(i)} + tv_0^{(i)}) + \cdots + c_{k_i}e^{\lambda t}(v_{k_i}^{(i)} + tv_{k_i-1}^{(i)} + \cdots + \frac{1}{k_i!}t^{k_i}v_0^{(i)}) = 0.$$ 

Since this must hold for all $t$, we can choose $t = 0$ and reduce this equation to

$$c_0v_0^{(i)} + c_1v_1^{(i)} + \cdots + c_{k_i}v_{k_i}^{(i)} = 0.$$ 

Since $\{v_0^{(i)}, v_1^{(i)}, \ldots, v_{k_i}^{(i)}\}$ is linearly independent, we conclude that $c_0 = c_1 = \cdots = c_{k_i} = 0$, as desired.

(e) We have

$$[x_j^{(i)}(t)]' = \lambda e^{\lambda t} \left( v_j^{(i)} + tv_{j-1}^{(i)} + \cdots + \frac{1}{j!}t^j v_0^{(i)} \right) + e^{\lambda t} \left( v_{j-1}^{(i)} + tv_{j-2}^{(i)} + \cdots + \frac{1}{(j-1)!}t^{j-1}v_0^{(i)} \right),$$

$$= e^{\lambda t} \left[ (A - \lambda I)v_j^{(i)} + (\lambda + t)v_{j-1}^{(i)} + \left( \frac{\lambda^2}{2} + t \right)v_{j-2}^{(i)} + \cdots + \left( \frac{\lambda^j}{j!} + \frac{t^{j-1}}{(j-1)!} \right)v_0^{(i)} \right].$$

$$= e^{\lambda t} \left[ (A - \lambda I)v_j^{(i)} + t(A - \lambda I)v_{j-1}^{(i)} + \frac{t^2}{2!}(A - \lambda I)v_{j-2}^{(i)} + \cdots + \frac{t^j}{j!}(A - \lambda I)v_0^{(i)} \right].$$

$$= Ae^{\lambda t} \left[ v_j^{(i)} + tv_{j-1}^{(i)} + \cdots + \frac{t^j}{j!}v_0^{(i)} \right],$$

as desired.
(f) Apply part (1) to each eigenvalue of $A$. For each eigenvalue, we obtain linearly independent solutions totalling the multiplicity of the eigenvalue. The sum of the multiplicities of all of the eigenvalues is $n$, thereby yielding $n$ solutions to $x' = Ax$. The solutions $x_i^{(t)}(t)$ comprising a given cycle of generalized eigenvectors are linearly independent by part (d), and vectors corresponding to different cycles and different eigenvalues are linearly independent from one another. Therefore, we have $n$ linearly independent solutions to $x' = Ax$.

Solutions to Section 7.6

True-False Review:

1. **FALSE.** The particular solution, given by Equation (7.6.4), depends on a fundamental matrix $X(t)$ for the nonhomogeneous linear system, and the columns of any fundamental matrix consist of linearly independent solutions to the corresponding homogeneous vector differential equation $x' = Ax$.

3. **TRUE.** By definition, we have $X = [x_1, x_2, \ldots, x_n]$, so
\[ X' = [x'_1, x'_2, \ldots, x'_n] = [Ax_1, Ax_2, \ldots, Ax_n] = AX. \]

5. **FALSE.** A formula for the particular solution is given in Theorem 7.6.1:
\[ x_p(t) = X(t) \int^t X^{-1}(s)b(s)ds. \]

Since an arbitrary integration constant can be applied to this expression, we have an arbitrary number of different choices for the particular solution $x_p(t)$.

Problems:

1. We have $\det(A - \lambda I) = 0$ if and only if
\[
\begin{vmatrix}
2 - \lambda & -1 \\
-1 & 2 - \lambda
\end{vmatrix}
= 0 \text{ if and only if } \lambda^2 - 4\lambda + 3 = 0 \text{ if and only if } (\lambda - 3)(\lambda - 1) = 0.
\]
This means that the eigenvalues of $A$ are $\lambda = 1$ and $\lambda = 3$.

Eigenvectors for $\lambda = 1$: We have $A - I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(1, 1)$, where $r \in \mathbb{R}$. Hence, one solution to the associated homogeneous system is $x_1(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = 3$: We have $A - 3I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(-1, 1)$, where $r \in \mathbb{R}$. Hence, a second solution to the associated homogeneous system is $x_2(t) = e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Therefore, the general solution to the associated homogeneous system is
\[ x_c(t) = c_1e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

and the fundamental matrix is
\[ X(t) = \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix}. \]

We have
\[ X^{-1}(t) = \frac{1}{2} \begin{bmatrix} e^{3t} & e^{-t} \\ -e^{3t} & e^{3t} \end{bmatrix}. \]
Therefore,

\[
\mathbf{u}(t) = \int_t^X \mathbf{X}^{-1}(s) \mathbf{b}(s) ds = \frac{1}{2} \int_t^t \begin{bmatrix} e^{-s} & e^{-s} \\ -e^{-3s} & e^{-3s} \end{bmatrix} \begin{bmatrix} 0 \\ 4e^s \end{bmatrix} ds = \frac{1}{2} \int_t^t \begin{bmatrix} 4 \\ 4e^{-2s} \end{bmatrix} ds = \begin{bmatrix} 2t \\ -e^{-2t} \end{bmatrix}.
\]

Hence, we obtain the particular solution

\[
\mathbf{x}_p(t) = \mathbf{X}(t) \mathbf{u}(t) = \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \begin{bmatrix} 2t \\ -e^{-2t} \end{bmatrix} = \begin{bmatrix} e^t(2t + 1) \\ e^t(2t - 1) \end{bmatrix}.
\]

Thus, the general solution to the nonhomogeneous system of differential equations is

\[
\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} e^t(2t + 1) \\ e^t(2t - 1) \end{bmatrix}.
\]

3. We have \(\det(A - \lambda I) = 0\) if and only if

\[
\begin{vmatrix} -1 - \lambda & 1 \\ 3 & 1 - \lambda \end{vmatrix} = 0 \text{ if and only if } \lambda^2 - 4 = 0.
\]

This means that the eigenvalues of \(A\) are \(\lambda = -2\) and \(\lambda = 2\).

Eigenvectors for \(\lambda = -2\): We have \(A + 2I = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\), which gives rise to corresponding eigenvectors of the form \(\mathbf{v} = r(-1, 1)\), where \(r \in \mathbb{R}\). Hence, one solution to the associated homogeneous system is \(\mathbf{x}_1(t) = e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}\).

Eigenvectors for \(\lambda = 2\): We have \(A - 2I = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}\), which gives rise to corresponding eigenvectors of the form \(\mathbf{v} = r(1, 3)\), where \(r \in \mathbb{R}\). Hence, a second solution to the associated homogeneous system is \(\mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}\).

Therefore, the general solution to the associated homogeneous system is

\[
\mathbf{x}_c(t) = c_1 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\]

and the fundamental matrix is

\[
\mathbf{X}(t) = \begin{bmatrix} -e^{-2t} & e^{2t} \\ e^{-2t} & 3e^{2t} \end{bmatrix}.
\]

We have

\[
\mathbf{X}^{-1}(t) = -\frac{1}{4} \begin{bmatrix} 3e^{2t} & -e^{2t} \\ -e^{-2t} & -e^{-2t} \end{bmatrix}.
\]
Therefore,
\[ u(t) = \int_{t_0}^t X^{-1}(s) b(s) \, ds \]

\[ = -\frac{1}{4} \int_{t_0}^t \begin{bmatrix} 3e^{2s} & -e^{2s} \\ -e^{-2s} & -e^{-2s} \end{bmatrix} \begin{bmatrix} 20e^{3s} \\ 12e^{-s} \end{bmatrix} \, ds \]

\[ = -\frac{1}{4} \int_{t_0}^t \begin{bmatrix} 60e^{5s} - 12e^{3s} \\ -20e^s - 12e^{-s} \end{bmatrix} \, ds = \int_{t_0}^t \begin{bmatrix} -15e^{5s} + 3e^{3s} \\ 5e^s + 3e^{-s} \end{bmatrix} \, ds \]

\[ = \begin{bmatrix} -3e^{5t} + e^{3t} \\ 5e^t - 3e^{-t} \end{bmatrix}. \]

Hence, we obtain the particular solution
\[ x_p(t) = X(t)u(t) = \begin{bmatrix} e^{-2t} \\ e^{2t} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 8e^{3t} - 4e^t \\ 12e^{3t} - 8e^t \end{bmatrix}. \]

Thus, the general solution to the nonhomogeneous system of differential equations is
\[ x(t) = x_c(t) + x_p(t) = e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 8e^{3t} - 4e^t \\ 12e^{3t} - 8e^t \end{bmatrix}. \]

5. We have \( \det(A - \lambda I) = 0 \) if and only if \( \begin{vmatrix} 2 - \lambda & 4 \\ -2 & -2 - \lambda \end{vmatrix} = 0 \). This means that the eigenvalues of \( A \) are \( \lambda = \pm 2i \).

Eigenvectors for \( \lambda = -2i \): We have \( A + 2iI = \begin{bmatrix} 2 + 2i & 4 \\ -2 & -2 + 2i \end{bmatrix} \), which gives rise to a complex-valued eigenvector of the form \( \mathbf{v} = r(-1 + i, 1) \), where \( r \in \mathbb{C} \). Therefore, we get the following complex-valued solution to the system:
\[ x(t) = e^{-2it} \begin{bmatrix} -1 + i \\ 1 \end{bmatrix} = (\cos 2t - i \sin 2t) \left( \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right). \]

By computing the real and imaginary parts of this expression, we obtain two real-valued solutions to the associated homogeneous system of differential equations:
\[ x_1(t) = \cos 2t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \sin 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos 2t + \sin 2t \\ \cos 2t \end{bmatrix} \]

and
\[ x_2(t) = \cos 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 2t + \sin 2t \\ -\sin 2t \end{bmatrix}. \]

Therefore, the general solution to the associated homogeneous system is
\[ x_c(t) = c_1 \begin{bmatrix} -\cos 2t + \sin 2t \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} \cos 2t + \sin 2t \\ -\sin 2t \end{bmatrix} \]

and the fundamental matrix is
\[ X(t) = \begin{bmatrix} -\cos 2t + \sin 2t & \cos 2t + \sin 2t \\ \cos 2t & -\sin 2t \end{bmatrix}. \]
We have
\[ X^{-1}(t) = \begin{bmatrix} \sin 2t & \cos 2t + \sin 2t \\ \cos 2t & \cos 2t - \sin 2t \end{bmatrix} . \]

Therefore
\[ u(t) = \int^t X^{-1}(s)b(s)ds \]
\[ = \int^t \begin{bmatrix} \sin 2s & \cos 2s + \sin 2s \\ \cos 2s & \cos 2s - \sin 2s \end{bmatrix} \begin{bmatrix} 8 \sin 2s \\ 8 \cos 2s \end{bmatrix} ds = \int^t \begin{bmatrix} 8 + 8 \sin 2s \cos 2s \\ 8 \cos 2s \end{bmatrix} ds \]
\[ = \begin{bmatrix} 8t + 2 \sin^2 2t \\ 4t + \sin 4t \end{bmatrix} . \]

Hence, we obtain the particular solution
\[ x_p(t) = X(t)u(t) = \begin{bmatrix} -\cos 2t + \sin 2t & \cos 2t + \sin 2t \\ \cos 2t & -\sin 2t \end{bmatrix} \begin{bmatrix} 8t + 2 \sin^2 2t \\ 4t + \sin 4t \end{bmatrix} \]
\[ = \begin{bmatrix} 12t \sin 2t - 4t \cos 2t + 2 \sin 2t \\ 8t \cos 2t - 4t \sin 2t \end{bmatrix} . \]

Thus, the general solution to the nonhomogeneous system of differential equations is
\[ x(t) = x_c(t) + x_p(t) = c_1 \begin{bmatrix} -\cos 2t + \sin 2t \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} \cos 2t + \sin 2t \\ -\sin 2t \end{bmatrix} + \begin{bmatrix} 12t \sin 2t - 4t \cos 2t + 2 \sin 2t \\ 8t \cos 2t - 4t \sin 2t \end{bmatrix} . \]

Note that we can also express
\[ x_p(t) = \begin{bmatrix} 12t \sin 2t - 4t \cos 2t + 2 \sin 2t \\ 8t \cos 2t - 4t \sin 2t \end{bmatrix} = \begin{bmatrix} (12t + 1) \sin 2t + (1 - 4t) \cos 2t \\ (8t - 1) \cos 2t - 4t \sin 2t \end{bmatrix} + \begin{bmatrix} \sin 2t - \cos 2t \\ \cos 2t \end{bmatrix} , \]
and since the latter term is a solution to the associated homogeneous system, we can choose \[ x_p(t) = \begin{bmatrix} (12t + 1) \sin 2t + (1 - 4t) \cos 2t \\ (8t - 1) \cos 2t - 4t \sin 2t \end{bmatrix} \] as an alternative particular solution instead of the one used above.

7. We have \[ \det(A - \lambda I) = 0 \] if and only if \[ \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 2 & -3 - \lambda & 2 \\ 1 & -2 & 2 - \lambda \end{vmatrix} = 0 \] if and only if \((1 - \lambda)(\lambda^2 + \lambda - 2) = 0.\) Therefore, the eigenvalues of \(A\) are \(\lambda = -2\) and \(\lambda = 1\) (with multiplicity 2).

Eigenvectors for \(\lambda = -2\): We have \(A + 2I = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 2 \\ 1 & -2 & 4 \end{bmatrix}\), which gives rise to corresponding eigenvectors of the form \(v = r(0, 2, 1),\) where \(r \in \mathbb{R}.\) Therefore, one solution of the associated homogeneous system is \(x_1(t) = e^{-2t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} .\)

Eigenvectors for \(\lambda = 1\): We have \(A - I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 2 \\ 1 & -2 & 1 \end{bmatrix}\), which gives rise to corresponding eigenvectors of the form \(v = r(2, 1, 0) + s(-1, 0, 1),\) where \(r, s \in \mathbb{R}.\) Therefore, we obtain two more linearly independent solutions.
to the corresponding homogeneous system: \( x_2(t) = e^{t} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \) and \( x_3(t) = e^{t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \). Consequently, the general solution to the associated homogeneous system is

\[
x_c(t) = c_1 e^{-2t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\]

and a fundamental matrix is

\[
X(t) = \begin{bmatrix} 0 & 2e^t & -e^t \\ 2e^{-2t} & e^t & 0 \\ e^{-2t} & 0 & e^t \end{bmatrix}.
\]

We have

\[
X^{-1}(t) = -\frac{1}{3} \begin{bmatrix} e^{2t} & -2e^{2t} & e^{2t} \\ -2e^{-t} & e^{-t} & -2e^{-t} \\ e^{-t} & 2e^{-t} & -4e^{-t} \end{bmatrix}.
\]

Therefore,

\[
u(t) = \int_{t}^{t} X^{-1}(s) b(s) ds
\]

\[
= -\frac{1}{3} \int_{t}^{t} \begin{bmatrix} e^{2s} & -2e^{2s} & e^{2s} \\ -2e^{-s} & e^{-s} & -2e^{-s} \\ e^{-s} & 2e^{-s} & -4e^{-s} \end{bmatrix} \begin{bmatrix} -e^s \\ -e^s \\ e^s \end{bmatrix} ds
\]

\[
= \int_{t}^{t} \begin{bmatrix} 4e^s \\ -2e^{-2s} \\ 1 - 4e^{-2s} \end{bmatrix} ds
\]

\[
= \begin{bmatrix} 4te^t \\ -2e^{-2t} \\ t + 2e^{-2t} \end{bmatrix}.
\]

Hence, we obtain the particular solution

\[
x_p(t) = X(t)u(t) = \begin{bmatrix} 0 & 2e^t & -e^t \\ 2e^{-2t} & e^t & 0 \\ e^{-2t} & 0 & e^t \end{bmatrix} \begin{bmatrix} 4e^t \\ e^{-2t} \\ t + 2e^{-2t} \end{bmatrix} = \begin{bmatrix} -te^t \\ 9e^{-t} \\ te^t + 6e^{-t} \end{bmatrix}.
\]

Therefore, the general solution to the nonhomogeneous system of differential equations is

\[
x(t) = x_c(t) + x_p(t) = c_1 e^{-2t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -te^t \\ 9e^{-t} \\ te^t + 6e^{-t} \end{bmatrix}.
\]

9. The solution to \( x' = A(t)x + b(t) \) can be written

\[
x(t) = X(t)c + X(t) \int_{t_0}^{t} X^{-1}(s)b(s) ds,
\]

where \( c = (c_1, c_2, \ldots, c_n) \) are arbitrary constants and \( X(t) \) is a fundamental matrix for the system. We can rewrite the equation above as

\[
x(t) = X(t)c + X(t) \int_{t_0}^{t} X^{-1}(s)b(s) ds.
\]
When \( t = t_0 \), this reduces to \( x(t_0) = X(t_0)c \), which means we can write \( c = X^{-1}(t_0)x(t_0) \). Substituting \( x(t_0) = x_0 \), we have \( c = X^{-1}(t_0)x_0 \). Therefore,

\[
x(t) = X(t)X^{-1}(t_0)x_0 + X(t) \int_{t_0}^{t} X^{-1}(s)b(s)ds.
\]

**Solutions to Section 7.7**

**True-False Review:**

1. **FALSE.** In order to solve a coupled spring-mass system with two masses and two springs as a first-order linear system, a \( 4 \times 4 \) linear system is required. More precisely, if the masses are \( m_1 \) and \( m_2 \), and the spring constants are \( k_1 \) and \( k_2 \), then the first-order linear system for the spring-mass system is \( x' = Ax \) where

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\frac{1}{m_1} (k_1 + k_2) & 0 & \frac{k_2}{m_1} & 0 \\
0 & 0 & 0 & 1 \\
\frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0
\end{bmatrix}.
\]

3. **FALSE.** The amount of chemical in a tank of solution is measured in grams in the metric system, but the concentration is measured in grams/L, since concentration is computed as the amount of substance per unit of volume.

5. **FALSE.** Although this is a closed system, chemicals still pass between the two tanks. So, for example, if tank 1 contained 100 grams of chemical and tank 2 contained no chemical, then as time passes and fluid flows between the two tanks, some of the chemical will move to tank 2. The amount of chemical in each tank changes over time.

**Problems:**

1. From Example 7.7.1, we have \( \det(A - \lambda I) = 0 \) if and only if \( \lambda^4 + 5\lambda^2 + 4 = (\lambda^2 + 4)(\lambda^2 + 1) = 0 \). Thus, we have \( \lambda = \pm i, \pm 2i \).

**Eigenvectors for \( \lambda = i \):** Here we have

\[
A - iI = \begin{bmatrix}
-i & 1 & 0 & 0 \\
-3 & -i & 1 & 0 \\
0 & 0 & -i & 1 \\
2 & 0 & -2 & -i
\end{bmatrix},
\]

and the reduced row-echelon form of the linear system \((A - iI)v_1 = 0\) is

\[
\begin{bmatrix}
1 & 0 & 0 & i/2 & 0 \\
0 & 1 & 0 & -1/2 & 0 \\
0 & 0 & 1 & i & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Consequently, \( v_1 = r(-i, 1, -2i, 2) \), where \( r \in \mathbb{C} \). It follows that a complex-valued solution to \( x' = Ax \) is

\[
u_1(t) = e^{it} \begin{bmatrix}
-i \\
1 \\
-2i \\
2
\end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix}
-i \\
1 \\
-2i \\
2
\end{bmatrix}.
\]
Taking the real and imaginary parts of this complex-valued solution yields two real-valued solutions,

\[
x_1(t) = \begin{bmatrix} \cos t \\ -\sin t \\ 2\sin t \\ 2\cos t \end{bmatrix} \quad \text{and} \quad x_2(t) = \begin{bmatrix} \sin t \\ \cos t \\ -2\cos t \\ 2\sin t \end{bmatrix}.
\]

Eigenvectors for \( \lambda = 2i \): Here we have

\[
A - 2iI = \begin{bmatrix} -2i & 1 & 0 & 0 \\ -3 & -2i & 1 & 0 \\ 0 & 0 & -2i & 1 \\ 2 & 0 & -2 & -2i \end{bmatrix},
\]

and the reduced row-echelon form of the linear system \((A - 2iI)v_2 = 0\) is

\[
\begin{bmatrix} -2i & 1 & 0 & 0 \\ -3 & -2i & 1 & 0 \\ 0 & 0 & -2i & 1 \\ 2 & 0 & -2 & -2i \end{bmatrix}
\]

Consequently, \(v_2 = s(i, -2, -i, 2)\), where \(s \in \mathbb{C}\). It follows that a complex-valued solution to \(x' = Ax\) is

\[
u_2(t) = e^{2it} \begin{bmatrix} i \\ -2 \\ -i \\ 2 \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} i \\ -2 \\ -i \\ 2 \end{bmatrix}.
\]

Taking the real and imaginary parts of this complex-valued solution yields two more real-valued solutions,

\[
x_3(t) = \begin{bmatrix} -\sin 2t \\ -2\cos 2t \\ \sin t \\ 2\cos 2t \end{bmatrix} \quad \text{and} \quad x_4(t) = \begin{bmatrix} \cos 2t \\ \sin 2t \\ -\cos 2t \\ 2\sin 2t \end{bmatrix}.
\]

3. The motion of the system is governed by the

\[
\begin{aligned}
x'' &= -3x + 4(y - x), \\
\frac{4}{3}y'' &= -4(y - x).
\end{aligned}
\]

Letting \(x_1 = x\), \(x_2 = x'_1\), \(x_3 = y\), and \(x_4 = y'\) yields the equivalent system

\[
\begin{aligned}
x'_1 &= x_2, \\
x'_2 &= -7x_1 + 4x_3, \\
x'_3 &= x_4, \\
x'_4 &= 3x_1 - 3x_3.
\end{aligned}
\]

Writing this problem as \(x' = Ax\), where \(x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\) and \(A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -7 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -3 & 0 \end{bmatrix}\), we find that

\[
\det(A - \lambda I) = 0 \text{ if and only if } \lambda^4 + 10\lambda^2 + 9 = 0 \text{ if and only if } (\lambda^2 + 1)(\lambda^2 + 9) = 0.
\]
Therefore, we obtain the eigenvalues \( \lambda = \pm i, \pm 3i \).

**Eigenvectors for \( \lambda = i \):** Here we have

\[
A - iI = \begin{bmatrix}
-i & 1 & 0 & 0 \\
-7 & -i & 4 & 0 \\
0 & 0 & -i & 1 \\
3 & 0 & -3 & -i
\end{bmatrix},
\]

and the reduced row-echelon form of the linear system \((A - iI)x = 0\) is

\[
\begin{bmatrix}
1 & -i & 0 & 0 \\
0 & 1 & -\frac{2i}{3} & 0 \\
0 & 0 & 1 & i \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

which gives rise to corresponding eigenvectors of the form \(v = r(-2i, 2, -3i, 3)\), where \(r \in \mathbb{C}\). The corresponding complex-valued solution to the system \(x' = Ax\) is

\[
u(t) = e^{it} v = (\cos t + i \sin t) \begin{bmatrix}
-2i \\
2 \\
-3i \\
3
\end{bmatrix}
\]

\[= (\cos t + i \sin t) \begin{bmatrix}
0 & 2 & 0 & -2 \\
2 & 0 & 0 & 0 \\
0 & -3 & 3 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}
\]

\[= \begin{bmatrix}
2 \sin t \\
2 \cos t \\
3 \sin t \\
3 \cos t
\end{bmatrix}
\]

\[+ i \begin{bmatrix}
-2 \cos t \\
2 \sin t \\
-3 \cos t \\
3 \sin t
\end{bmatrix},
\]

which gives rise to the two real-valued solutions

\[
x_1(t) = \begin{bmatrix}
2 \sin t \\
2 \cos t \\
3 \sin t \\
3 \cos t
\end{bmatrix}
\]

and \(x_2(t) = \begin{bmatrix}
-2 \cos t \\
2 \sin t \\
-3 \cos t \\
3 \sin t
\end{bmatrix}.
\]

**Eigenvectors for \( \lambda = 3i \):** Here we have

\[
A - 3iI = \begin{bmatrix}
-3i & 1 & 0 & 0 \\
-7 & -3i & 4 & 0 \\
0 & 0 & -3i & 1 \\
3 & 0 & -3 & -3i
\end{bmatrix},
\]
and the reduced row-echelon form of the linear system $(A - 3iI)x = 0$ is

\[
\begin{pmatrix}
1 & \frac{i}{3} & 0 & 0 \\
0 & 1 & -3i & 3 \\
0 & 0 & 1 & \frac{i}{3} \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

gives rise to corresponding eigenvectors of the form $v = r(2i, -6, -i, 3)$, where $r \in \mathbb{C}$. The corresponding complex-valued solution to the system $x' = Ax$ is

\[
\begin{aligned}
u(t) &= e^{3it}v \\
&= (\cos 3t + i \sin 3t) \begin{pmatrix} 2i \\ -6 \\ -i \\ 3 \end{pmatrix} + i (\cos 3t \begin{pmatrix} 0 \\ -6 \\ 0 \\ 3 \end{pmatrix} + \sin 3t \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}) \\
&= \begin{pmatrix} \cos 3t \\ -6 \cos 3t \\ -6 \sin 3t \\ 3 \cos 3t \end{pmatrix} + i \begin{pmatrix} -2 \sin 3t \\ -6 \cos 3t \\ -6 \sin 3t \\ 3 \sin 3t \end{pmatrix},
\end{aligned}
\]

which gives rise to the two real-valued solutions

\[
x_3(t) = \begin{pmatrix} -2 \sin 3t \\ -6 \cos 3t \\ \sin 3t \\ 3 \cos 3t \end{pmatrix} \quad \text{and} \quad x_4(t) = \begin{pmatrix} 2 \cos 3t \\ -6 \sin 3t \\ -\cos 3t \\ 3 \sin 3t \end{pmatrix}.
\]

Consequently, the general solution to the linear system is

\[
x(t) = c_1x_1(t) + c_2x_2(t) + c_3x_3(t) + c_4x_4(t)
\]

\[
= c_1 \begin{pmatrix} 2 \sin t \\ 2 \cos t \\ 3 \sin t \\ 3 \cos t \end{pmatrix} + c_2 \begin{pmatrix} -2 \cos t \\ 2 \sin t \\ -3 \cos t \\ 3 \sin t \end{pmatrix} + c_3 \begin{pmatrix} -2 \sin 3t \\ -6 \cos 3t \\ \sin 3t \\ 3 \cos 3t \end{pmatrix} + c_4 \begin{pmatrix} 2 \cos 3t \\ -6 \sin 3t \\ -\cos 3t \\ 3 \sin 3t \end{pmatrix}.
\]

Since $x = x_1$ and $y = x_3$, it follows that the motion of the spring-mass system is given by

\[
x(t) = 2(c_1 \sin t - c_2 \cos t - c_3 \sin 3t + c_4 \cos 3t) \quad \text{and} \quad y(t) = 3c_1 \sin t - 3c_2 \cos t + c_3 \sin 3t - c_4 \cos 3t.
\]
5. We have

\[ \det (A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ -k_1 + k_2 & -\lambda & k_2 \\ m_1 & m_1 & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ m_2 & 0 & -k_2 & m_2 \\ \end{vmatrix} = 0 \]

\[ \iff -\lambda \left[ -\lambda (\lambda^2 + \frac{k_2}{m_2}) - \left( -\frac{k_1 + k_2}{m_1} (\lambda^2 + \frac{k_2}{m_2}) + \frac{k_2 k_2}{m_1 m_2} \right) \right] = 0 \]

\[ \iff \lambda^4 + \frac{k_2}{m_2} \lambda^2 + \frac{k_1 + k_2}{m_1} \lambda^2 - \left( -\frac{k_1 + k_2}{m_1} \frac{k_2}{m_2} \right) = 0 \]

\[ \iff \lambda^4 + \left( \frac{k_2}{m_2} + \frac{k_1 + k_2}{m_1} \right) \lambda^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \]

\[ \iff m_1 m_2 \lambda^2 + \left[ m_1 k_1 + m_2 (k_1 + k_2) \right] \lambda^2 + k_1 k_2 = 0. \tag{5.1} \]

The discriminant of Equation (5.1) is given by

\[ [m_1 k_1 + m_2 (k_1 + k_2)]^2 - 4 m_1 m_2 k_1 k_2 = (m_1 k_1 - m_2 k_2)^2 + m_2^2 k_1^2 + 2 m_1 m_2 k_1^2 + 2 m_2 k_1 k_2, \]

which is strictly positive. Thus, Equation (5.1) has two distinct real roots for \( \lambda^2 \). Since all coefficients are positive, both roots must be negative, so letting \( \omega_1 = \sqrt{-r_1} \) and \( \omega_2 = \sqrt{-r_2} \), we obtain \( (\lambda^2 + \omega_1^2)(\lambda^2 + \omega_2^2) = 0 \) which implies that \( \lambda = \pm i \omega_1, \ \ \lambda = \pm i \omega_2. \)

7. The assumption that \( V_1 \) and \( V_2 \) are constant implies that the net rate of change is zero. Thus,

\[ \begin{cases} r_{in} + r_{12} - r_{21} = 0, \\ r_{21} - r_{12} - r_{out} = 0, \end{cases} \quad \text{or} \quad \begin{cases} r_{in} + r_{12} = r_{21}, \\ r_{12} + r_{out} = r_{21}. \end{cases} \]

9. From the given information in the problem we see that the volume of solution in both tanks remains constant at 60 liters. Thus \( V_1 = 60 = V_2. \) Further, during a short time interval \( \Delta t, \ \Delta A_1 \approx \left( 12 + 2 \frac{A_2}{V_2} - 8 \frac{A_1}{V_1} \right) \Delta t, \ \Delta A_2 \approx \left( 8 \frac{A_2}{V_2} - 6 \frac{A_2}{V_2} \right) \Delta t. \) Substituting in for \( V_1 \) and \( V_2, \) dividing by \( \Delta t, \) and taking the limit as \( \Delta t \to 0, \) we have

\[ \frac{dA_1}{dt} = -\frac{2}{15} A_1 + \frac{1}{30} A_2 + 12 \quad \text{and} \quad \frac{dA_2}{dt} = \frac{2}{15} A_1 - \frac{2}{15} A_2. \]

That is \( \mathbf{x}' = A \mathbf{x} + \mathbf{b}, \) where

\[ \mathbf{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{2}{15} & \frac{1}{30} \\ \frac{2}{15} & -\frac{2}{15} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 12 \\ 0 \end{bmatrix}. \]

Now,

\[ \det (A - \lambda I) = 0 \iff \begin{vmatrix} -\frac{2}{15} - \lambda & \frac{1}{30} \\ \frac{2}{15} & -\frac{2}{15} - \lambda \end{vmatrix} = 0 \iff \left( \lambda + \frac{2}{15} \right)^2 - \left( \frac{1}{15} \right)^2 = 0 \iff \lambda \in \left\{ -\frac{1}{5}, -\frac{1}{15} \right\}. \]
When \( \lambda = -\frac{1}{5} \): \( A + \frac{1}{5} I = \begin{bmatrix} 1 & 1 \\ 15 & 30 \\ 2 & 1 \\ 15 & 15 \end{bmatrix} \), so that the corresponding eigenvectors are of the form \( v_1 = r(-1, 2) \), where \( r \in \mathbb{R} \).

When \( \lambda = -\frac{1}{15} \): \( A + \frac{1}{15} I = \begin{bmatrix} -\frac{1}{15} & 1 \\ 15 & 30 \\ 2 & 1 \\ 15 & 15 \end{bmatrix} \), so that the corresponding eigenvectors are of the form \( v_2 = s(1, 2) \), where \( s \in \mathbb{R} \). Consequently two linearly independent solutions to the associated homogeneous system are \( x_1(t) = e^{-t/5} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \) and \( x_2(t) = e^{-t/15} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). According to the variation-of-parameters technique, a particular solution to the non-homogeneous equation \( x' = Ax + b \) is \( x_p = Xu \), where \( u \) is determined by solving \( Xu' = b \) with \( X = \begin{bmatrix} -e^{-t/5} & e^{-t/15} \\ 2e^{-t/5} & 2e^{-t/15} \end{bmatrix} \). The reduced row echelon form of the augmented matrix corresponding to \( Xu' = b \) is \( \begin{bmatrix} 1 & 0 & -6e^{t/5} \\ 0 & 1 & 6e^{t/15} \end{bmatrix} \), so that \( u' = \begin{bmatrix} -6e^{t/5} \\ 6e^{t/15} \end{bmatrix} \), which implies that \( u = \begin{bmatrix} -30e^{t/15} \\ 90e^{t/15} \end{bmatrix} \). Hence, \( x_p(t) = \begin{bmatrix} -e^{-t/5} & e^{-t/15} \\ 2e^{-t/5} & 2e^{-t/15} \end{bmatrix} \begin{bmatrix} -30e^{t/5} \\ 90e^{t/15} \end{bmatrix} = \begin{bmatrix} 120 \\ 120 \end{bmatrix} \). Thus the general solution to the non-homogeneous system is \( x(t) = c_1 e^{-t/5} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-t/15} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 120 \\ 120 \end{bmatrix} \). Imposing the initial condition \( x(0) = \begin{bmatrix} 60 \\ 200 \end{bmatrix} \) yields \( c_1 = 50 \) and \( c_2 = -10 \), so that

\[
x(t) = 50e^{-t/5} \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 10e^{-t/15} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 120 \\ 120 \end{bmatrix}.
\]

Thus,

\[
A_1(t) = 120 - 50e^{-t/5} - 10e^{-t/15} \quad \text{and} \quad A_2(t) = 120 + 100e^{-t/5} - 20e^{-t/15}.
\]

11.

(a) In the closed system, \( r_{in} = r_{out} = 0 \) liters/minute and \( r_{12} = r_{21} \) so the system

\[
\begin{align*}
\frac{dA_1}{dt} &= -\frac{r_{21}}{V_1} A_1 + \frac{r_{12}}{V_2} A_2 + c_{in} r_{in}, \\
\frac{dA_2}{dt} &= \frac{r_{21}}{V_1} A_1 - \frac{r_{21}}{V_2} A_2,
\end{align*}
\]

becomes

\[
\begin{align*}
\frac{dA_1}{dt} &= -\frac{r_{21}}{V_1} A_1 + \frac{r_{21}}{V_2} A_2, \\
\frac{dA_2}{dt} &= \frac{r_{21}}{V_1} A_1 - \frac{r_{21}}{V_2} A_2.
\end{align*}
\]

(b) Replace \( V_2 \) by \( \beta V_1 \) and write the system in part (a) as \( x' = Ax \), where \( x = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \) and \( A = \begin{bmatrix} -\frac{r_{21}}{V_1} & \frac{r_{21}}{V_2} \\ \frac{r_{21}}{V_1} & -\frac{r_{21}}{V_2} \end{bmatrix} \).
\[
\begin{bmatrix}
\frac{-r_{21}}{V_1} & \frac{r_{21}}{\beta V_1} \\
\frac{r_{21}}{V_1} & \frac{-r_{21}}{\beta V_1}
\end{bmatrix}
\]. Then

\[
\det (A - \lambda I) = 0 \iff \det
\begin{bmatrix}
\frac{-r_{21}}{V_1} - \lambda & \frac{r_{21}}{\beta V_1} \\
\frac{r_{21}}{V_1} & \frac{-r_{21}}{\beta V_1} - \lambda
\end{bmatrix}
= 0 \iff \lambda^2 + \frac{(1 + \beta) r_{21}}{\beta V_1} \lambda = 0
\iff \lambda \in \{0, -\frac{(1 + \beta)}{\beta V_1} r_{21}\}.
\]

(c) When \(\lambda_1 = 0\), then the corresponding eigenvectors take the form \(v = r(1, \beta)\), where \(r \in \mathbb{R}\). When \(\lambda_2 = -\frac{(1 + \beta)}{\beta V_1} r_{21}\), then the corresponding eigenvectors take the form \(v = s(-1, 1)\), where \(s \in \mathbb{R}\). The general solution for the system is

\[
x(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.
\]

Now

\[
x(0) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \implies \begin{cases} A_1(t) = \frac{\alpha_1 + \alpha_2}{1 + \beta} - \frac{\alpha_2 - \beta \alpha_1}{1 + \beta} e^{\lambda_2 t}, \\
A_2(t) = \frac{\beta \alpha_1 + \alpha_2}{1 + \beta} + \frac{\alpha_2 - \beta \alpha_1}{1 + \beta} e^{\lambda_2 t}.
\end{cases}
\]

(d) Since \(\lambda_2 < 0\), \(\lim_{t \to \infty} e^{\lambda_2 t} = 0\). Therefore, \(\lim_{t \to \infty} \frac{A_1}{V_1} = \frac{\alpha_1 + \alpha_2}{(1 + \beta) V_1}\) and \(\lim_{t \to \infty} \frac{A_2}{V_2} = \lim_{t \to \infty} \frac{A_2}{\beta V_1} = \frac{\alpha_1 + \alpha_2}{(1 + \beta) V_1}\).

Yes, the result is reasonable, because as the time increases without bound the concentrations in the tanks represented by the two fractions, \(\frac{A_1}{V_1}\) and \(\frac{A_2}{V_2}\), will become equal.

Solutions to Section 7.8

True-False Review:

1. TRUE. This is precisely the content of Equation (7.8.2).

3. TRUE. For each generalized eigenvector \(v\), \(e^{At}v\) is a solution to \(x' = Ax\). If \(A\) is an \(n \times n\) matrix, then \(A\) has \(n\) linearly independent generalized eigenvectors, and therefore, the general solution to \(x' = Ax\) can be formulated as linear combinations of \(n\) linearly independent solutions of the form \(e^{At}v\). As usual, application of the initial condition yields the unique solution to the given initial-value problem.

5. TRUE. This follows immediately from Equations (7.8.4) and (7.8.6).

Problems:

1. If \(X(t)\) is a fundamental matrix for \(x' = Ax\), then \(X(t)B\), where \(B\) is an invertible matrix, is also a fundamental matrix. Clearly, \(X^{-1}(0)\) is invertible, and its inverse is \(X(0)\). Thus, \(X_0 = X(t)^{-1}X(0)\) is a fundamental matrix. Since \(X_0(0) = X(0)^{-1}X(0) = I_n\), \(X_0\) is the transition matrix, based at \(t = 0\), for \(x' = Ax\).

3. \det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 1 \\ 0 & 2 - \lambda \end{vmatrix} = 0 \iff (\lambda - 2)^2 = 0\), which implies that the only eigenvalue of \(A\) is \(\lambda = 2\) (with multiplicity 2).
We have \( A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), which gives rise to corresponding eigenvectors of the form \( v = r(1,0) \), where \( r \in \mathbb{R} \). Thus, one solution of the linear system \( \mathbf{x}' = A\mathbf{x} \) is given by \( \mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{2t} \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} \).

We will determine a second linearly independent solution of the form

\[
\mathbf{x}_2(t) = e^{2t}(\mathbf{v}_1 + t\mathbf{v}_0)
\]

where \( \mathbf{v}_1 \) and \( \mathbf{v}_0 \) are determined from

\[
(A - 2I)^2\mathbf{v}_1 = 0, \quad (A - 2I)\mathbf{v}_1 \neq 0, \quad \mathbf{v}_0 = (A - 2I)\mathbf{v}_1.
\]

In this case, we have \( A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) and \( (A - 2I)^2 = 0_2 \). Therefore, we may choose \( \mathbf{v}_1 \) to be any vector such that \( (A - 2I)\mathbf{v}_1 \neq 0 \). For simplicity, we take \( \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Thus, \( \mathbf{v}_0 = (A - 2I)\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). From the expressions for \( \mathbf{v}_0 \) and \( \mathbf{v}_1 \), we can write down a second linearly independent solution to the vector differential equation:

\[
\mathbf{x}_2(t) = e^{2t}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = e^{2t}\begin{bmatrix} t \\ 1 \end{bmatrix} = e^{2t}\begin{bmatrix} te^{2t} \\ e^{2t} \end{bmatrix}.
\]

A fundamental matrix for the system \( \mathbf{x}' = A\mathbf{x} \) is therefore given by \( X(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \). Therefore, \( X^{-1}(t) = \begin{bmatrix} e^{-2t} & -te^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \), and hence, \( X^{-1}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \). Thus,

\[
e^{At} = X(t)X^{-1}(0) = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e^{2t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.
\]

5. \( \det(A - \lambda I) = 0 \iff 3 - \lambda \frac{1}{4} - \frac{1}{1 - \lambda} = 0 \iff \lambda^2 - 2\lambda + 1 = 0 \iff (\lambda - 1)^2 = 0. \) Therefore, the only eigenvalue of \( A \) is \( \lambda = 1 \) (with multiplicity 2).

Eigenvectors for \( \lambda = 1 \): We have \( A - I = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \), which gives rise to corresponding eigenvectors of the form \( \mathbf{v} = r(1,2) \), where \( r \in \mathbb{R} \). Thus, one solution of the linear system \( \mathbf{x}' = A\mathbf{x} \) is given by \( \mathbf{x}_1(t) = e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} = e^t \begin{bmatrix} e^t \\ 2e^t \end{bmatrix} \). We will determine a second linearly independent solution corresponding to \( \lambda = 1 \) of the form \( \mathbf{x}_2(t) = e^t(\mathbf{v}_1 + t\mathbf{v}_0) \), where \( \mathbf{v}_1 \) and \( \mathbf{v}_0 \) are determined from

\[
(A - I)^2\mathbf{v}_1 = 0, \quad (A - I)\mathbf{v}_1 \neq 0, \quad \mathbf{v}_0 = (A - I)\mathbf{v}_1.
\]

In this case, we have \( A - I = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \) and \( (A - I)^2 = 0_2 \). Therefore, we may choose \( \mathbf{v}_1 \) to be any vector such that \( (A - I)\mathbf{v}_1 \neq 0 \). For simplicity, we choose \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Then \( \mathbf{v}_0 = (A - I)\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \). It follows that

\[
\mathbf{x}_2(t) = e^t\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\} = e^t\begin{bmatrix} 1 + 2t \\ 4t \end{bmatrix} = e^t\begin{bmatrix} (1 + 2t)e^t \\ 4te^t \end{bmatrix}.
\]
A fundamental matrix for the system is therefore given by $X(t) = \begin{bmatrix} e^t & (1 + 2t)e^t \\ 2e^t & 4te^t \end{bmatrix}$. Therefore, $X^{-1}(t) = \begin{bmatrix} -2te^{-t} & \frac{1}{2}(1 + 2t)e^{-t} \\ e^{-t} & -\frac{1}{2}e^{-t} \end{bmatrix}$. Thus, $X^{-1}(0) = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$. Thus,

$$e^{At} = X(t)X^{-1}(0) = \begin{bmatrix} e^t & (1 + 2t)e^t \\ 2e^t & 4te^t \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} (1 + 2t)e^t & -te^t \\ 4te^t & e^t - 2te^t \end{bmatrix} = e^t \begin{bmatrix} 1 + 2t & -t \\ 4t & 1 - 2t \end{bmatrix}.$$

7. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & -8 \\ 0 & 2 & -7 - \lambda \end{vmatrix} = 0 \iff (2 - \lambda)(\lambda^2 + 6\lambda + 9) = 0 \iff (2 - \lambda)(\lambda + 3)^2 = 0.$

Therefore, the eigenvalues of $A$ are $\lambda = 2$ and $\lambda = -3$ (with multiplicity 2).

Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & -8 \\ 0 & 2 & -4 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(1, 0, 0)$, where $r \in \mathbb{R}$. Thus, one solution of the linear system $x' = Ax$ is given by $x_1(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 0 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = -3$: We have $A + 3I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & -8 \\ 0 & 2 & -4 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(0, 2, 1)$, where $r \in \mathbb{R}$. Thus, a second solution of the linear system $x' = Ax$ is given by $x_2(t) = e^{-3t} \begin{bmatrix} 0 \\ 2e^{-3t} \\ -3 \end{bmatrix}$.

We will determine a second linearly independent solution corresponding to $\lambda = -3$ of the form $x_3(t) = e^{-3t}(v_1 + tv_0)$, where $v_1$ and $v_0$ are determined from

$$(A + 3I)^2v_1 = 0, \quad (A + 3I)v_1 \neq 0, \quad v_0 = (A + 3I)v_1.$$

In this case, we have $A + 3I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & -8 \\ 0 & 2 & -4 \end{bmatrix}$ and $(A + 3I)^2 = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let us choose $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Then $v_0 = (A + 3I)v_1 = \begin{bmatrix} 0 \\ -8 \\ -4 \end{bmatrix}$. From the expressions for $v_0$ and $v_1$, we can write down a second linearly independent solution to the vector differential equation corresponding to $\lambda = -3$:

$$x_3(t) = e^{-3t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ -8 \\ -4 \end{bmatrix} = e^{-3t} \begin{bmatrix} 0 \\ -8t \\ 1 - 4t \end{bmatrix} = \begin{bmatrix} 0 \\ -8te^{-3t} \\ (1 - 4t)e^{-3t} \end{bmatrix}.$$

A fundamental matrix for the system $x' = Ax$ is therefore given by $X(t) = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & 2e^{2t} & -8te^{-3t} \\ 0 & e^{-3t} & (1 - 4t)e^{-3t} \end{bmatrix}$.
We have eigenvectors for of the form $v$ of the form $X$.

From the characteristic polynomial, we find that the eigenvalues of $A$ are $\lambda = -1$ and $\lambda = 3$ (with multiplicity 2).

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & -2 \\ 1 & 1 & 3 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(-7, 4, 1)$, where $r \in \mathbb{R}$. Thus, one solution of the linear system $x' = Ax$ is given by $x_1(t) = e^{-t} \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -7e^{-t} \\ 4e^{-t} \\ e^{-t} \end{bmatrix}$.

Eigenvectors for $\lambda = 3$: We have $A - 3I = \begin{bmatrix} -3 & 1 & 3 \\ 2 & 0 & -2 \\ 1 & 1 & -1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = s(1, 0, 1)$, where $s \in \mathbb{R}$. Thus, a second linearly independent solution of the linear system $x' = Ax$ is given by $x_2(t) = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}$.

We will determine a second linearly independent solution corresponding to $\lambda = 3$ of the form $x_3(t) = e^{3t}(v_1 + tv_0)$, where $v_1$ and $v_0$ are determined from

$$(A - 3I)^2 v_1 = 0, \quad (A - 3I)v_1 \neq 0, \quad v_0 = (A - 3I)v_1.$$ 

In this case, we have $A - 3I = \begin{bmatrix} -3 & 1 & 3 \\ 2 & 0 & -2 \\ 1 & 1 & -1 \end{bmatrix}$ and $(A - 3I)^2 = \begin{bmatrix} 14 & 0 & -14 \\ -8 & 0 & 8 \\ -2 & 0 & 2 \end{bmatrix}$. Let us choose $v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then $v_0 = (A - 3I)v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. From the expressions for $v_0$ and $v_1$, we can write down a second linearly independent solution to the vector differential equation corresponding to $\lambda = 3$:

$$x_3(t) = e^{3t} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} te^{3t} \\ e^{3t} \\ e^{3t} \end{bmatrix}.$$ 

Therefore, the general solution to $x' = Ax$ is given by

$$x(t) = c_1 \begin{bmatrix} -7e^{-t} \\ 4e^{-t} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} + c_3 \begin{bmatrix} te^{3t} \\ e^{3t} \\ e^{3t} \end{bmatrix}.$$ 

From the characteristic polynomial, we find that the eigenvalues of $A$ are $\lambda = i$ (with multiplicity 2) and $\lambda = -i$ (with multiplicity 2).
Eigenvectors for $\lambda = i$: In this case, $A - iI = \begin{bmatrix} -i & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 0 & -i & -1 \\ 0 & 1 & 1 & -i \end{bmatrix}$ and $(A - iI)^2 = \begin{bmatrix} -2 & 2i & 0 & 0 \\ -2i & -2 & 0 & 0 \\ -2i & -2 & -2 & 2i \\ 2 & -2i & -2i & -2 \end{bmatrix}$.

Now $(A - iI)^2v = 0$ has two linearly independent solutions. Solving the system we obtain

$$v = r(i, 1, 0, 0) + s(0, 0, i, 1),$$

where $r, s \in \mathbb{C}$. Let $v_1 = (i, 1, 0, 0)$, so that

$$u_1(t) = e^{At}v_1 = e^{it}[v_1 + t(A - iI)v_1]$$

$$= e^{it}\begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -i & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 0 & -i & -1 \\ 0 & 1 & 1 & -i \end{bmatrix}\begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= e^{it}\begin{bmatrix} i \\ 1 \\ it \end{bmatrix} = (\cos t + i \sin t)\begin{bmatrix} i \\ 1 \\ it \end{bmatrix} = \begin{bmatrix} -\sin t & \cos t \\ \cos t & -t \sin t \\ t \cos t & t \sin t \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ 

Hence, two linearly independent solutions are given by

$$x_1(t) = \begin{bmatrix} -\sin t \\ \cos t \\ -t \sin t \\ t \cos t \end{bmatrix} \quad \text{and} \quad x_2(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \cos t \\ t \sin t \end{bmatrix}.$$ 

Next, let $v_2 = (0, 0, i, 1)$, so that

$$u_2(t) = e^{At}v_2 = e^{it}[v_2 + t(A - iI)v_2]$$

$$= e^{it}\begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} + t\begin{bmatrix} -i & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 0 & -i & -1 \\ 0 & 1 & 1 & -i \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$$

$$= e^{it}\begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} = (\cos t + i \sin t)\begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ 

Hence, two further linearly independent real-valued solutions are given by

$$x_3(t) = \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad x_4(t) = \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix}.$$ 

Solutions to Section 7.9
True-False Review:

1. **FALSE.** The trajectories need not approach the equilibrium point as $t \to \infty$. For instance, Figures 7.9.4 and 7.9.8 show equilibrium points for which not all solution trajectories approach the origin as $t \to \infty$.

3. **TRUE.** This is case (b), described below Theorem 7.9.3.

5. **TRUE.** This is essentially the definition of a stable point.

Problems:

1. Equilibrium solutions arise when $x(x - y + 1) = 0$ and $y(y + 2x) = 0$. Hence, either $y = 0$, in which case $x = 0$ or $-1$, or $y = -2x$, in which case $x = -\frac{1}{3}$. Therefore, the equilibrium solutions are $(0,0)$, $(-1,0)$, and $\left(-\frac{1}{3}, \frac{2}{3}\right)$.

3. Equilibrium solutions arise when $x(x^2 + y^2 - 1) = 0$ and $y(xy - 1) = 0$. Two cases arise:
   (a) $y = 0 \implies x(x^2 - 1) = 0 \implies x = 0, \pm 1$. Hence the corresponding equilibrium solutions are $(0,0)$, $(1,0)$ and $(-1,0)$.
   (b) $y \neq 0 \implies y = \frac{1}{x} \implies x(x^2 + \frac{1}{x^2} - 1) = 0 \implies \frac{1}{x^2}(x^4 - x^2 + 1) = 0$. Since this equation has no real solution, there are no corresponding equilibrium solutions to the system of differential equations.

5. $A$ has eigenvalues $\lambda = \pm 2i$. Hence $(0,0)$ is a stable center.

![Figure 0.0.49: Figure for Exercise 5](image)

7. $A$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with corresponding linearly independent eigenvectors $v_1 = (-3, 1)$ and $v_2 = (1, -1)$. Hence $(0,0)$ is a saddle point.

![Figure 0.0.50: Figure for Exercise 7](image)
9. $A$ has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$ with corresponding linearly independent eigenvectors $v_1 = (-1, 1)$ and $v_2 = (1, 1)$, respectively. Hence $(0, 0)$ is a stable node.

![Figure 0.0.51: Figure for Exercise 9](image)

11. $A$ has eigenvalues $\lambda = \pm i$. Hence, $(0, 0)$ is a stable center.

![Figure 0.0.52: Figure for Exercise 11](image)

13. $A$ has eigenvalues $\lambda = 2 \pm i$. Hence $(0, 0)$ is an unstable spiral.

![Figure 0.0.53: Figure for Exercise 13](image)

15. $A$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$ with corresponding linearly independent eigenvectors $v_1 = (1, -1)$ and $v_2 = (1, 3)$, respectively. Hence $(0, 0)$ is an unstable node.

17. $A$ has a single eigenvalue $\lambda = -2$ with a corresponding eigenvector $v_1 = (1, -3)$. Hence $(0, 0)$ is a stable degenerate node.
19. \( A \) has a single eigenvalue \( \lambda = 3 \) with corresponding linearly independent eigenvectors \( \mathbf{v}_1 = (1, 0) \) and \( \mathbf{v}_2 = (0, 1) \), respectively. Hence, \((0, 0)\) is an unstable proper node.

21. Setting \( u = y \) and \( v = \frac{dy}{dt} \) yields the system
\[
\frac{du}{dt} = v, \quad \frac{dv}{dt} = -9u - 6v.
\]
This can be written as \( \mathbf{x}' = A\mathbf{x} \), where \( A = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} \). The matrix \( A \) has eigenvalue \( \lambda = -3 \) with corresponding eigenvector \( \mathbf{v} = (1, -3) \). Hence, the equilibrium point \((0, 0)\) is a stable degenerate node.

23. Setting \( u = y \) and \( v = \frac{dy}{dt} \) yields the system
\[
\frac{du}{dt} = v, \quad \frac{dv}{dt} = -5u - 4v.
\]
This can be written as \( \mathbf{x}' = A\mathbf{x} \), where \( A = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \). The matrix \( A \) has eigenvalues \( \lambda = -2 \pm i \). Hence, the equilibrium point \((0, 0)\) is a stable spiral point.

25. Setting \( u = y \) and \( v = \frac{dy}{dt} \) yields the system
\[
\frac{du}{dt} = v, \quad \frac{dv}{dt} = -ku - 2cv.
\]
This can be written as $\mathbf{x}' = A\mathbf{x}$, where 

$$A = \begin{bmatrix} 0 & 1 \\ -k & -2c \end{bmatrix}.$$  

The matrix $A$ has eigenvalues 

$$\lambda = -c \pm \sqrt{c^2 - k}.$$  

Consequently there are three cases: 

(a) If $c^2 - k > 0$ then there are two real distinct and negative eigenvalues. Consequently, $(0, 0)$ is a stable node. Since $y = u$, the system passes through equilibrium whenever the corresponding trajectory crosses the $v$-axis. From the phase portrait we see that this happens at most once.

(b) If $c^2 - k = 0$ then $A$ has the real repeated negative eigenvalue $\lambda = -c$ with a single linearly independent eigenvector. Therefore, $(0, 0)$ is a stable degenerate node. The behavior of the physical system is similar to that in (a).
(c) If \( c^2 - k < 0 \) then \( A \) has complex conjugate eigenvalues with negative real part. Consequently, \((0,0)\) is a stable spiral. In this case the physical system oscillates about \( y = 0 \) with decreasing amplitude and velocity. The spring-mass system eventually comes to rest at its equilibrium position.

\[
\begin{align*}
\text{Figure 0.0.59: Figure for Exercise 25(c)}
\end{align*}
\]

27. Write \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). Then

\[
\det(A - \lambda I) = \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc).
\]

According to the quadratic equation, the roots of this characteristic equation are

\[
\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}.
\]

(a) With the assumption that \( A \) has a repeated eigenvalue, we conclude that \( \sqrt{(a + d)^2 - 4(ad - bc)} = 0 \). Therefore, \( \lambda = (a + d)/2 \) is the repeated eigenvalue.

(b) Since \( A \) has two linearly independent eigenvectors corresponding to \( \lambda \), we know that every nonzero vector in \( \mathbb{R}^2 \) is an eigenvector of \( A \). Therefore, with \( e_1 \) and \( e_2 \) denoting the standard basis vectors on \( \mathbb{R}^2 \), we have

\[
(A - \lambda I)e_1 = 0 \quad \text{and} \quad (A - \lambda I)e_2 = 0.
\]

This implies that the \( A - \lambda I \) must be the zero matrix. That is, \( A = \lambda I \), which means that \( A \) is a scalar matrix.

(c) By part (b), if \( A \) is a 2 \times 2 matrix of real constants with a repeated eigenvalue \( \lambda \) that is not a scalar multiple of the identity matrix, then \( A \) cannot have two linearly independent eigenvectors corresponding to \( \lambda \). That is, \( A \) is defective.

**Solutions to Section 7.10**

True-False Review:

1. **TRUE.** This is just the definition of the Jacobian matrix given in the text.

3. **TRUE.** From Equations (7.10.2) and (7.10.3), we find that the Jacobian of the system is

\[
J(x, y) = \begin{bmatrix} a - by & -bx \\ cy & cx - d \end{bmatrix}.
\]
so that \( J(0, 0) = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix} \). Since \( a, d > 0 \), we have one positive and one negative eigenvalue, and therefore, the equilibrium point \((0, 0)\) is a saddle point.

**5. FALSE.** The equilibrium point is only an unstable spiral if \( \mu < 2 \). In this equation, \( \mu = 3 \), in which case the equilibrium point is an unstable node, as discussed in the text.

**Problems:**

1. The equilibrium points are obtained by solving

\[
y(3x - 2) = 0, \quad 2x + 9y^2 = 0.
\]

The only solution to the system is \( x = y = 0 \), and hence the only equilibrium point is \((0, 0)\). The Jacobian for the given system is

\[
J(x, y) = \begin{bmatrix} 3y & 3x - 2 \\ 2 & 18y \end{bmatrix} \implies J(0, 0) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.
\]

The eigenvalues of \( J(0, 0) \) are \( \lambda = \pm 2i \), so that the equilibrium point is either a center or a spiral point.

3. The equilibrium points are obtained by solving

\[
x - y^2 = 0, \quad y(9x - 4) = 0.
\]

The solutions to this system are \( x = y = 0 \), and \( x = \frac{4}{9}, \ y = \pm \frac{2}{3} \). The Jacobian for the given system is

\[
J(x, y) = \begin{bmatrix} 1 & -2y \\ 9y & 9x - 4 \end{bmatrix}.
\]

Consequently, \( J(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} \) with eigenvalues \( \lambda = 1, -4 \implies (0, 0) \) is a saddle point.

\[
J\left(\frac{4}{9}, \frac{2}{3}\right) = \begin{bmatrix} 1 & -\frac{4}{3} \\ 6 & 0 \end{bmatrix}
\]

with eigenvalues \( \lambda = \frac{1}{2}(1 \pm i\sqrt{31}) \implies \left(\frac{4}{9}, \frac{2}{3}\right) \) is an unstable spiral point.

\[
J\left(\frac{4}{9}, \frac{-2}{3}\right) = \begin{bmatrix} 1 & \frac{4}{3} \\ -6 & 0 \end{bmatrix}
\]

with eigenvalues \( \lambda = \frac{1}{2}(1 \pm i\sqrt{31}) \implies \left(\frac{4}{9}, \frac{-2}{3}\right) \) is an unstable spiral point.

5. The equilibrium points are obtained by solving

\[
2x + 5y^2 = 0, \quad y(3 - 4x) = 0.
\]

The only solution to this system is \( x = y = 0 \), and hence the only equilibrium point is \((0, 0)\). The Jacobian for the given system is

\[
J(x, y) = \begin{bmatrix} 2 & 10y \\ -4y & 3 \end{bmatrix} \implies J(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.
\]

The eigenvalues of \( J(0, 0) \) are \( \lambda = 2, 3 \) so that the equilibrium point is an unstable node.

7. The equilibrium points are obtained by solving

\[
x - 2y + 5xy = 0, \quad 2x + y = 0.
\]
The solutions to this system are \( x = 0, \, y = 0 \), and \( x = \frac{1}{2}, \, y = -1 \). The Jacobian for the given system is
\[
J(x, y) = \begin{bmatrix}
1 + 5y & -2 + 5x \\
2 & 1
\end{bmatrix}.
\]

Consequently, \( J(0, 0) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \) with eigenvalues \( \lambda = 1 \pm 2i \Rightarrow (0, 0) \) is an unstable spiral point.

\( J\left(\frac{1}{2}, -1\right) = \begin{bmatrix} -4 & 1 \\ 2 & 1 \end{bmatrix} \) with eigenvalues \( \lambda = \frac{1}{2}(1 \pm \sqrt{29}) \Rightarrow \left(\frac{1}{2}, -1\right) \) is a saddle point.

9. The equilibrium points are obtained by solving
\[
4x - y - y\sin x = 0, \; x + 2y = 0.
\]
The only solution to this system is \( x = y = 0 \). Hence, there is only one equilibrium point, namely \((0, 0)\). The Jacobian for the given system is
\[
J(x, y) = \begin{bmatrix}
4 - y\cos x & -1 - \sin x \\
1 & 2
\end{bmatrix} \Rightarrow J(0, 0) = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}.
\]
The eigenvalues of \( J(0, 0) \) are \( \lambda = 3 \) of multiplicity 2. Since we have a repeated eigenvalue, the type and stability of the equilibrium point of the nonlinear system is indeterminate.

11. The phase portrait verifies the result obtained in Problem 6, namely that \((0, 0)\) is an unstable spiral point.

13. From the phase portrait, we see the center and two saddle points identified in Problem 2.

15. From the phase portrait, we see the saddle point behavior of the trajectories around \((0, 0)\).

17. The saddle point and unstable spiral point found in Problem 7 are clearly visible in the phase portrait.

19. In the case of the initial condition \( x(0) = 1, \; y(0) = 0.1 \), the initial predator population is only 10% of the initial population. Consequently the prey population grows rapidly initially. The increased number of prey enable the predator population to grow with a corresponding decrease in the prey population. The
lack of prey causes the predator population to decrease until we return to the initial situation and the cycle then repeats.

In the case of the initial condition $x(0) = 1, y(0) = 1$, the general behavior is similar to the preceding case. However, the initial predator and prey populations are equal, and therefore there is not such a wide variation in the predator or prey populations.

21. 

(a) If we let $u = y$ and $v = \frac{dv}{dt}$, then the given differential equation is replaced by the system

$$\frac{du}{dt} = v, \quad \frac{dv}{dt} = -u - 0.1(u - 4)(u + 1)v.$$
The only equilibrium point of this system is \((0, 0)\), and

\[
J(0, 0) = \begin{bmatrix}
0 & 1 \\
-1 & 0.4
\end{bmatrix}
\]

which has eigenvalues \(\lambda = \frac{1}{5}(1 \pm 2i\sqrt{6})\). Consequently the equilibrium point \((0, 0)\) is an unstable spiral point.

(b) The sketch of the phase plane on the square \(-2 \leq u \leq 2, \ -2 \leq v \leq 2\) suggests that the differential equation does not have a limit cycle.

(c) The sketch of the phase plane on the square \(-8 \leq u \leq 8, \ -8 \leq v \leq 8\) suggests that our conclusion in
(b) was incorrect, and that in fact the differential equation does have a limit cycle.

\[ \begin{bmatrix} 2t - 1 & 0 \\ e^{t^2-t} & 1 \end{bmatrix} \begin{bmatrix} e^{t^2-t} \\ -1 \end{bmatrix} = \begin{bmatrix} (2t - 1)e^{t^2-t} \\ 0 \end{bmatrix} = x'_1(t) \]

and

\[ \begin{bmatrix} 2t - 1 & 0 \\ e^{t^2-t} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2e^t \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^t \end{bmatrix} = x'_2(t). \]

Therefore, \( x_1(t) \) and \( x_2(t) \) are indeed solutions to \( \mathbf{x}' = \mathbf{Ax} \). Furthermore, we can use Theorem 7.2.4 to show that \( \{x_1(t), x_2(t)\} \) is linearly independent. To do so, we compute the Wronskian of the given vector functions:

\[ W[x_1, x_2](t) = \det \begin{bmatrix} e^{t^2-t} & 0 \\ -1 & 2e^t \end{bmatrix} = 2e^{t^2} \neq 0. \]

Therefore, \( \{x_1(t), x_2(t)\} \) is linearly independent by Theorem 7.2.4. Therefore, by Theorem 7.3.2, since the dimension of the solution space of \( \mathbf{x}' = \mathbf{Ax} \) is 2 in this case, \( \{x_1(t), x_2(t)\} \) is a basis for the solution space. Hence, the general solution to \( \mathbf{x}' = \mathbf{Ax} \) is

\[ \mathbf{x}(t) = c_1 \begin{bmatrix} e^{t^2-t} \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2e^t \end{bmatrix}. \]

3. We begin by finding the eigenvalues of \( A \). We have

\[ \det(A - \lambda I) = \det \begin{bmatrix} -6 - \lambda & 1 \\ 1 & -5 - \lambda \end{bmatrix} = (-6 - \lambda)(-5 - \lambda) - 6 = \lambda^2 + 11\lambda + 24 = (\lambda + 3)(\lambda + 8). \]

Therefore, the eigenvalues of \( A \) are \( \lambda = -3 \) and \( \lambda = -8 \).

\underline{Eigenvectors for \( \lambda = -3 \)}: We have \( A + 3I = \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix} \), which gives rise to corresponding eigenvectors of the form \( \mathbf{v} = r(1, 3) \), where \( r \in \mathbb{R} \). Therefore, we obtain the solution \( x_1(t) = e^{-3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) to the system \( \mathbf{x}' = \mathbf{Ax} \).
Eigenvectors for $\lambda = -8$: We have $A + 8I = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(-1, 2)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $x_2(t) = e^{-8t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ to the system $x' = Ax$.

Putting the results above together, we obtain the general solution to the linear system:

$$x(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-8t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$ 

5. We begin by finding the eigenvalues of $A$. We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 10 - \lambda & -4 \\ 4 & 2 - \lambda \end{bmatrix} = (10 - \lambda)(2 - \lambda) + 16 = \lambda^2 - 12\lambda + 36 = (\lambda - 6)^2.$$

Therefore, the only eigenvalue of $A$ is $\lambda = 6$ (with multiplicity 2).

Eigenvectors for $\lambda = 6$: We have $A - 6I = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(1, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $x_1(t) = e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to the system $x' = Ax$.

We will determine a second linearly independent solution of the form

$$x_2(t) = e^{6t}(v_1 + tv_0)$$

where $v_1$ and $v_0$ are determined from

$$(A - 6I)^2 v_1 = 0, \quad (A - 6I)v_1 \neq 0, \quad v_0 = (A - 6I)v_1.$$

In this case, $(A - 6I)^2 = 0_2$. Therefore, we may choose $v_1$ to be any vector such that $(A - 6I)v_1 \neq 0$. For simplicity, we take $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, $v_0 = (A - 6I)v_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. From the expressions for $v_0$ and $v_1$, we can write down a second (linearly independent) solution to the vector differential equation:

$$x_2(t) = e^{6t} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right\} = e^{6t} \begin{bmatrix} 1 + 4t \\ 4t \end{bmatrix}.$$

Consequently, the general solution to the vector differential equation is

$$x(t) = c_1 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 1 + 4t \\ 4t \end{bmatrix}.$$ 

7. We begin by finding the eigenvalues of $A$. We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 0 & 4 \\ 0 & 2 - \lambda & 0 \\ -4 & 0 & -5 - \lambda \end{bmatrix} = (2-\lambda)[(3 - \lambda)(-5 - \lambda) + 16] = (2-\lambda)[\lambda^2+2\lambda+1] = (2-\lambda)(\lambda+1)^2.$$

Therefore, the eigenvalues of $A$ are $\lambda = 2$ and $\lambda = -1$ (with multiplicity 2).
Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \\ -4 & 0 & -7 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(0, 1, 0)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $x_1(t) = e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ to the system $x' = Ax$.

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 0 \\ -4 & 0 & -4 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(-1, 0, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $x_2(t) = e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ to the system $x' = Ax$.

We will determine a second linearly independent solution corresponding to $\lambda = -1$ in the form $x_3(t) = e^{-t}(v_1 + tv_0)$ where $v_1$ and $v_0$ are determined from

$$(A + I)^2v_1 = 0, \quad (A + I)v_1 \neq 0, \quad v_0 = (A + I)v_1.$$ 

In this case, $(A + I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let us take $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Thus, $v_0 = (A + I)v_1 = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$. From the expressions for $v_0$ and $v_1$, we can write down a second (linearly independent) solution to the vector differential equation corresponding to $\lambda = -1$:

$$x_3(t) = e^{-t} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} \right\} = e^{-t} \begin{bmatrix} 1 + 4t \\ 0 \\ -4t \end{bmatrix}.$$ 

Consequently, the general solution to the vector differential equation is

$$x(t) = c_1e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3e^{-t} \begin{bmatrix} 1 + 4t \\ 0 \\ -4t \end{bmatrix}.$$ 

9. We begin by finding the eigenvalues of $A$. We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 13 \\ -1 & -3 - \lambda \end{bmatrix} = (3 - \lambda)(-3 - \lambda) + 13 = \lambda^2 + 4.$$

Therefore, the eigenvalues of $A$ are $\lambda = \pm 2i$.

Eigenvectors for $\lambda = 2i$: We have $A - 2iI = \begin{bmatrix} 3 - 2i & 13 \\ -1 & -3 - 2i \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(1, 1, 0)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $x_1(t) = e^{2it} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ to the system $x' = Ax$.
vectors of the form \( \mathbf{v} = r(3 + 2i, -1) \), where \( r \in \mathbb{C} \). Therefore, we obtain solutions of the form

\[
\mathbf{x}(t) = e^{2it} \begin{bmatrix} 3 + 2i \\ -1 \end{bmatrix}
\]

\[
= (\cos 2t + i \sin 2t) \left( \begin{bmatrix} 3 \\ -1 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} 3 \cos 2t - 2 \sin 2t \\ -\cos 2t \end{bmatrix} + i \begin{bmatrix} 2 \cos 2t + 3 \sin 2t \\ -\sin 2t \end{bmatrix}.
\]

Taking the real and imaginary parts of \( \mathbf{x}(t) \), we obtain the two real-valued solutions

\[
\mathbf{x}_1(t) = \begin{bmatrix} 3 \cos 2t - 2 \sin 2t \\ -\cos 2t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} 2 \cos 2t + 3 \sin 2t \\ -\sin 2t \end{bmatrix}.
\]

Therefore, the general solution to the linear system \( \mathbf{x}' = A\mathbf{x} \) is

\[
\mathbf{x}(t) = c_1 \begin{bmatrix} 3 \cos 2t - 2 \sin 2t \\ -\cos 2t \end{bmatrix} + c_2 \begin{bmatrix} 2 \cos 2t + 3 \sin 2t \\ -\sin 2t \end{bmatrix}.
\]

11. We begin by finding the eigenvalues of \( A \). We have

\[
det(A - \lambda I) = \det \begin{bmatrix} -1 - \lambda & -5 & 1 \\ 4 & -9 - \lambda & -1 \\ 0 & 0 & 3 - \lambda \end{bmatrix} = (3 - \lambda)(\lambda^2 + 10\lambda + 29).
\]

Therefore, the eigenvalues of \( A \) are \( \lambda = 3 \) and \( \lambda = -5 \pm 2i \).

- **Eigenvectors for \( \lambda = 3 \):** We have \( A - 3I = \begin{bmatrix} -4 & -5 & 1 \\ 4 & -12 & -1 \\ 0 & 0 & -2 \end{bmatrix} \), which gives rise to corresponding eigenvectors of the form \( \mathbf{v} = r(1, 0, 4) \), where \( r \in \mathbb{C} \). Therefore, we obtain the solution \( \mathbf{x}_1(t) = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \) to the system \( \mathbf{x}' = A\mathbf{x} \).

- **Eigenvectors for \( \lambda = -5 + 2i \):** We have \( A - (-5 + 2i)I = \begin{bmatrix} 4 - 2i & -5 & 1 \\ 4 & -4 - 2i & -1 \\ 0 & 0 & 8 - 2i \end{bmatrix} \), which gives rise to corresponding eigenvectors of the form \( \mathbf{v} = r(4 + 2i, 4, 0) \), where \( r \in \mathbb{C} \). Therefore, we obtain solutions of...
the form

\[
x(t) = e^{(-5+2i)t} \begin{bmatrix} 4 + 2i \\ 4 \\ 0 \end{bmatrix}
\]

\[
e^{-5t}(\cos 2t + i \sin 2t) \left( \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right)
\]

\[
e^{-5t} \left( \cos 2t \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right) + ie^{-5t} \left( \cos 2t \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \sin 2t \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \right)
\]

\[
e^{-5t} \begin{bmatrix} 4 \cos 2t - 2 \sin 2t \\ 4 \cos 2t \\ 0 \end{bmatrix} + ie^{-5t} \begin{bmatrix} 2 \cos 2t + 4 \sin 2t \\ 4 \sin 2t \\ 0 \end{bmatrix}.
\]

Taking the real and imaginary parts of \(x(t)\), we obtain the two real-valued solutions

\[
x_2(t) = e^{-5t} \begin{bmatrix} 4 \cos 2t - 2 \sin 2t \\ 4 \cos 2t \\ 0 \end{bmatrix}
\]

and

\[
x_3(t) = e^{-5t} \begin{bmatrix} 2 \cos 2t + 4 \sin 2t \\ 4 \sin 2t \\ 0 \end{bmatrix}.
\]

Therefore, the general solution to the linear system \(x' = Ax\) is

\[
x(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 4 \cos 2t - 2 \sin 2t \\ 4 \cos 2t \\ 0 \end{bmatrix} + c_3 e^{-5t} \begin{bmatrix} 2 \cos 2t + 4 \sin 2t \\ 4 \sin 2t \\ 0 \end{bmatrix}.
\]

13. In the hint, we are given the eigenvalues of \(A\): \(\lambda = 2\), \(\lambda = -2\), and \(\lambda = -5\).

Eigenvectors for \(\lambda = 2\): We have \(A - 2I = \begin{bmatrix} 0 & -2 & 1 \\ 1 & -6 & 1 \\ 2 & 2 & -5 \end{bmatrix}\), which gives rise to corresponding eigenvectors of the form \(v = r(4, 1, 2)\), where \(r \in \mathbb{R}\). Therefore, we obtain the solution \(x_1(t) = e^{2t} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}\).

Eigenvectors for \(\lambda = -2\): We have \(A + 2I = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -2 & 1 \\ 2 & 2 & -1 \end{bmatrix}\), which gives rise to corresponding eigenvectors of the form \(v = r(0, 1, 2)\), where \(r \in \mathbb{R}\). Therefore, we obtain the solution \(x_2(t) = e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\).

Eigenvectors for \(\lambda = -5\): We have \(A + 5I = \begin{bmatrix} 7 & -2 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\), which gives rise to corresponding eigenvectors of the form \(v = r(1, 2, -3)\), where \(r \in \mathbb{R}\). Therefore, we obtain the solution \(x_3(t) = e^{-5t} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}\).
Putting the results above together, we obtain the general solution to the linear system:

\[
\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 e^{-5t} \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}.
\]

15. In the hint, we are given the eigenvalues of \( A \): \( \lambda = 4 \) and \( \lambda = -3 \). Since \( \det(A) = -48 \), we use the fact that the \( \det(A) \) is the product of the eigenvalues of \( A \) (see Section 5.7, Problem 29(c)) to deduce that \( \lambda = 4 \) occurs with multiplicity 2.

Eigenvectors for \( \lambda = 4 \): We have \( A - 4I = \begin{bmatrix} -21 & 0 & -42 \\ -7 & 0 & -14 \\ 7 & 0 & 14 \end{bmatrix} \), which gives rise to eigenvectors of the form \( \mathbf{v} = r(0, 1, 0) + s(-2, 0, 1) \), where \( r, s \in \mathbb{R} \). Therefore, we obtain the solutions \( \mathbf{x}_1(t) = e^{4t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \) and \( \mathbf{x}_2(t) = e^{4t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \) to the system \( \mathbf{x}' = A\mathbf{x} \).

Eigenvectors for \( \lambda = -3 \): We have \( A + 3I = \begin{bmatrix} -14 & 0 & -42 \\ -7 & 7 & -14 \\ 7 & 0 & 21 \end{bmatrix} \), which gives rise to corresponding eigenvectors of the form \( \mathbf{v} = r(-3, -1, 1) \), where \( r \in \mathbb{R} \). Therefore, we obtain the solution \( \mathbf{x}_3(t) = e^{3t} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \) to the system \( \mathbf{x}' = A\mathbf{x} \).

Putting the results above together, we obtain the general solution to the linear system:

\[
\mathbf{x}(t) = c_1 e^{4t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}.
\]

17. In the hint, we are given the eigenvalues of \( A \): \( \lambda = 0 \) and \( \lambda = -3 \). According to Section 5.7, Problem 29(c), the sum of the eigenvalues of \( A \) is \( \text{tr}(A) = -3 \). Therefore, we conclude that \( \lambda = 0 \) occurs with multiplicity 2.

Eigenvectors for \( \lambda = 0 \): The matrix \( A - 0I = A \) gives rise to corresponding eigenvectors of the form \( \mathbf{v} = r(-1, 0, 1) \), where \( r \in \mathbb{R} \). Therefore, we obtain the solution \( \mathbf{x}_1(t) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \) to \( \mathbf{x}' = A\mathbf{x} \).

We will determine a second linearly independent solution corresponding to \( \lambda = 0 \) in the form \( \mathbf{x}_3(t) = \mathbf{v}_1 + t\mathbf{v}_0 \), where \( \mathbf{v}_1 \) and \( \mathbf{v}_0 \) are determined from

\[
A^2\mathbf{v}_1 = \mathbf{0}, \quad A\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = A\mathbf{v}_1.
\]
In this case, \( A^2 = \begin{bmatrix} 18 & 18 & 18 \\ 9 & 9 & 9 \\ -18 & -18 & -18 \end{bmatrix} \). Let us choose \( \mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \). Then \( \mathbf{v}_0 = A\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \). From the expressions for \( \mathbf{v}_0 \) and \( \mathbf{v}_1 \), we can write down a second (linearly independent) solution to the vector differential equation corresponding to \( \lambda = 0 \):

\[
x_3(t) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -t \\ -1 \\ 1 + t \end{bmatrix}.
\]

Eigenvectors for \( \lambda = -3 \): We have \( A + 3I = \begin{bmatrix} -4 & -6 & -7 \\ -3 & 0 & -3 \\ 7 & 6 & 10 \end{bmatrix} \), which gives rise to corresponding eigenvectors of the form \( \mathbf{v} = r(-2, -1, 2) \), where \( r \in \mathbb{R} \). Therefore, we obtain the solution \( bfx_3(t) = e^{-3t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \) to the system \( \mathbf{x}' = A\mathbf{x} \).

Putting the results above together, we obtain the general solution to the linear system:

\[
x(t) = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -t \\ -1 \\ 1 + t \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}.
\]

19. In the hint, we are given the eigenvalues of \( A \): \( \lambda = -1 \) and \( \lambda = 1 \pm 2i \).

Eigenvectors for \( \lambda = -1 \): The matrix \( A + I = \begin{bmatrix} 0 & -4 & -2 \\ -4 & -4 & -6 \\ 4 & 8 & 8 \end{bmatrix} \), which gives rise to corresponding eigenvectors of the form \( \mathbf{v} = r(-2, -1, 2) \), where \( r \in \mathbb{C} \). Therefore, we obtain the solution \( \mathbf{x}_1(t) = e^{-t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \) to the system \( \mathbf{x}' = A\mathbf{x} \).

Eigenvectors for \( \lambda = 1 + 2i \): The matrix \( A - (1 + 2i)I = \begin{bmatrix} -2 - 2i & -4 & -2 \\ -4 & -6 - 2i & -6 \\ 4 & 8 & 6 - 2i \end{bmatrix} \), which gives rise to corresponding eigenvectors of the form \( \mathbf{v} = r(-1, i, 1 - i) \), where \( r \in \mathbb{C} \). Therefore, we obtain solutions of
the form
\[
x(t) = e^{(1+2i)t} \begin{bmatrix} -1 \\ i \\ 1 - i \end{bmatrix}
\]
\[
= e^t (\cos 2t + i \sin 2t) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + i e^t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}
\]
\[
= e^t \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} - i e^t \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}
\]
\[
= e^t \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} + i e^t \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix}.
\]

Taking the real and imaginary parts of \( x(t) \), we obtain the two real-valued solutions
\[
x_2(t) = e^t \begin{bmatrix} -\cos 2t \\ -\sin 2t \end{bmatrix} \quad \text{and} \quad x_3(t) = e^t \begin{bmatrix} \sin 2t \\ -\cos 2t \end{bmatrix}.
\]

Putting together the above results, we obtain the general solution to the system \( x' = Ax \):
\[
x(t) = c_1 e^{-t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + c_2 e^t \begin{bmatrix} -\cos 2t \\ -\sin 2t \\ \cos 2t + \sin 2t \end{bmatrix} + c_3 e^t \begin{bmatrix} \sin 2t \\ \cos 2t \\ -\cos 2t + \sin 2t \end{bmatrix}.
\]

21. We begin by finding the eigenvalues of \( A \). We have
\[
det(A - \lambda I) = \det \begin{bmatrix} -3 - \lambda & -1 & -2 \\ 1 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix} = -1 - 2\lambda - \lambda(\lambda^2 + 3\lambda + 1) = -\lambda^3 - 3\lambda^2 - 3\lambda - 1 = -(\lambda + 1)^3,
\]
where we have used cofactor expansion along the third row to compute the determinant. Therefore, we have a single eigenvalue, \( \lambda = -1 \), with multiplicity 3.

**Eigenvectors for \( \lambda = -1 \):** We have \( A + I = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \), which gives rise to corresponding eigenvectors of the form \( \mathbf{v} = r(-1, 0, 1) \), where \( r \in \mathbb{R} \). Therefore, we obtain the solution \( x_0(t) = e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \) to the system \( \mathbf{x}' = A\mathbf{x} \).

According to Theorem 7.5.4, there exist two further linearly independent solutions to \( \mathbf{x}' = A\mathbf{x} \) of the form
\[
x_1(t) = e^{-t}(\mathbf{v}_1 + t\mathbf{v}_0),
\]
\[
x_2(t) = e^{-t}(\mathbf{v}_2 + t\mathbf{v}_1 + \frac{1}{2!}t^2\mathbf{v}_0),
\]
where \((A + I)^3 v_2 = 0, (A + I)^2 v_2 \neq 0\),

and

\[ v_1 = (A + I)v_2, \quad v_0 = (A + I)^2 v_2. \]

A short calculation shows that

\[ A + I = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad (A + I)^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}, \quad (A + I)^3 = 0. \]

Therefore, we may take

\[ v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_1 = (A + I)v_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad v_0 = (A + I)^2 v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}. \]

Hence, we obtain

\[ x_1(t) = e^{-t} \left( \begin{bmatrix} -2 + t \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} -2 + t \\ 1 \\ 1 - t \end{bmatrix} \]

and

\[ x_2(t) = e^{-t} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 + t \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2!} t^2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} 1 - 2t + t^2/2 \\ t \\ t - t^2/2 \end{bmatrix}. \]

Therefore, the general solution to the vector differential equation is

\[ x(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2 + t \\ 1 \\ 1 - t \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 - 2t + t^2/2 \\ t \\ t - t^2/2 \end{bmatrix} \]

\[ = e^{-t} \left\{ c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 + t \\ 1 \\ 1 - t \end{bmatrix} + c_3 \begin{bmatrix} 1 - 2t + t^2/2 \\ t \\ t - t^2/2 \end{bmatrix} \right\}. \]

23. We begin by finding the eigenvalues of \(A\). We have

\[ \det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 13 & 0 & 0 \\ -1 & -2 - \lambda & 0 & 0 \\ 0 & 0 & 2 - \lambda & 4 \\ 0 & 0 & 0 & 2 - \lambda \end{bmatrix} = (2 - \lambda)^2(\lambda^2 + 9), \]

so the eigenvalues of \(A\) are \(\lambda = 2\) (with multiplicity 2) and \(\lambda = \pm 3i\).

**Eigenvectors for \(\lambda = 2\):** We have \(A - 2I = \)

\[ \begin{bmatrix} 0 & 13 & 0 & 0 \\ -1 & -4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

which gives rise to corresponding eigenvectors of the form \(v = r(0, 0, 1, 0)\), where \(r \in \mathbb{C}\). Therefore, we obtain the solution \(x_1(t) = e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\).
We will determine a second linearly independent solution corresponding to $\lambda = 2$ in the form $x_2(t) = v_1 + tv_0$, where $v_1$ and $v_0$ are determined from

$$(A - 2I)^2v_1 = 0, \quad (A - 2I)v_1 \neq 0, \quad v_0 = (A - 2I)v_1.$$ 

In this case, $(A - 2I)^2 = \begin{bmatrix} -13 & -52 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so we can choose $v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Then $v_0 = (A - 2I)v_1 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}$. From the expressions for $v_0$ and $v_1$, we can write down a second (linearly independent) solution to the vector differential equation corresponding to $\lambda = 2$:

$$x_2(t) = e^{2t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} = e^{2t} \begin{bmatrix} 0 \\ 0 \\ 4t \\ 0 \end{bmatrix}. $$

Eigenvectors for $\lambda = 3i$: We have $A - 3iI = \begin{bmatrix} 2 - 3i & 13 & 0 & 0 \\ -1 & -2 - 3i & 0 & 0 \\ 0 & 0 & 2 - 3i & 0 \\ 0 & 0 & 0 & 2 - 3i \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(2 + 3i, -1, 0, 0)$, where $r \in \mathbb{C}$. Hence, we obtain solutions of the form

$$x(t) = e^{3it} \begin{bmatrix} 2 + 3i \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$= (\cos 3t + i \sin 3t) \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cos 3t - 3 \sin 3t \\ -\cos 3t \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 3 \cos 3t + 2 \sin 3t \\ -\sin 3t \\ 0 \\ 0 \end{bmatrix}.$$ 

Taking the real and imaginary parts of $x(t)$ yields the two real-valued solutions

$$x_3(t) = \begin{bmatrix} 2 \cos 3t - 3 \sin 3t \\ -\cos 3t \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x_4(t) = \begin{bmatrix} 3 \cos 3t + 2 \sin 3t \\ -\sin 3t \\ 0 \\ 0 \end{bmatrix}. $$
Putting together the four solutions $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$ to the linear system $x' = Ax$ obtained above, we obtain the general solution to the system:

$$x(t) = c_1 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 4t \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \cos 3t - 3 \sin 3t \\ -\cos 3t \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 3 \cos 3t + 2 \sin 3t \\ -\sin 3t \\ 0 \\ 0 \end{bmatrix}.$$

25. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -6 - \lambda & 1 \\ 6 & -5 - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 + 11\lambda + 24 = 0$ if and only if $(\lambda - 3)(\lambda + 8) = 0$. This means that the eigenvalues of $A$ are $\lambda = 3$ and $\lambda = -8$.

Eigenvectors for $\lambda = -3$: We have $A + 3I = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(1,3)$, where $r \in \mathbb{R}$. Hence, one solution to the associated homogeneous system is $x_1(t) = e^{-3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Eigenvectors for $\lambda = -8$: We have $A + 8I = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $v = r(-1, 2)$, where $r \in \mathbb{R}$. Hence, a second solution to the associated homogeneous system is $x_2(t) = e^{-8t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Putting the two solutions obtained above together, we obtain the general solution to the associated homogeneous system:

$$x_c(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-8t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$ 

The fundamental matrix is

$$X(t) = \begin{bmatrix} e^{-3t} & -e^{-8t} \\ 3e^{-3t} & 2e^{-8t} \end{bmatrix}.$$ 

We have

$$X^{-1}(t) = \frac{1}{5} \begin{bmatrix} 2e^{3t} & e^{3t} \\ -3e^{8t} & e^{8t} \end{bmatrix}.$$ 

Therefore,

$$u(t) = \int^t X^{-1}(s)b(s)ds$$

$$= \frac{1}{5} \int^t \begin{bmatrix} 2e^{3s} & e^{3s} \\ -3e^{8s} & e^{8s} \end{bmatrix} \begin{bmatrix} 1 \\ e^{-s} \end{bmatrix} ds$$

$$= \frac{1}{5} \int^t \begin{bmatrix} 2e^{3s} + e^{2s} \\ -3e^{8s} + e^{7s} \end{bmatrix} ds$$

$$= \frac{1}{5} \begin{bmatrix} \frac{2}{3}e^{3t} + \frac{1}{2}e^{2t} \\ -\frac{3}{8}e^{8t} + \frac{1}{4}e^{7t} \end{bmatrix}.$$ 

Hence, we obtain the particular solution

$$x_p(t) = X(t)u(t) = \begin{bmatrix} e^{-3t} & -e^{-8t} \\ 3e^{-3t} & 2e^{-8t} \end{bmatrix} \frac{1}{5} \begin{bmatrix} \frac{2}{3}e^{3t} + \frac{1}{2}e^{2t} \\ -\frac{3}{8}e^{8t} + \frac{1}{4}e^{7t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{3} + \frac{1}{4}e^{-t} \\ \frac{5}{4} + \frac{1}{4}e^{-t} \end{bmatrix}.$$
Thus, the general solution to the nonhomogeneous system of differential equations is
\[
x(t) = x_c(t) + x_p(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-8t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{-t}.
\]

27. We have \(\det(A - \lambda I) = 0\) if and only if
\[
\begin{vmatrix}
10 - \lambda & -4 \\
-2 & 2 - \lambda
\end{vmatrix} = 0
\]
if and only if \(\lambda^2 - 12\lambda + 36 = 0\) if and only if \((\lambda - 6)^2 = 0\). This means that the only eigenvalue of \(A\) is \(\lambda = 6\), with multiplicity 2.

Eigenvectors for \(\lambda = 6\): We have \(A - 6I = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\), which gives rise to corresponding eigenvectors of the form \(v = r(1, 1)\), where \(r \in \mathbb{R}\). Hence, one solution to the associated homogeneous system is \(x_1(t) = e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\). We will determine a second linearly independent solution of the form
\[
x_2(t) = e^{6t}(v_1 + tv_0)
\]
where \(v_1\) and \(v_0\) are determined from
\[
(A - 6I)^2v_1 = 0, \quad (A - 6I)v_1 \neq 0, \quad v_0 = (A - 6I)v_1.
\]
In this case, we have \(A - 6I = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\) and \((A - 6I)^2 = 0_2\). Therefore, we may choose \(v_1\) to be any vector such that \((A - 6I)v_1 \neq 0\). For simplicity, we take \(v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\). Thus, \(v_0 = (A - 6I)v_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}\). From the expressions for \(v_0\) and \(v_1\), we can write down a second solution to the associated homogeneous system:
\[
x_2(t) = e^{6t}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 4 \end{bmatrix}\right\} = e^{6t}\begin{bmatrix} 1 + 4t \\ 4t \end{bmatrix}.
\]

Putting the two solutions obtained above together, we obtain the general solution to the associated homogeneous system:
\[
x(t) = c_1 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 1 + 4t \\ 4t \end{bmatrix}.
\]
The fundamental matrix is
\[
X(t) = \begin{bmatrix} e^{6t} & (1 + 4t)e^{6t} \\ e^{6t} & 4te^{6t} \end{bmatrix}.
\]
We have
\[
X^{-1}(t) = -e^{-12t} \begin{bmatrix} 4te^{6t} & -(1 + 4t)e^{6t} \\ -e^{6t} & e^{6t} \end{bmatrix} = \begin{bmatrix} -4te^{-6t} & (1 + 4t)e^{-6t} \\ e^{-6t} & -e^{-6t} \end{bmatrix}.
\]
Therefore,
\[
u(t) = \int_0^t X^{-1}(s)b(s)ds
= \int_0^t \begin{bmatrix} -4se^{-6s} & (1 + 4s)e^{-6s} \\ e^{-6s} & -e^{-6s} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{6s} ds
= \int_0^t \begin{bmatrix} (1 + 4s)/s \\ -1/s \end{bmatrix} ds
= \begin{bmatrix} \ln |t| + 4t \\ -\ln |t| \end{bmatrix}.
\]
Hence, we obtain the particular solution
\[ x_p(t) = X(t)u(t) = \begin{bmatrix} e^{6t} & (1 + 4t)e^{6t} \\ e^{6t} & 4te^{6t} \end{bmatrix} \begin{bmatrix} \ln|t| + 4t \\ -\ln|t| \end{bmatrix} = e^{6t} \begin{bmatrix} 4t(1 - \ln|t|) \\ \ln|t| + 4t(1 - \ln|t|) \end{bmatrix}. \]

Thus, the general solution to the nonhomogeneous system of differential equations is
\[ x(t) = x_c(t) + x_p(t) = c_1e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{6t} \begin{bmatrix} 1 + 4t \\ 4t \end{bmatrix} + e^{6t} \begin{bmatrix} 4t(1 - \ln|t|) \\ \ln|t| + 4t(1 - \ln|t|) \end{bmatrix}. \]

29. Using the hint from Problem 13, the eigenvalues of \( A \) are \( \lambda = 2, \lambda = -2, \) and \( \lambda = -5. \)

Eigenvectors for \( \lambda = 2: \) We have \( A - 2I = \begin{bmatrix} 0 & -2 & 1 \\ 1 & -6 & 1 \\ 2 & 2 & -5 \end{bmatrix}, \) which gives rise to corresponding eigenvectors of the form \( v = r(4, 1, 2), \) where \( r \in \mathbb{R}. \) Therefore, we obtain the solution \( x_1(t) = e^{2t} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}. \)

Eigenvectors for \( \lambda = -2: \) We have \( A + 2I = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -2 & 1 \\ 2 & 2 & -1 \end{bmatrix}, \) which gives rise to corresponding eigenvectors of the form \( v = r(0, 1, 2), \) where \( r \in \mathbb{R}. \) Therefore, we obtain the solution \( x_2(t) = e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}. \)

Eigenvectors for \( \lambda = -5: \) We have \( A + 5I = \begin{bmatrix} 7 & -2 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \) which gives rise to corresponding eigenvectors of the form \( v = r(1, 1, -3), \) where \( r \in \mathbb{R}. \) Therefore, we obtain the solution \( x_3(t) = e^{-5t} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}. \)

Putting the results above together, we obtain the general solution to the associated homogeneous system:
\[ x_c(t) = c_1e^{2t} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} + c_2e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3e^{-5t} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}. \]

A fundamental matrix for this system is
\[ X(t) = \begin{bmatrix} 4e^{2t} & 0 & e^{-5t} \\ e^{2t} & e^{-2t} & 2e^{-5t} \\ 2e^{2t} & 2e^{-2t} & -3e^{-5t} \end{bmatrix}. \]

Using the adjoint method, we determine
\[ X^{-1}(t) = \frac{1}{28}e^{5t} \begin{bmatrix} -7e^{-7t} & 2e^{-7t} & -e^{-7t} \\ 7e^{-3t} & -14e^{-3t} & -7e^{-3t} \\ 0 & -8 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^{-2t} & -\frac{1}{4}e^{-2t} & \frac{1}{2}e^{-2t} \\ -\frac{1}{4}e^{2t} & \frac{3}{4}e^{2t} & \frac{1}{2}e^{2t} \\ 0 & \frac{3}{2}e^{5t} & -\frac{3}{2}e^{5t} \end{bmatrix}. \]
Therefore,

\[ u(t) = \int^t X^{-1}(s)b(s)ds \]

\[ = \int^t \begin{bmatrix} \frac{1}{4}e^{-2s} & -\frac{1}{14}e^{-2s} & \frac{3}{4}e^{-2s} \\ -\frac{4}{9}e^{2s} & \frac{4}{9}e^{2s} & \frac{4}{9}e^{2s} \\ 0 & \frac{7}{9}e^{5s} & -\frac{1}{7}e^{5s} \end{bmatrix} \begin{bmatrix} s \\ 0 \\ 1 \end{bmatrix} ds \]

\[ = \int^t \begin{bmatrix} \frac{1}{3}s + \frac{1}{9}s \end{bmatrix} e^{-2s} \]

\[ = \begin{bmatrix} (\frac{1}{3}t - \frac{9}{17})e^{-2t} \\ (\frac{1}{3}t + \frac{7}{10})e^{2t} \\ -\frac{1}{3}e^{5t} \end{bmatrix}. \]

Hence, we obtain the particular solution

\[ x_p(t) = X(t)u(t) = \begin{bmatrix} 4e^{2t} \\ e^{2t} \\ 2e^{2t} \end{bmatrix} \begin{bmatrix} 0 \\ e^{-5t} \\ 2e^{-5t} \end{bmatrix} + \begin{bmatrix} \frac{9}{8}t - \frac{9}{17} \\ \frac{9}{8}t + \frac{15}{18} \\ \frac{1}{3}e^{5t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8}t - \frac{7}{20} \\ -\frac{1}{8}t + \frac{33}{20} \\ -\frac{1}{2}t + \frac{3}{10} \end{bmatrix}. \]

Thus, the general solution to the nonhomogeneous system of differential equations is

\[ x(t) = x_e(t) + x_p(t) = c_1e^{2t} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} + c_2e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3e^{-5t} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8}t - \frac{7}{20} \\ -\frac{1}{8}t + \frac{33}{20} \\ -\frac{1}{2}t + \frac{3}{10} \end{bmatrix}. \]

31. TRUE. We have

\[ x'' = (x_0)' = (Ax_0)' = Ax_0 = A(Ax_0) = A^2x_0, \]

which shows that \( x_0 \) is a solution to the linear system \( x'' = A^2x \).

33.

(a) By setting \( x_1 = y \), \( x_2 = y' \), and \( x_3 = y'' \), we obtain the system \( x' = Ax \), where

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{bmatrix}. \]

(b) Supposing that \( y_1 = f_1(t) \), \( y_2 = f_2(t) \), and \( y_3 = f_3(t) \) are solutions to (7.11.8), we have that

\[ \frac{d^3y_1}{dt^3} + a\frac{d^2y_1}{dt^2} + b\frac{dy_1}{dt} + cy_1 = 0, \quad \frac{d^3y_2}{dt^3} + a\frac{d^2y_2}{dt^2} + b\frac{dy_2}{dt} + cy_2 = 0, \quad \frac{d^3y_3}{dt^3} + a\frac{d^2y_3}{dt^2} + b\frac{dy_3}{dt} + cy_3 = 0, \]

which gives

\[ f''_1(t) + af''_1(t) + bf'_1(t) + cf_1(t) = 0, \quad f''_2(t) + af''_2(t) + bf'_2(t) + cf_2(t) = 0, \quad f''_3(t) + af''_3(t) + bf'_3(t) + cf_3(t) = 0. \]

Substituting \( x_1(t) = \begin{bmatrix} f_1(t) \\ f'_1(t) \\ f''_1(t) \end{bmatrix} \) into \( x' = Ax \), we obtain

\[ \begin{bmatrix} f_1(t) \\ f'_1(t) \\ f''_1(t) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f''_1(t) \\ -cf_1(t) - bf'_1(t) - af''_1(t) \end{bmatrix}. \]
which corresponds precisely to the differential equation for \( f_1 \) above. Likewise, substituting \( x_2(t) = \begin{bmatrix} f_2(t) \\ f'_{2}(t) \\ f''_{2}(t) \end{bmatrix} \) into \( x' = Ax \), we obtain
\[
\begin{bmatrix} f'_{2}(t) \\ f''_{2}(t) \\ f''_{2}(t) \end{bmatrix} = \begin{bmatrix} f'_{2}(t) \\ f''_{2}(t) \\ -c f_2(t) - b f'_{2}(t) - a f''_{2}(t) \end{bmatrix},
\]
which corresponds precisely to the differential equation for \( f_2 \) above. Finally, substituting \( x_3(t) = \begin{bmatrix} f_3(t) \\ f'_{3}(t) \\ f''_{3}(t) \end{bmatrix} \) into \( x' = Ax \), we obtain
\[
\begin{bmatrix} f'_{3}(t) \\ f''_{3}(t) \\ f''_{3}(t) \end{bmatrix} = \begin{bmatrix} f'_{3}(t) \\ f''_{3}(t) \\ -c f_3(t) - b f'_{3}(t) - a f''_{3}(t) \end{bmatrix},
\]
which corresponds precisely to the differential equation for \( f_3 \) above.

(c) Using the Wronskian in \( V_3(I) \) we have
\[
W[x_1, x_2, x_3](t) = f_{1} f_{2} f_{3} f'_{1} f'_{2} f'_{3} f''_{1} f''_{2} f''_{3}.
\]
Similarly, using the Wronskian in \( C^1(I) \),
\[
W[y_1, y_2, y_3](t) = \begin{vmatrix} f_{1} & f_{2} & f_{3} \\ f'_{1} & f'_{2} & f'_{3} \\ f''_{1} & f''_{2} & f''_{3} \end{vmatrix}.
\]

35. The matrix \( A \) has eigenvalues \( \lambda_1 = -2 \) and \( \lambda_2 = -3 \) with corresponding nonproportional eigenvectors \( v_1 = (3, 1) \) and \( v_2 = (2, 1) \), respectively. Since both eigenvalues are real and negative, the equilibrium point is a stable node.

37. The matrix \( A \) has a single eigenvalue \( \lambda = -4 \) (with multiplicity 2) with corresponding nonproportional eigenvectors \( v_1 = (1, 0) \) and \( v_2 = (0, 1) \). In this case, the equilibrium point is called a proper node, with all trajectories approaching the equilibrium point.

39. The matrix \( A \) has eigenvalues \( \lambda = -5 \pm i \) with corresponding nonproportional eigenvectors \( v_1 = (2 + i, 1) \) and \( v_2 = (2 - i, 1) \), respectively. Since the eigenvalues are complex with negative real part, the equilibrium point is a stable spiral.

41. The matrix \( A \) has a single eigenvalue \( \lambda = 6 \) (with multiplicity 2) with a single corresponding eigenvector \( v = (2, 1) \). Since \( \lambda > 0 \), the equilibrium point in this case is an unstable degenerate node.

43. The first-order system is
\[
\begin{align*}
u' &= v \\
v' &= -25u.
\end{align*}
\]
The corresponding matrix formulation of the linear system is \( x' = Ax \), where \( x = \begin{bmatrix} u \\ v \end{bmatrix} \) and \( A = \begin{bmatrix} 0 & 1 \\ -25 & 0 \end{bmatrix} \). The eigenvalues of \( A \) are \( \lambda = \pm 5i \), and since they are pure imaginary, we conclude that the equilibrium point \((0, 0)\) is a stable center.
True-False Review:

1. **FALSE.** This function has a discontinuity at every integer \( n \neq 0, 1 \), and therefore, it has infinitely many discontinuities and cannot be piecewise continuous.

3. **TRUE.** If \( f \) has discontinuities at \( a_1, a_2, \ldots, a_r \) in \( I \) and \( g \) has discontinuities at \( b_1, b_2, \ldots, b_s \) in \( I \), then \( f + g \) has at most \( r + s \) discontinuities in \( I \). Therefore, we can divide \( I \) into finitely many subintervals so that \( f + g \) is continuous on each subinterval and \( f \) approaches a finite limit as the endpoints of each subinterval are approached from within. Hence, from Definition 8.1.4, \( f \) is piecewise continuous on the interval \( I \).

5. **FALSE.** The lower limit of the integral defining the Laplace transform must be 0, not 1 (see Definition 8.1.1).

7. **FALSE.** We have \( L[2 \cos 3t] = 2L[\cos 3t] \), and \( L[\cos 3t] = \frac{s^2}{s^2 + 9} \) provided that \( s > 0 \), according to Equation (8.1.4). So this Laplace transform is defined for \( s > 0 \), not just \( s > 3 \).

9. **FALSE.** For instance, if we take \( f(t) = e^t \), then \( L[f] = \frac{1}{s-1} \), but \( L[f^2] = L[e^{2t}] = \frac{1}{s-2} \neq \left(\frac{1}{s-1}\right)^2 \).

Problems:

1. For \( s > 2 \),

\[
F(s) = \int_0^\infty e^{-st} e^{2t} dt = \lim_{n \to \infty} \int_0^n e^{(2-s)t} dt \\
= \frac{1}{2-s} \lim_{n \to \infty} \left[ e^{(2-s)t} \right]_0^n \\
= \frac{1}{2 - s}.
\]

3. For \( s > 0 \),

\[
F(s) = \int_0^\infty e^{-st} \sin bt dt = \lim_{n \to \infty} \int_0^n e^{-st} \sin bt dt \\
= \lim_{n \to \infty} \left[ \frac{e^{-st}(- \sin bt - b \cos bt)}{s^2 + b^2} \right]_0^n \\
= \frac{b}{s^2 + b^2}.
\]

5. For \( s > |b| \),

\[
F(s) = \int_0^\infty e^{-st} \cosh bt dt = \lim_{n \to \infty} \int_0^n e^{-st} \left( \frac{e^{bt} + e^{-bt}}{2} \right) dt \\
= \lim_{n \to \infty} \left[ \frac{e^{(b-s)t}}{2(b-s)} - \frac{e^{-(b+s)t}}{2(b+s)} \right]_0^n \\
= \lim_{n \to \infty} \left[ \frac{e^{(b-s)n} - 1}{2(b-s)} - \frac{e^{-(b+s)n} + 1}{2(b+s)} \right] \\
= \frac{s}{s^2 - b^2}.
\]
7. For $s > 0$,

$$F(s) = \int_0^\infty e^{-st}t \, dt = 2 \lim_{n \to \infty} \int_0^n e^{-st}t \, dt$$

$$= 2 \lim_{n \to \infty} \left[ \frac{e^{-st}(-st - 1)}{s^2} \right]_0^n$$

$$= 2 \lim_{n \to \infty} \left[ \frac{-n}{se^{sn}} - \frac{1}{s^2e^{sn}} + \frac{1}{s^2} \right]$$

$$= \frac{2}{s^2}.$$

9. For $s > 0$,

$$F(s) = \int_0^2 e^{-st}(1) \, dt + \int_2^\infty e^{-st}(-1) \, dt$$

$$= \left[ \frac{e^{-st}}{-s} \right]_0^2 + \lim_{n \to \infty} \int_2^n (-e^{-st}) \, dt$$

$$= \left[ -\frac{e^{-2s}}{s} + \frac{1}{s} \right] + \lim_{n \to \infty} \left[ \frac{e^{-st}}{s} \right]_2^n$$

$$= \frac{1}{s} - \frac{1}{se^{2s}} + \lim_{n \to \infty} \left[ \frac{1}{se^{sn}} - \frac{1}{se^{2s}} \right]$$

$$= \frac{1}{s}(1 - 2e^{-2s}).$$

11. For $s > 1$,

$$F(s) = \int_0^\infty e^{-s(t - e^t \sin t)} \, dt$$

$$= \lim_{n \to \infty} \int_0^n e^{(1-s)t} \sin t \, dt$$

$$= \lim_{n \to \infty} \left[ \frac{e^{(1-s)t}(\sin t - \cos t)}{(s-1)^2 + 1} \right]_0^n$$

$$= \lim_{n \to \infty} \left[ \frac{e^{(1-2n)(\sin t - \cos n)}}{(s-1)^2 + 1} + \frac{1}{(s-1)^2 + 1} \right]$$

$$= \frac{1}{(s-1)^2 + 1}.$$

13. For $s > 3$,

$$L[2t - e^{3t}] = L[2t] - L[e^{3t}] = 2L[t] - L[e^{3t}]$$

$$= 2 \left( \frac{1}{s^2} \right) - \frac{1}{s - 3}$$

$$= \frac{2}{s^2} - \frac{1}{s - 3}.$$
15. For $s > |b|$, 
\[
L[\cosh bt] = L \left[ \frac{1}{2} e^{bt} + \frac{1}{2} e^{-bt} \right] = \frac{1}{2} L[e^{bt}] + \frac{1}{2} L[e^{-bt}]
\]
\[
= \frac{1}{2} \left( \frac{1}{s - b} \right) + \frac{1}{2} \left( \frac{1}{s + b} \right)
\]
\[
= \frac{s}{s^2 - b^2}.
\]

17. For $s > 0$, 
\[
L[3t^2 - 5 \cos 2t + \sin 3t] = 3L[t^2] - 5L[\cos 2t] + L[\sin 3t]
\]
\[
= 3 \left( \frac{2t}{s^3} \right) - 5 \left( \frac{s}{s^2 + 4} \right) + \frac{3}{s^2 + 9}
\]
\[
= \frac{6s}{s^3} - \frac{5s}{s^2 + 4} + \frac{3}{s^2 + 9}.
\]

19. For $s > 1$, 
\[
L[2e^{-3t} + 4e^t - 5 \sin t] = 2L[e^{-3t}] + 4L[e^t] - 5L[\sin t]
\]
\[
= \frac{2}{s + 3} + \frac{4}{s - 1} - \frac{5}{s^2 + 1}.
\]

21. For $s > 0$, 
\[
L[4 \cos^2 bt] = 2L[2 \cos^2 bt] = 2L[\cos 2bt + 1]
\]
\[
= 2L[\cos 2bt] + 2L[1]
\]
\[
= 2 \left( \frac{s}{4b^2 + s^2} \right) + 2 \left( \frac{1}{s} \right)
\]
\[
= \frac{4s^2 + 2b^2}{s(s^2 + 4b^2)}.
\]

23. The graph of $f$ is piece-wise continuous on $[0, \infty)$.

25. The graph of $f$ is not piece-wise continuous on $[0, \infty)$, because $\lim_{t \to 1^+} f(t) = \infty$, i.e. the right-hand limit as $t$ approaches 1 from above is not finite.

27. The graph of $f$ is piece-wise continuous on $[0, \infty)$.

29. The graph of $f$ is not piece-wise continuous on $[0, \infty)$.

31. We have, 
\[
F(s) = \int_0^1 e^{-st}tdt + \int_1^\infty e^{-st}(0)dt
\]
\[
= \left[ \frac{e^{-st}}{s^2}(-st - 1) \right]_1^1
\]
\[
= \frac{1}{s^2}[1 - e^{-s}(s + 1)].
\]
33. We have,

\[ F(s) = \int_0^1 e^{-st}(0)dt + \int_1^2 e^{-st}(t)dt + \int_2^\infty e^{-st}(0)dt \]

\[ = \left[ \frac{1}{s^2} e^{-st}(-st - 1) \right]_2^1 \]

\[ = \frac{1}{s^2} e^{-2s} \left[ e^s(s + 1) - (2s + 1) \right]. \]
35. We have,

\[ L[e^{ibt}] = \frac{1}{s - bi} = \frac{s + bi}{s^2 + b^2} = \frac{s}{s^2 + b^2} + \frac{b}{s^2 + b^2} i. \]  \hspace{1cm} (0.0.34)
But

\[ e^{ibt} = \cos bt + i\sin bt; \]

hence,

\[ L[e^{ibt}] = L[\cos bt + i\sin bt] = L[\cos bt] + iL[\sin bt]. \]  \hspace{1cm} (0.0.35)

From (0.0.34) and (0.0.35) we have, on equating real and imaginary parts,

\[ L[\cos bt] = \frac{s}{s^2 + b^2} \quad \text{and} \quad L[\sin bt] = \frac{b}{s^2 + b^2}. \]

37. If \( n = 1 \), then

\[ L[t^n] = L[t] = \int_0^\infty e^{-st}tdt = \lim_{n \to \infty} \int_0^n e^{-st}tdt \]
\[ = \lim_{n \to \infty} \left\{ \left[ -\frac{te^{-st}}{s} \right]_0^n + \frac{1}{s} \int_0^n e^{-st}dt \right\} \]
\[ = \lim_{n \to \infty} \left[ -\frac{ne^{-sn}}{s} - \frac{e^{-sn}}{s^2} + \frac{1}{s^2} \right] = \frac{1}{s^2} = \frac{n!}{s^{n+1}}. \]
so the proposition is true for \( n = 1 \). Now suppose that for some positive integer \( k \), it is true that

\[ L[t^k] = \frac{k!}{s^{k+1}}. \]

Then integration by parts yields

\[
L[t^{k+1}] = \int_0^\infty e^{-st} t^{k+1} dt
= \lim_{n \to \infty} \left\{ \left. \left[ -\frac{e^{-st} t^{k+1}}{s} \right]_0^n + \frac{k+1}{s} \int_0^n e^{-st} t^k dt \right\} \right.
= \frac{k+1}{s} \lim_{n \to \infty} \int_0^n e^{-st} t^k dt
= \frac{k+1}{s} \cdot \frac{k!}{s^{k+1}} = \frac{(k+1)!}{s^{(k+1)+1}}.
\]

Thus the proposition is true for every positive integer \( k + 1 \) given that it is true for the positive integer \( k \). Consequently, by mathematical induction it follows that

\[ L[t^n] = \frac{n!}{s^{n+1}} \text{ for every positive integer } n. \]

**Solutions to §8.2**

**True-False Review:**

1. **TRUE.** If \( f \) is of exponential order, then by Definition 8.2.1, we have \( |f(t)| \leq M e^{\alpha t} \) \((t > 0)\) for some constants \( M \) and \( \alpha \). But then \( |g(t)| < f(t) \leq |f(t)| \leq M e^{\alpha t} \) for all \( t > 0 \), and so \( g \) is also of exponential order.

3. **FALSE.** Looking at the Comparison Test for Improper Integrals, we should conclude from the assumptions that if \( \int_0^\infty H(t) dt \) converges, then so does \( \int_0^\infty G(t) dt \). As a specific counterexample, we could take \( G(t) = 0 \) and \( H(t) = t \). Then although \( G(t) \leq H(t) \) for all \( t \geq 0 \) and \( \int_0^\infty G(t) dt = 0 \) converges, \( \int_0^\infty H(t) dt \) does not converge.

5. **FALSE.** From the formulas preceding Example 8.2.6, we see that the inverse Laplace transform of \( \frac{s}{s^2 + 9} \) is \( f(t) = \cos 3t \).

**Problems:**

1. \( |f(t)| = |\cos 2t| \leq 1e^t \) for all \( t > 0 \) so \( \cos 2t \) is of exponential order.

3. \( |f(t)| = |e^{3t} \sin 4t| \leq 1e^{3t} \) for all \( t > 0 \) so \( e^{3t} \sin 4t \) is of exponential order.

5. \( |f(t)| = |t^n e^{at}| \leq (e^t)^n e^{at} \) for all \( t > 0 \) so \( t^n e^{at} \) is of exponential order.

7. \( L^{-1} \left[ \frac{2}{s} \right] = 2L^{-1} \left[ \frac{1}{s} \right] = (2)(1) = 2. \)

9. \( L^{-1} \left[ \frac{5}{s + 3} \right] = 5L^{-1} \left[ \frac{1}{s + 3} \right] = 5e^{-3t}. \)

11. \( L^{-1} \left[ \frac{2s}{s^2 + 9} \right] = 2L^{-1} \left[ \frac{s}{s^2 + 9} \right] = 2 \cos 3t. \)
13. \[ L^{-1} \left[ \frac{s + 6}{s^2 + 1} \right] = 2L^{-1} \left[ \frac{s}{s^2 + 1} \right] + 6L^{-1} \left[ \frac{1}{s^2 + 1} \right] = \cos t + 6 \sin t. \]

15. \[ L^{-1} \left[ \frac{2 - 3}{s - 1} \right] = 2L^{-1} \left[ \frac{1}{s} \right] - 3L^{-1} \left[ \frac{1}{s + 1} \right] = 2 - 3e^{-t}. \]

17. \[ L^{-1} \left[ \frac{1}{s(s + 1)} \right] = L^{-1} \left[ \frac{1}{s} - \frac{1}{s + 1} \right] = L^{-1} \left[ \frac{1}{s} \right] - L^{-1} \left[ \frac{1}{s + 1} \right] = 1 - e^{-t}. \]

19. Using the linearity of the inverse Laplace transform,
\[
L^{-1} \left[ \frac{2s + 3}{(s - 2)(s^2 + 1)} \right] = L^{-1} \left[ \frac{1}{5} \right] \left[ \frac{7}{s - 2} - \frac{17s + 4}{5s^2 + 1} \right]
= \frac{1}{5} \left( 7L^{-1} \left[ \frac{1}{s - 2} \right] - 7L^{-1} \left[ \frac{3(s^2 + 1)}{s^2 + 1} \right] + 4L^{-1} \left[ \frac{1}{s^2 + 1} \right] \right)
= \frac{1}{5} (7e^{2t} - 7 \cos t - 4 \sin t).
\]

21. Using the linearity of the inverse Laplace transform,
\[
L^{-1} \left[ \frac{2s + 3}{(s - 2)(s^2 + 1)} \right] = L^{-1} \left[ -\frac{2s + 3}{3(s^2 + 4)} \right] + L^{-1} \left[ \frac{2s + 3}{3(s^2 + 1)} \right]
= -\frac{2}{3} \left[ L^{-1} \left[ \frac{s}{s^2 + 4} \right] - \frac{1}{2} L^{-1} \left[ \frac{2}{s^2 + 4} \right] + \frac{2}{3} L^{-1} \left[ \frac{s}{s^2 + 1} \right] + L^{-1} \left[ \frac{3}{s^2 + 1} \right] \right]
= -\frac{2}{3} \cos 2t - \frac{1}{2} \sin 2t + \frac{2}{3} \cos 2t + \sin t
= \frac{1}{6} (4 \cos t + 6 \sin 2t + \frac{2}{3} \cos 2t - 3 \sin 2t).
\]

**True-False Review:**

1. **FALSE.** The period of \( f \) is required to be the *smallest* positive real number \( T \) such that \( f(t + T) = f(t) \) for all \( t \geq 0 \). There can be only one *smallest* positive real number \( T \), so the period, if it exists, is uniquely determined.

3. **FALSE.** The function \( f(t) = \cos(2t) \) has period \( \pi \), not \( \pi/2 \).

5. **FALSE.** Even a continuous function need not be periodic. For instance, the function \( f(t) = \sin(t^2) \) in the previous item is continuous, hence piecewise continuous, but not periodic.

7. **FALSE.** If \( f(t) = \cos t \) and \( g(t) = \sin t \), then \( f \) and \( g \) are periodic with period \( m = n = 2\pi \). However, \( (fg)(t) = \cos t \sin t \) is periodic with period \( 2\pi \), not period \( (2\pi)^2 \).

**Problems:**

1. We have,
\[
L[f] = \frac{1}{1 - e^{-s}} \int_0^1 e^{-st}tdt = \frac{1}{1 - e^{-s}} \left[ \frac{e^{-st}}{s^2} (-st - 1) \right]_0^1
= \frac{1}{s^2(1 - e^{-s})} \left[ 1 - e^{-s} (s + 1) \right].
\]
3. We have,

\[ L[f] = \frac{1}{1 - e^{-\pi s}} \int_0^\pi e^{-st} \sin t \, dt = \frac{1}{1 - e^{-\pi s}} \left[ e^{-st}(-s \sin t - \cos t) \right]_0^\pi = \frac{1 + e^{-\pi s}}{(1 - e^{-\pi s})(s^2 + 1)}. \]

5. We have,

\[ L[f] = \frac{1}{1 - e^{-s}} \int_0^1 e^{-st} \, dt = \frac{1}{1 - e^{-s}} \int_0^1 e^{(1-s)t} \, dt = \frac{1}{(1 - e^{-s})(1 - s)} \left[ e^{(1-s)t} \right]_0^1 = e^{1-s} - 1. \]

7. We have,

\[ L[f] = \frac{1}{1 - e^{-\pi s}} \left[ \int_0^{\pi/2} e^{-st} \, dt + \int_{\pi/2}^\pi e^{-st} \sin t \, dt \right] = \frac{1}{1 - e^{-\pi s}} \left[ \frac{2}{\pi} \left[ e^{-s(-st - 1)} \right]_0^{\pi/2} + \frac{e^{-st}}{s^2 + 1} \left[ -s \sin t - \cos t \right]_0^{\pi/2} \right] = \frac{1}{1 - e^{-\pi s}} \left[ \frac{2 - e^{-\pi s/2}(\pi s + 2)}{\pi s^2} + \frac{s e^{-\pi s/2} - e^{-\pi s}}{s^2 + 1} \right]. \]

9. We have,

\[ L[f] = \frac{1}{1 - e^{-2as}} \left[ \int_0^a e^{-st} \, dt + \int_a^{2a} e^{-st} \left( \frac{2a - t}{a} \right) \, dt \right] = \frac{1}{1 - e^{-2as}} \left[ e^{-st}(-st - 1) \right]_0^a - \frac{2}{s} \left[ e^{-st} \right]_a^{2a} + \frac{1}{a} \left[ \frac{e^{-st}(-st - 1)}{s^2} \right]_a^{2a} = \frac{1 - 2e^{-as} + e^{-2as}}{as^2(1 - e^{-2as})} = \frac{e^{as/2} - e^{-as/2}}{as^2(e^{as/2} + e^{-as/2})} = \frac{1}{as^2} \tanh \left( \frac{as}{2} \right). \]

11. We have,

\[ L[f] = \frac{1}{1 - e^{-2\pi s/a}} \int_0^{2\pi/a} e^{-st} \cos at \, dt = \frac{1}{1 - e^{-2\pi s/a}} \left[ e^{-st}(-s \cos at + a \sin at) \right]_0^{2\pi/a} = \frac{s}{s^2 + a^2}. \]

**Solutions to Section 8.4**

True-False Review:

1. **FALSE.** The Laplace transform of \( f \) may not exist unless we assume additionally that \( f \) is of exponential order.

3. **TRUE.** If we leave the initial conditions as arbitrary constants, the solution technique presented in this section will result in the general solution to the differential equation.

Problems:
1. We apply the Laplace transform to both sides of the differential equation:

\[ L[y'] + L[y] = 8L[e^{3t}] \]

Using the rule for the transform of the derivative, we obtain

\[ sY(s) - y(0) + Y(s) = \frac{8}{s - 3} \]

Substituting \( y(0) = 2 \), this becomes

\[ Y(s)(s + 1) - 2 = \frac{8}{s - 3} \]

Solving for \( Y(s) \), we have

\[ Y(s) = \frac{8}{(s - 3)(s + 1)} + \frac{2}{s + 1} \]

which simplifies to

\[ Y(s) = \frac{2}{s - 3} \]

Taking the inverse Laplace transform of both sides of this equation yields,

\[ L^{-1}[Y(s)] = 2L^{-1}\left(\frac{1}{s - 3}\right) \]

so that

\[ y(t) = 2e^{3t} \]

3. We apply the Laplace transform to both sides of the differential equation:

\[ L[y'] + 2L[y] = 4L[t] \]

Using the rule for the transform of the derivative, we obtain

\[ sY(s) - y(0) + 2Y(s) = \frac{4}{s^2} \]

Substituting \( y(0) = 1 \), this becomes

\[ Y(s)(s + 2) - 1 = \frac{4}{s^2} \]

Solving for \( Y(s) \), we have

\[ Y(s) = -\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s + 2} \]

Taking the inverse Laplace transform of both sides of this equation yields,

\[ L^{-1}[Y(s)] = -L^{-1}\left(\frac{2}{s^2}\right) + L^{-1}\left(\frac{2}{s + 2}\right) \]

so that

\[ y(t) = 2t - 1 + 2e^{-2t} \]

5. We apply the Laplace transform to both sides of the differential equation:

\[ L[y'] - L[y] = 5L[\sin 2t] \]
Using the rule for the transform of the derivative, we obtain

\[ sY(s) - y(0) - Y(s) = 5 \frac{2}{s^2 + 4}. \]

Substituting \( y(0) = -1 \), and solving for \( Y(s) \), we have

\[ Y(s) = \frac{10}{(s^2 + 4)(s - 1)} - \frac{1}{s - 1}. \]

Decomposing the right-hand side of this equation into partial fractions yields

\[ Y(s) = \frac{1}{s - 1} - \frac{2s}{s^2 + 4} - \frac{2}{s^2 + 4}. \]

Taking the inverse Laplace transform of both sides of this equation gives,

\[ L^{-1}[Y(s)] = L^{-1} \left[ \frac{1}{s - 1} \right] - 2L^{-1} \left[ \frac{s}{s^2 + 4} \right] - L^{-1} \left[ \frac{2}{s^2 + 4} \right] \]

so that

\[ y(t) = e^t - \sin 2t - 2 \cos 2t. \]

7. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

\[ [s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] - 2Y(s) = 0. \]

Substituting \( y(0) = 1 \), \( y'(0) = 4 \) gives

\[ [s^2Y(s) - s - 4] + [sY(s) - 1] - 2Y(s) = 0, \]

so that

\[ Y(s) = \frac{s + 5}{(s - 1)(s + 2)}. \]

Decomposing the right-hand side of this equation into partial fractions yields

\[ Y(s) = \frac{2}{s - 1} - \frac{1}{s + 2}. \]

Taking the inverse Laplace transform of both sides of this equation gives

\[ y(t) = 2e^t - e^{-2t}. \]

9. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

\[ [s^2Y(s) - 1] - 3sY(s) + 2Y(s) = \frac{4}{s}, \]

where we have incorporated the initial conditions \( y(0) = 0 \), \( y'(0) = 1 \). Solving for \( Y(s) \) yields

\[ Y(s) = \frac{4}{s(s - 1)(s - 2)} + \frac{1}{(s - 1)(s - 2)}. \]
Decomposing the right-hand side of this equation into partial fractions gives

\[ Y(s) = \frac{2}{s} - \frac{5}{s-1} + \frac{3}{s-2}. \]

Taking the inverse Laplace transform of both sides of this equation gives

\[ L^{-1}[Y(s)] = 2L^{-1}\left[\frac{1}{s}\right] - 5L^{-1}\left[\frac{1}{s-1}\right] + 3L^{-1}\left[\frac{1}{s-2}\right], \]

so that

\[ y(t) = 2 - 5e^t + 3e^{2t}. \]

11. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

\[ [s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] - 2Y(s) = \frac{10}{s+1}. \]

Imposing the initial conditions \( y(0) = 0, \ y'(0) = 1 \) and solving for \( Y(s) \) yields

\[ Y(s) = \frac{4}{(s+1)(s-1)(s+2)}, \]

or equivalently,

\[ Y(s) = \frac{2}{s-1} - \frac{5}{s+1} + \frac{3}{s+2}. \]

Taking the inverse Laplace transform of both sides of this equation gives

\[ L^{-1}[Y(s)] = 2L^{-1}\left[\frac{1}{s-1}\right] - 5L^{-1}\left[\frac{1}{s+1}\right] + 3L^{-1}\left[\frac{1}{s+2}\right], \]

so that

\[ y(t) = 2e^t - 5e^{-t} + 3e^{-2t}. \]

13. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

\[ [s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] - 2Y(s) = \frac{30}{s+3}. \]

Imposing the initial conditions \( y(0) = 1, \ y'(0) = 0 \) and solving for \( Y(s) \) yields

\[ Y(s) = -\frac{4}{s} + \frac{3}{s-2} + \frac{2}{s+3}. \]

Taking the inverse Laplace transform of both sides of this equation gives

\[ y(t) = -4 + 3e^{2t} + 2e^{-3t}. \]

15. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

\[ [s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = \frac{10}{s+1}. \]
Imposing the initial conditions $y(0) = 4$, $y'(0) = 0$ and solving for $Y(s)$ yields

$$Y(s) = \frac{2 + 2s}{s^2 + 4} + \frac{2}{s + 1}.$$  

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 2 \cos 2t + \sin 2t + 2e^{-t}.$$  

17. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$\left[ s^2 Y(s) - sy(0) - y'(0) \right] - Y(s) = \frac{6s}{s^2 + 1}.$$  

Imposing the initial conditions $y(0) = 0$, $y'(0) = 4$ and solving for $Y(s)$ yields

$$Y(s) = \frac{7}{2(s - 1)} - \frac{1}{2(s + 1)} - \frac{3s}{s^2 + 1}.$$  

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = \frac{7}{2}e^t - \frac{1}{2}e^{-t} - 3 \cos t.$$  

19. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$\left[ s^2 Y(s) - sy(0) - y'(0) \right] - Y(s) = \frac{8}{s^2 + 1} - \frac{6s}{s^2 + 1}.$$  

Imposing the initial conditions $y(0) = 2$, $y'(0) = -1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{1}{s - 1} - \frac{2}{s + 1} + \frac{3s}{s^2 + 1} - \frac{4}{s^2 + 1}.$$  

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = e^t - 2e^{-t} + 3 \cos t - 4 \sin t.$$  

21. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$\left[ s^2 Y(s) - sy(0) - y'(0) \right] + 5\left[ sY(s) - y(0) \right] + 4Y(s) = \frac{40}{s^2 + 4}.$$  

Imposing the initial conditions $y(0) = -1$, $y'(0) = 2$ and solving for $Y(s)$ yields

$$Y(s) = \frac{2}{s + 1} - \frac{1}{s + 4} - \frac{2s}{s^2 + 4}.$$  

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 2e^{-t} - e^{-4t} - 2 \cos 2t.$$
23. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain
\[ [s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] + 2Y(s) = \frac{3s}{s^2 + 1} + \frac{1}{s^2 + 1}. \]
Imposing the initial conditions \( y(0) = 1, \ y'(0) = 1 \) and solving for \( Y(s) \) yields
\[ Y(s) = \frac{7}{5(s - 2)} - \frac{1}{s - 1} + \frac{3s - 4}{5(s^2 + 1)}. \]
Taking the inverse Laplace transform of both sides of this equation gives
\[ y(t) = \frac{1}{5}(7e^{2t} - 5e^t + 3 \cos t - 4 \sin t). \]

25. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain
\[ [s^2Y(s) - sy(0) - y'(0)] + Y(s) = \frac{6s}{s^2 + 4}. \]
Imposing the initial conditions \( y(0) = 0, \ y'(0) = 2 \) and solving for \( Y(s) \) yields
\[ Y(s) = \frac{2}{s^2 + 1} + \frac{2s}{s^2 + 1} - \frac{2s}{s^2 + 4}. \]
Taking the inverse Laplace transform of both sides of this equation gives
\[ y(t) = 2(\cos t + \sin t - \cos 2t). \]

27. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain
\[ [s^2Y(s) - sy(0) - y'(0)] - Y(s) = 0. \]
Solving for \( Y(s) \) yields
\[ Y(s) = \frac{sy(0) + y'(0)}{(s + 1)(s - 1)}, \]
which can be written in the equivalent form
\[ Y(s) = \frac{y(0) - y'(0)}{2(s - 1)} + \frac{y(0) - y'(0)}{2(s + 1)}. \]
Taking the inverse Laplace transform of both sides of this equation gives
\[ y(t) = \frac{y(0) + y'(0)}{2}e^t + \frac{y(0) - y'(0)}{2}e^{-t}. \]

29. (a) We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain
\[ sI(s) - i(0) + \frac{R}{L}I(s) = \frac{E_0}{L} - \frac{1}{s}. \]
Imposing the initial conditions \( i(0) = 0 \), and solving for \( I(s) \) yields

\[
I(s) = \frac{E_0}{Ls(s + R/L)},
\]

which can be written in the equivalent form

\[
I(s) = \frac{E_0}{Rs} - \frac{E_0}{R(s + R/L)}.
\]

Taking the inverse Laplace transform of both sides of this equation gives

\[
i(t) = \frac{E_0}{R} - \frac{E_0}{R} e^{-Rt/L} = \frac{E_0}{R} \left( 1 - e^{-Rt/L} \right).
\]

(b) We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

\[
sI(s) - i(0) + \frac{R}{L} I(s) = \frac{E_0}{L} \cdot \frac{\omega}{s^2 + \omega^2}.
\]

Imposing the initial conditions \( i(0) = 0 \), and solving for \( I(s) \) yields

\[
I(s) = \frac{E_0\omega}{(s^2 + \omega^2)(Ls + R)},
\]

which can be written in the equivalent form

\[
I(s) = \frac{E_0\omega}{R^2 + L^2\omega^2} \left( \frac{R}{s^2 + \omega^2} - \frac{Ls}{s^2 + \omega^2} + \frac{L}{s + R/L} \right).
\]

Taking the inverse Laplace transform of both sides of this equation gives

\[
i(t) = \frac{E_0\omega}{R^2 + L^2\omega^2} \left( \frac{R}{\omega} \sin \omega t - L \cos \omega t + L e^{-Rt/L} \right),
\]

that is,

\[
i(t) = \frac{E_0}{R^2 + L^2\omega^2} \left( R \sin \omega t - L \omega \cos \omega t + L \omega e^{-Rt/L} \right).
\]

(0.0.36)

We now define the acute angle \( \theta \) by

\[
\tan \theta = \frac{\omega L}{R},
\]

in which case

\[
\cos \theta = \frac{R}{\sqrt{R^2 + \omega^2L^2}}, \quad \sin \theta = \frac{\omega L}{\sqrt{R^2 + \omega^2L^2}}.
\]

Solving the preceding equations for \( R \) and \( \omega L \) yields

\[
R = \sqrt{R^2 + \omega^2L^2} \cos \theta, \quad \omega L = \sqrt{R^2 + \omega^2L^2} \sin \theta.
\]

We now insert these results into (0.0.36) to obtain

\[
i(t) = \frac{E_0}{R^2 + L^2\omega^2} \left( \sqrt{R^2 + \omega^2L^2} \cos \theta \sin \omega t - \sqrt{R^2 + \omega^2L^2} \sin \theta \cos \omega t + L \omega e^{-Rt/L} \right),
\]
or, equivalently,
\[
i(t) = \frac{E_0}{\sqrt{R^2 + L^2} \omega^2} \left( \sin \omega t \cos \theta - \cos \omega t \sin \theta + L \omega e^{-Rt/L} \right)
\]
\[
= \frac{E_0}{\sqrt{R^2 + L^2} \omega^2} \left[ \sin (\omega t - \theta) + L \omega e^{-Rt/L} \right].
\]

31. Taking the Laplace transform of the given system yields

\[sX_1 = -4X_1 - 2X_2, \quad sX_2 - 1 = X_1 - X_2,\]

that is,
\[(s + 4)X_1 + 2X_2 = 0, \quad -X_1 + (s + 1)X_2 = 1.\]

Solving this linear algebraic system using Cramer’s rule yields

\[X_1 = \frac{-2}{s^2 + 5s + 6} = \frac{-2}{(s + 2)(s + 3)} = 2 \left( \frac{1}{s + 3} - \frac{1}{s + 2} \right),\]
\[X_2 = \frac{s + 4}{s^2 + 5s + 6} = \frac{s + 4}{(s + 2)(s + 3)} = \frac{2}{s + 2} - \frac{1}{s + 3}.\]

Taking the inverse Laplace transform of these equation we obtain
\[x_1(t) = 2(e^{-3t} - e^{-2t}), \quad x_2(t) = 2e^{-2t} - e^{-3t}.\]

33. Let \(P(n)\) be the statement:

\[L[f^{(n)}] = s^n L[f] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).\]

Then we have established in Theorem 8.4.1 that

\[L[f'] = sL[f] - f(0),\]

which means that \(P(1)\) is true. Let \(k\) be an arbitrary positive integer, and assume that

\[P(k) : L[f^{(k)}] = s^k L[f] - s^{k-1} f(0) - s^{k-2} f'(0) - \cdots - s f^{(k-2)}(0) - f^{(k-1)}(0)\quad(0.0.37)\]

is true. Then, since \(f^{(k+1)} = (f^{(k)})'\), we have

\[L[f^{(k+1)}] = L \left[ (f^{(k)})' \right] = sL[f^{(k)}] - f^{(k)}(0),\]

where we have applied \(P(1)\) in the second step. Inserting the expression for \(L[f^{(k)}]\) given in (0.0.37) into the preceding equation yields

\[L[f^{(k+1)}] = s \left\{ s^k L[f] - s^{k-1} f(0) - s^{k-2} f'(0) - \cdots - s f^{(k-2)}(0) - f^{(k-1)}(0) \right\} - f^{(k)}(0)
\]
\[= s^{k+1} L[f] - s^k f(0) - s^{k-1} f'(0) - \cdots - s f^{(k-1)}(0) - f^{(k)}(0),\]

so that \(P(k + 1)\) is true. We have therefore established that
1. \( P(1) \) is true.

2. For each positive integer \( k \), if \( P(k) \) is true then \( P(k + 1) \) is true.

It follows from mathematical induction, that \( P(n) \) is true for all positive integers \( n \).

**Solutions to Section 8.5**

**True-False Review:**

1. **TRUE.** This follows at once from the first formula in the First Shifting Theorem, with \( a \) replaced by \( -a \).

2. **TRUE.** The formula for \( f(t) \) is obtained from the formula for \( f(t+2) \) by replacing \( t+2 \) with \( t \) (and hence \( t+3 \) with \( t+1 \) and \( t \) with \( t-2 \)).

3. **FALSE.** The correct formula is \( L[e^{-t} \sin 2t] = \frac{2}{(s+1)^2 + 4} \).

4. **TRUE.** The inverse Laplace transform of \( F(s) = \frac{s}{s^2 + 9} \) is \( f(t) = \cos 3t \), and so the First Shifting Theorem with \( a = -4 \) gives the indicated inverse Laplace transform.

**Problems:**

1. \( f(t-a) = f(t-1) = t-1 \).

2. \( f(t-a) = f(t+2) = (t+2)^2 - 2(t+2) = t(t+2) \).

3. \( f(t-a) = f(t-\pi) = e^{2(t-\pi)} \cos (t-\pi) = -e^{2(t-\pi)} \cos t \).

4. \( f(t-a) = f(t-\pi/6) = e^{-(t-\pi/6)} \sin [2(t-\pi/6)] = e^{-(t-\pi/6)} [\sin (2t) \cos (\pi/3) - \cos (2t) \sin (\pi/3)] = e^{-(t-\pi/6)} [\frac{1}{2} \sin 2t - \frac{\sqrt{3}}{2} \cos 2t] = \frac{e^{-(t-\pi/6)}}{2} (\sin 2t - \sqrt{3} \cos 2t) \).

5. \( f(t-a) = f(t-2) = \frac{(t-2) + 1}{(t-2)^2 - 2(t-2) + 2} = \frac{t-1}{t^2 - 6t + 10} \).

6. \( f(t-1) = (t-1)^2 \). Replace \( t-1 \) by \( t \); \( f(t) = t^2 \).

7. \( f(t-2) = (t-2) e^{2(t-2)} \). Replace \( t-2 \) by \( t \): \( f(t) = te^{3t} \).

8. \( f(t-3) = te^{-(t-3)} \). Replace \( t \) by \( t+3 \): \( f(t) = (t+3)e^{-t} \).

9. \( L[\cos 4t] = \frac{s}{s^2 + 16} \Rightarrow L[e^{3t} \cos 4t] = \frac{s - 3}{(s-3)^2 + 16} \).

10. \( L[t] = \frac{1}{s^2} \Rightarrow L[te^{2t}] = \frac{1}{(s+1)^2} \).

11. \( L[t^3] = \frac{3!}{s^4} \Rightarrow L[e^{-4t}t^3] = \frac{3!}{(s+4)^4} \).
23. 
\[ L[2e^{3t} \sin t + 4e^{3t} \cos 3t] = 2L[e^{3t} \sin t] + 4L[\cos 3t] \]
\[ = 2 \left[ \frac{1}{(s - 3)^2 + 1} \right] + 4 \left[ \frac{s + 1}{(s + 1)^2 + 9} \right] \]
\[ = \frac{2[2s^3 - 9s^2 + 10s + 30]}{[(s - 3)^2 + 1][(s + 1)^2 + 9]} . \]

25. \( L[t^2(e^t - 3)] = L[e^t t^2] - 3L[t^2] = \frac{2}{(s - 1)^3} - \frac{6}{s^3} . \)

27. \( L^{-1} \left[ \frac{1}{s^2} \right] = t \implies L^{-1} \left[ \frac{1}{(s - 3)^2} \right] = te^{3t} . \)

29. Since \( L^{-1} [\sqrt{\pi/s}] = t^{-1/2} \)

it follows that \( L^{-1} \left[ \frac{1}{s^{1/2}} \right] = \pi^{-1/2}t^{-1/2} . \)

Therefore,

\( L^{-1} \left[ \frac{2}{s^{1/2}} \right] = \frac{2}{\sqrt{\pi}}t^{-1/2} . \)

Consequently,

\( L^{-1} \left[ \frac{2}{(s + 3)^{1/2}} \right] = \frac{2}{\sqrt{\pi}}e^{-3t}t^{-1/2} . \)

31. \( L^{-1} \left[ \frac{s}{s^2 + 9} \right] = \cos 3t \implies L^{-1} \left[ \frac{s + 2}{(s + 2)^2 + 9} \right] = e^{-2t} \cos 3t . \)

33. \( L^{-1} \left[ \frac{5}{(s - 2)^2 + 16} \right] = \frac{5}{4} L^{-1} \left[ \frac{4}{(s - 2)^2 + 16} \right] = \frac{5}{4} e^{2t} \sin 4t . \)

35. 
\[ L^{-1} \left[ \frac{s - 2}{s^2 + 2s + 26} \right] = L^{-1} \left[ \frac{s - 2}{(s + 1)^2 + 25} \right] \]
\[ = L^{-1} \left[ \frac{s + 1}{(s + 1)^2 + 25} - \frac{3}{(s + 1)^2 + 25} \right] \]
\[ = L^{-1} \left[ \frac{s + 1}{(s + 1)^2 + 25} \right] + L^{-1} \left[ \frac{3}{(s + 1)^2 + 25} \right] \]
\[ = e^{-t} \cos 5t - \frac{3}{5} e^{-t} \sin 5t = \frac{1}{5} e^{-t}(5 \cos 5t - 3 \sin 5t) . \]
37. 
\[ L^{-1}\left[ \frac{s}{(s + 1)^2 + 4} \right] = L^{-1}\left[ \frac{s + 1}{(s + 1)^2 + 4} - \frac{1}{(s + 1)^2 + 4} \right] \]
\[ = L^{-1}\left[ \frac{s + 1}{(s + 1)^2 + 4} \right] - \frac{1}{2} L^{-1}\left[ \frac{2}{(s + 1)^2 + 4} \right] \]
\[ = e^{-t} \cos 2t - \frac{e^{-t}}{2} \sin 2t = \frac{1}{2} e^{-t}(2 \cos 2t - \sin 2t). \]

39. 
\[ L^{-1}\left[ \frac{4}{s(s + 2)^2} \right] = L^{-1}\left[ \frac{1}{s} - \frac{1}{s + 2} - \frac{2}{(s + 2)^2} \right] \]
\[ = L^{-1}\left[ \frac{1}{s} \right] - L^{-1}\left[ \frac{1}{s + 2} \right] - 2L^{-1}\left[ \frac{1}{(s + 2)^2} \right] \]
\[ = 1 - e^{-2t} - 2te^{-2t} = 1 - e^{-2t}(1 + 2t). \]

41. 
\[ L^{-1}\left[ \frac{2s + 3}{s(s^2 - 2s + 5)} \right] = L^{-1}\left[ \frac{3}{5s} - \frac{3s - 16}{5[(s - 1)^2 + 4]} \right] \]
\[ = L^{-1}\left[ \frac{3}{5s} - \frac{3(s - 1)}{5[(s - 1)^2 + 4]} + \frac{26}{10[(s - 1)^2 + 4]} \right] \]
\[ = \frac{3}{5} L^{-1}\left[ \frac{1}{s} \right] - \frac{3}{5} L^{-1}\left[ \frac{s - 1}{(s - 1)^2 + 4} \right] + \frac{13}{10} L^{-1}\left[ \frac{2}{(s - 1)^2 + 4} \right] \]
\[ = \frac{3}{5} - \frac{3}{5} e^{t} \cos 2t + \frac{13}{10} e^{t} \sin 2t = \frac{1}{10}[6 + e^{t}(13 \sin 2t - 6 \cos 2t)]. \]

43. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain
\[ s^2 Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{12}{s - 2}. \]
Imposing the initial conditions \( y(0) = 2, \ y'(0) = 3 \) and solving for \( Y(s) \) yields
\[ Y(s) = \frac{2s^2 - s + 6}{(s + 2)(s - 2)^2}, \]
which can be written in the equivalent form
\[ Y(s) = \frac{3}{(s - 2)^2} + \frac{1}{s - 2} + \frac{1}{s + 2}. \]
Taking the inverse Laplace transform of both sides of this equation gives
\[ y(t) = 3te^{2t} + e^{2t} + e^{-2t} = e^{2t}(1 + 3t) + e^{-2t}. \]

45. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain
\[ s^2 Y(s) - sy(0) - y'(0) + sY(s) - y(0) - 2Y(s) = \frac{3}{s + 1}. \]
Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{3s^2 + 8s + 7}{(s + 2)^2(s - 1)},$$

which can be written in the equivalent form

$$Y(s) = \frac{2}{s - 1} - \frac{1}{(s + 2)^2} + \frac{1}{s + 2}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 2e^t - te^{-2t} + e^{-2t} = 2e^t + e^{-2t}(1 - t).$$

47. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + Y(s) = \frac{2}{s + 1}.$$

Imposing the initial conditions $y(0) = 2$, $y'(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{1}{(s + 1)^2} \left( \frac{2}{s + 1} + 2s + 5 \right),$$

which can be written in the equivalent form

$$Y(s) = \frac{2}{(s + 1)^3} + \frac{3}{(s + 1)^2} + \frac{2}{s + 1},$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = t^2 e^{-t} + 3te^{-t} + 2e^{-t} = e^{-t}(2 + 3t + t^2).$$

49. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \frac{12}{(s - 2)^2}.$$

Imposing the initial conditions $y(0) = 0$, $y'(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{s^2 - 4s + 16}{(s - 2)^2(s + 1)(s + 2)},$$

which can be written in the equivalent form

$$Y(s) = \frac{1}{(s - 2)^2} - \frac{7}{12(s - 2)} + \frac{7}{3(s + 1)} - \frac{7}{4(s + 2)}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = te^{2t} - \frac{7}{12} e^{2t} + \frac{7}{3} e^{-t} - \frac{7}{4} e^{-2t} = \frac{7}{3} e^{-t} - \frac{7}{4} e^{-2t} + \frac{1}{12} e^{2t}(12t - 7).$$
51. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

\[ s^2Y(s) - sy(0) - y'(0) - Y(s) = \frac{16}{(s-1)^2 + 4}. \]

Imposing the initial conditions \( y(0) = 2, \ y'(0) = -2 \) and solving for \( Y(s) \) yields

\[ Y(s) = \frac{16}{(s+1)(s-1)(s^2 - 2s + 5)} + \frac{2}{s+1}, \]

which can be written in the equivalent form

\[ Y(s) = \frac{2}{s-1} + \frac{1}{s+1} - \frac{s-1}{(s-1)^2 + 4} - \frac{2}{(s-1)^2 + 4}. \]

Taking the inverse Laplace transform of both sides of this equation gives

\[ y(t) = 2e^t + e^{-t} - e^t(\cos 2t + \sin 2t). \]

53. Taking the Laplace transform of each differential equation and using the given initial conditions yields:

\[ sX_1(s) - 1 = 2X_1(s) - X_2(s), \quad sX_2(s) = X_1(s) + 2X_2(s), \]

or equivalently,

\[ (2-s)X_1(s) - X_2(s) = -1, \quad X_1(s) + (2-s)X_2(s) = 0. \]

Solving this algebraic system of equations we obtain:

\[ X_1(s) = \frac{s-2}{s^2 - 4s + 5} = \frac{s-2}{(s-2)^2 + 1}, \quad X_2(s) = \frac{1}{(s-2)^2 + 1}. \]

Consequently,

\[ x_1(t) = e^{2t}\cos t, \quad x_2(t) = e^{2t}\sin t. \]

Solutions to Section 8.6

True-False Review:

1. FALSE. The given function is not well-defined at \( t = a \). The unit step function is 0 for \( 0 \leq t < a \), not \( 0 \leq t \leq a \).

3. FALSE. For values of \( t \) with \( a < t < b \), we have \( u_a(t) = 1 \) and \( u_b(t) = 0 \), so the given inequality does not hold for such \( t \).

Problems:

1. \( f(t) = 2u_1(t) - 4u_3(t) = \begin{cases} 0, & \text{if } 0 \leq t < 1, \\ 2, & \text{if } 1 \leq t < 3, \\ -2, & \text{if } 3 \leq t. \end{cases} \)

3. \( f(t) = t(1-u_1(t)) = \begin{cases} t, & \text{if } 0 \leq t < 1, \\ 0, & \text{if } 1 \leq t. \end{cases} \)
5. Let \([t]\) represent the greatest integer equal to or less than the real number \(t\). Extending problem number four:

\[
f(t) = \sum_{i=1}^{\infty} u_i(t) = [t], \ i - 1 \leq t < i \text{ where } i \in \{1, 2, 3, \ldots\}.
\]

7. \(f(t) = [3 - 3u_1(t)] + (-1)u_1(t) \implies f(t) = 3 - 4u_1(t)\).

9. \(f(t) = [2 - 2u_2(t)] + [u_2(t) - u_4(t)] - u_4(t) = 2 - u_2(t) - 2u_4(t)\).

11. \(f(t) = [t - tu_3(t)] + [(6 - t)u_3(t) - (6 - t)u_6(t)] = t + 2(3-t)u_3(t) - (6-t)u_6(t)\).

13. \[
f(t) = [1 - u_{\pi/2}(t)] + \sin t \cdot u_{\pi/2}(t) - \sin t \cdot u_{3\pi/2}(t) + (-1)u_{3\pi/2}(t)
= 1 + (\sin t - 1)u_{\pi/2}(t) - (\sin t + 1)u_{3\pi/2}(t).
\]
Solutions to Section 8.7

True-False Review:

1. **FALSE.** According to Equation (8.7.2), the inverse Laplace transform of $e^{at}F(s)$ is $u_a(t)f(t+a)$. Note that this requires $a < 0$.

2. **TRUE.** This is an immediate consequence of the Second Shifting Theorem.
Figure 0.0.78: Figure for Exercise 9

Figure 0.0.79: Figure for Exercise 11

5. FALSE. We have
\[ L[u_3(t)e^t] = L[u_3(t)e^{(t+3)-3}] = e^{-3s}L[e^{t+3}] = e^{-3s}e^{3} \frac{1}{s-1} = \frac{e^{3}}{e^{3s}(s-1)}. \]

7. FALSE. The correct formula is
\[ L^{-1}\left[ \frac{1}{se^{2s}} \right] = u_2(t). \]

Problems:

1. Letting \( g(t) = t \), we have \( L[f(t)] = L[(t-1)u_1(t)] = L[u_1(t)g(t-1)] = e^{-s}L[g(t)] = e^{-s}L[t] = \frac{e^{-s}}{s^2}. \)
3. Letting $g(t) = \sin t$, we have $L[f(t)] = L[u_{\pi/4}(t)g(t - \pi/4)] = e^{-\pi s/4}L[g(t)] = e^{-\pi s/4}L[\sin t] = \frac{e^{-\pi s/4}}{s^2 + 1}$.

5. Letting $g(t) = t^2$, we have $L[f(t)] = L[u_2(t)g(t - 2)] = e^{-2s}L[g(t)] = e^{-2s}L[t^2] = \frac{2e^{-2s}}{s^3}$.

7. Letting $g(t) = (t + 1)^2$, we have

$$L[f(t)] = L[u_2(t)g(t - 2)]$$
$$= e^{-2s}L[g(t)]$$
$$= e^{-2s}L[(t + 1)^2]$$
$$= e^{-2s}L[t^2 + 2t + 1]$$
$$= e^{-2s} \left( \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right)$$
$$= \frac{2 + 2s + s^2}{s^3}e^{-2s}.$$

9. Letting $g(t) = e^{-2t} \sin 3t$, we have

$$L[f(t)] = L[u_1(t)e^{-2(t-1)} \sin [3(t-1)]]$$
$$= L[u_1(t)g(t - 1)]$$
$$= e^{-s}L[g(t)]$$
$$= e^{-s}L[e^{-2t} \sin 3t]$$
$$= e^{-s} \frac{3}{(s + 2)^2 + 9}.$$

11. $L^{-1}[F(s)] = L^{-1} \left[ \frac{e^{-2s}}{s^2} \right] = L^{-1}[e^{-2s}L[t]] = u_2(t)(t - 2)$.

13. $L^{-1}[F(s)] = L^{-1} \left[ \frac{e^{-3s}}{s + 4} \right] = L^{-1}[e^{-3s}L[e^{-4t}]] = u_3(t)e^{-4(t-3)}.$
15. \( L^{-1}[F(s)] = L^{-1} \left[ \frac{e^{-3s}}{s^2 + 1} \right] = L^{-1}[e^{-3s}L[\sin t]] = u_3(t) \sin (t - 3) \).

17.

\[
L^{-1}[F(s)] = L^{-1} \left[ \frac{e^{-s}}{(s + 1)(s - 4)} \right] \\
= L^{-1} \left[ \frac{1}{5} e^{-s} \left( \frac{1}{s - 4} - \frac{1}{s + 1} \right) \right] \\
= \frac{1}{5} L^{-1}[e^{-s}L[e^{4t}] - e^{-s}L[e^{-t}]] \\
= \frac{1}{5} u_1(t)[e^{4(t-1)} - e^{-(t-1)}].
\]

19.

\[
L^{-1}[F(s)] = L^{-1} \left[ \frac{e^{-s}(s + 6)}{s^2 + 9} \right] \\
= L^{-1} \left[ \frac{se^{-s}}{s^2 + 9} + \frac{6e^{-s}}{s^2 + 9} \right] \\
= L^{-1}[e^{-s}L[\cos 3t] + 2e^{-s}L[\sin 3t]] \\
= u_1(t)[\cos 3(t - 1) + 2 \sin 3(t - 1)].
\]

21. \( L^{-1}[F(s)] = L^{-1} \left[ \frac{e^{-2s}}{(s - 3)^3} \right] = \frac{1}{2} L^{-1}[e^{-2s}L[e^{3t}2]] = \frac{1}{2} u_2(t)e^{3(t-2)}(t - 2)^2. \)

23.

\[
L^{-1}[F(s)] = L^{-1} \left[ \frac{e^{-s}(2s - 1)}{s^2 + 4s + 5} \right] \\
= L^{-1} \left[ \frac{e^{-s}(2s - 1)}{(s + 2)^2 + 1} \right] \\
= L^{-1} \left[ \frac{2e^{-s}(s + 2)}{(s + 2)^2 + 1} - \frac{5e^{-s}}{(s + 2)^2 + 1} \right] \\
= 2L^{-1} \left[ \frac{e^{-s}(s + 2)}{(s + 2)^2 + 1} \right] - 5L^{-1} \left[ \frac{e^{-s}}{(s + 2)^2 + 1} \right] \\
= 2L^{-1}[e^{-s}L[e^{-2t} \cos t]] - 5L^{-1}[e^{-s}L[e^{-2t} \sin t]] \\
= u_1(t) \left( 2e^{-2(t-1)} \cos (t - 1) - 5e^{-2(t-1)} \sin (t - 1) \right) \\
= u_1(t)e^{-2(t-1)} \left( 2 \cos (t - 1) - 5 \sin (t - 1) \right).
\]
25. Applying the Laplace transform to both sides of the differential equation \( y' - 2y = u_2(t)e^{-t^2} \), we have \( sY - y(0) - 2Y = e^{-2s} \). Solving for \( Y(s) \) and substituting the initial values, we have

\[
Y(s) = \frac{e^{-2s}}{s-1} + \frac{2}{s-2} \implies Y(s) = e^{-2s} \left( \frac{1}{s-2} - \frac{1}{s-1} \right) + \frac{2}{s-2}.
\]

Taking the inverse Laplace transform, we conclude that

\[
y(t) = L^{-1}[Y(s)] = 2e^{2t} - u_2(t)[e^{-t^2} - e^{-2(t-2)}].
\]

27. Applying the Laplace transform to both sides of the differential equation \( y' + 2y = u_\pi(t) \sin 2t \), we have \( sY(s) - y(0) + 2Y(s) = \frac{e^{-2s}}{s-1} \). Solving for \( Y(s) \) and substituting the initial values, we have

\[
sY(s) - y(0) + 2Y(s) = L[u_\pi(t) \sin 2(t-\pi)] = L[u_\pi(t) \sin 2t] = \frac{e^{-2s}}{s-1}.
\]

Solving for \( Y(s) \) and substituting the initial values, we have

\[
Y(s)(s+2) = \frac{3}{s^2 + 4} + \frac{2e^{-\pi s}}{4(s^2 + 4)(s+2)} \implies Y(s) = \frac{3}{s+2} + e^{-\pi s} \left( \frac{1}{4(s+2)} - \frac{s-2}{4(s^2 + 4)} \right).
\]

Therefore, taking the inverse Laplace transform, we conclude that

\[
y(t) = 3e^{-2t} + \left[ \frac{1}{4} e^{2(t-\pi)} - \frac{1}{4} \cos(2(t-\pi)) + \frac{1}{4} \sin(2(t-\pi)) \right] u_\pi(t) = 3e^{-2t} + \frac{1}{4} u_\pi(t) \left[ e^{2(t-\pi)} - \cos 2t + \sin 2t \right].
\]

31. We have \( f(t) = \sin t + (1 - \sin t)u_{\pi/2}(t) = \sin t + u_{\pi/2}(t) - u_{\pi/2}(t) \cos (t - \pi/2). \) Thus, taking the Laplace transform of each of the differential equation \( y' - 3y = \sin t + u_{\pi/2}(t) - u_{\pi/2}(t) \cos (t - \pi/2) \), we obtain

\[
sY(s) - y(0) - 3Y(s) = \frac{1}{s^2 + 1} + \frac{e^{-\pi s/2}}{s} - \frac{6e^{-\pi s/2}}{s^2 + 1}.
\]

Solving for \( Y(s) \) and substituting the initial values, we have

\[
Y(s) = \frac{1}{(s^2 + 1)(s-3)} + \frac{e^{-\pi s/2}}{s(s-3)} - \frac{6e^{-\pi s/2}}{30(s-3)} - \frac{e^{-\pi s/2}}{3s} + \frac{e^{-\pi s/2}(3s-1)}{10(s^2 + 1)}.
\]
Taking the inverse Laplace transform, we conclude that
\[
y(t) = \frac{21}{10} e^{3t} - \frac{1}{10} \cos t - \frac{3}{10} \sin t + \left[ \frac{1}{30} e^{3(t-\pi/2)} - \frac{1}{3} + \frac{3}{10} \cos (t-\pi/2) - \frac{1}{10} \sin (t-\pi/2) \right] u_{\pi/2}(t)
\]
\[
= \frac{1}{10} \left\{ 21 e^{3t} - \cos t - 3 \sin t + \frac{1}{3} \left[ e^{3(t-\pi/2)} - 10 + 9 \sin t + 3 \cos t \right] u_{\pi/2}(t) \right\}.
\]

33. Taking the Laplace transform of both sides of the differential equation \( y'' - y = u_1(t) \), we obtain \( s^2 Y(s) - s y(0) - y'(0) - Y(s) = \frac{e^{-s}}{s} \). Solving for \( Y(s) \) and substituting the initial values, we have
\[
Y(s) = e^{-s} \frac{1}{s(s^2 - 1)} + \frac{2s}{s^2 - 1} = e^{-s} \left[ \frac{2}{2(s - 1)} + \frac{1}{2(s + 1)} - \frac{1}{s} \right] + \frac{1}{s + 1} + \frac{1}{s - 1}.
\]
Taking the inverse Laplace transform, we obtain
\[
y(t) = \left( \frac{e^{-t} - e^{-t+1}}{2} - 1 \right) u_1(t) + e^{-t} + e^t = u_1(t) [\cosh (t - 1) - 1] + 2 \cosh t.
\]

35. Taking the Laplace transform of both sides of the differential equation \( y'' - 4y = u_1(t) - u_2(t) \), we obtain \( s^2 Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{e^{-s} - e^{-2s}}{s} \). Solving for \( Y(s) \) and substituting the initial values, we have
\[
Y(s) = \frac{4}{s^2 - 4} + \frac{e^{-s}}{s(s^2 - 4)} - \frac{e^{-2s}}{s(s^2 - 4)}
\]
\[
= \frac{1}{s - 2} - \frac{1}{s + 2} + \frac{e^{-s}}{4} \left[ -\frac{1}{s} + \frac{1}{2(s - 2)} + \frac{1}{2(s + 2)} \right] - \frac{e^{-2s}}{4} \left[ -\frac{1}{s} + \frac{1}{2(s - 2)} + \frac{1}{2(s + 2)} \right].
\]
Taking the inverse Laplace transform, we obtain
\[
y(t) = e^{2t} - e^{-2t} - \frac{1}{4} u_1(t) + \frac{1}{8} e^{2(t-1)} u_1(t) + \frac{1}{8} e^{-2(t-1)} u_1(t) + \frac{1}{4} u_2(t) - \frac{1}{8} e^{2(t-2)} u_2(t) - \frac{1}{8} e^{-2(t-2)} u_2(t)
\]
\[
= 2 \sinh 2t + \frac{1}{4} u_1(t) [\cosh 2(t - 1) - 1] - \frac{1}{4} u_2(t) [\cosh 2(t - 2) - 1].
\]

37. Taking the Laplace transform of both sides of the differential equation \( y'' + 3y' + 2y = 10 u_{\pi/4}(t) \sin (t - \pi/4) \), we obtain \( s^2 Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \frac{10 e^{-\pi s/4}}{s^2 + 1} \). Solving for \( Y(s) \) and substituting the initial values, we have
\[
Y(s) = \frac{s + 3}{s^2 + 3s + 2} + \frac{10 e^{-\pi s/4}}{(s^2 + 1)(s^2 + 3s + 2)} = \frac{2}{s + 1} - \frac{1}{s + 2} + \frac{10 e^{-\pi s/4}}{(s^2 + 1)(s^2 + 3s + 2)}.
\]
Taking the inverse Laplace transform, we obtain
\[
y(t) = 2 e^{-t} - e^{-2t} + 5 e^{-(t-\pi/4)} u_{\pi/4}(t) - 2 e^{-2(t-\pi/4)} u_{\pi/4}(t) + \sin (t - \pi/4) u_{\pi/4}(t) - 3 \cos (t - \pi/4) u_{\pi/4}(t)
\]
\[
= 2 e^{-t} - e^{-2t} + u_{\pi/4}(t) \left[ 5 e^{-(t-\pi/4)} - 2 e^{-2(t-\pi/4)} - 3 \cos (t - \pi/4) + \sin (t - \pi/4) \right].
\]
39. Taking the Laplace transform of both sides of the differential equation \( y'' + 4y' + 5y = 5u_3(t) \), we obtain
\[ s^2Y(s) - sy(0) - y'(0) + 4sY(s) - y(0)] + 5Y(s) = \frac{5e^{-3s}}{s} \]. Solving for \( Y(s) \) and substituting the initial values, we have
\[ Y(s) = \frac{9 + 2s + 5e^{-3s}}{s^2 + 4s + 5} = \frac{2s + 4}{(s + 2)^2 + 1} + \frac{5}{(s + 2)^2 + 1} + e^{-3s} \left[ \frac{1}{s} - \frac{s + 2}{(s + 2)^2 + 1} - \frac{2}{(s + 2)^2 + 1} \right] \].

Therefore, taking the inverse Laplace transform, we obtain
\[ y(t) = 2e^{-2t} \cos t + 5e^{-2t} \sin t + L^{-1}[e^{-3s}L[1] - L^{-1}[e^{-3s}L[e^{-2t} \cos t]] - 2L^{-1}[e^{-3s}L[e^{-2t} \sin t]]
= e^{-2t}(2 \cos t + 5 \sin t) + u_3(t) \left\{ 1 - e^{-2(t-3)} \left[ \cos (t-3) - 2 \sin (t-3) \right] \right\} . \]

41. We have
\[ f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ t - 1, & 1 \leq t < 2, \\ 3 - t, & 2 \leq t < 3, \\ 0, & 3 \leq t. \end{cases} \]

Thus,
\[ f(t) = [(t-1)u_1(t) - (t-1)u_2(t)] + [(3-t)u_2(t) - (3-t)u_3(t)] 
= u_1(t)(t-1) - 2u_2(t)(t-2) + u_3(t)(t-3). \]

Taking the Laplace transform of both sides of the differential equation
\[ y' + y = u_1(t)(t-1) - 2u_2(t)(t-2) + u_3(t)(t-3), \]
we obtain \( sY(s) - y(0) + Y(s) = \frac{e^{-s}}{s} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2} \). Solving for \( Y(s) \) and substituting the initial values, we obtain
\[ Y(s) = \frac{2}{s + 1} + \left( e^{-s} - 2e^{-2s} + e^{-3s} \right) \frac{1}{s^2(s + 1)} = \frac{2}{s + 1} + \left( e^{-s} - 2e^{-2s} + e^{-3s} \right) \left( \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s + 1} \right) . \]

Taking the inverse Laplace transform, we have
\[ y(t) = 2e^{-t} + [(t-1) - 1 + e^{-(t-1)}]u_1(t) - 2[(t-2) - 1 + e^{-(t-2)}]u_2(t) + [(t-3) - 1 + e^{-(t-3)}]u_3(t)
= 2e^{-t} + u_1(t)[e^{-(t-1)} + t - 2] - 2u_2(t)[e^{-(t-2)} + t - 3] + u_3(t)[e^{-(t-3)} + t - 4] . \]

43. We have
\[ f(t) = (t - tu_1(t)) + u_1(t)e^{-(t-1)} 
= t + (e^{-(t-1)} - t)u_1(t) 
= t + e^{-(t-1)}u_1(t) - (t-1)u_1(t) - u_1(t) . \]

Taking the Laplace transform of both sides of the differential equation
\[ y' - y = t + e^{-(t-1)}u_1(t) - (t-1)u_1(t) - u_1(t) \]
yields
\[ sY(s) - 2 - Y(s) = \frac{1}{s^2} + \frac{e^{-s}}{s + 1} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} . \]
Solving for $Y(s)$ and substituting the initial values, we obtain

$$Y(s) = \frac{2}{s-1} + \frac{1}{s^2(s-1)} + e^{-s}\left[\frac{1}{(s+1)(s-1)} - \frac{1}{s^2(s-1)} - \frac{1}{s-1}\right]$$

$$= \frac{3}{s-1} - \frac{1}{s^2} - \frac{1}{s} + e^{-s}\left[\frac{3}{2(s-1)} - \frac{1}{2(s+1)} + \frac{1}{s^2} + \frac{1}{s}\right].$$

Taking the inverse Laplace transform, we obtain

$$y(t) = 3e^t - t - 1 - \frac{1}{2}\mu_1(t)\left[3e^{t-1} + e^{-(t-1)} - 2t\right].$$

45. We have $f(t) = \begin{cases} 2, & \text{if } 0 \leq t < 1, \\ -1, & \text{if } 1 \leq t. \end{cases}$ Thus, $f(t) = 2 - 3\mu_1(t)$.

(a) Taking the Laplace transform of the differential equation $y' - y = 2 - 3\mu_1(t)$, we get

$$sY(s) - y(0) - Y(s) = \frac{2}{s} - \frac{3e^{-s}}{s}.$$

Solving for $Y(s)$ and substituting the initial values, we have

$$Y(s) = \frac{1}{s-1} + \frac{2}{s(s-1)} - \frac{3e^{-s}}{s(s-1)}$$

$$= \frac{3}{s-1} - \frac{2}{s} + 3e^{-s}\left[\frac{1}{s} - \frac{1}{s-1}\right].$$

Taking the inverse Laplace transform, we obtain

$$y(t) = 3e^t - 2 + 3(1 - e^{t-1})\mu_1(t).$$

(b) In order to use the techniques of Chapter 1 we must solve the differential equation on the interval $[0, 1)$ and $[1, \infty)$ separately.

On $[0, 1)$ we must solve $y' - y = 2$, where $y(0) = 1$. An integrating factor is $I(t) = e^{-t}$ so

$$\frac{d}{dt}(e^{-t}y) = 2e^{-t} \implies y(t) = -2 + c_1e^t.$$

Substituting $y(0) = 1$, we have $c_1 = 3$. Thus, $y(t) = 3e^t - 2$ on $[0, 1)$.

Now consider $y' - y = -1$ on $[1, \infty)$. Again, an integrating factor is $I(t) = e^{-t}$. Thus,

$$\frac{d}{dt}(e^{-t}y) = -e^{-t} \implies y(t) = 1 + ce^t$$

on $[1, \infty)$. If the solutions are continuous at $t = 1$, then \(\lim_{{t \to 1}} y(t) = y(1)\). From $y(t) = 3e^t - 2$ on $[0, 1)$, we have that $y(1) = 3e - 2$ whereas from $y(t) = 1 + ce^t$ on $[1, \infty)$, we have $\lim_{{t \to 1}} y(t) = 1 + ce = 3e - 2 \implies c = 3 - 3e^{-1}$ so from $y(t) = 1 + ce^t$, we obtain $y(t) = 1 + 3e^t - 3e^{t-1}$ on $[1, \infty)$. Thus

$$y(t) = \begin{cases} 3e^t - 2, & \text{if } 0 \leq t < 1, \\ 1 + 3e^t - 3e^{t-1}, & \text{if } 1 \leq t. \end{cases}$$
47. We have \( E(t) = \begin{cases} 20, & 0 \leq t < 10, \\ 20e^{-(t-10)}, & 10 \leq t. \end{cases} \) Thus,

\[ E(t) = 20(1 - u_{10}(t)) + 20u_{10}(t)e^{-(t-10)}. \]

Substituting this into the given differential equation, we have

\[ \frac{dq}{dt} + \frac{1}{RC}q = \frac{20}{R} \left[ 1 - u_{10}(t) + u_{10}(t)e^{-(t-10)} \right]. \]

Taking the Laplace transform of both sides and substituting the initial values, we have

\[ sQ(s) + \frac{1}{RC}Q(s) = \frac{20}{R} \left[ \frac{1}{s} + e^{-10s} \left( \frac{1}{s+1} - \frac{1}{s} \right) \right]. \]

Solving for \( Q(s) \), we have

\[ Q(s) = \frac{20}{R} \left[ \frac{1}{s(s+a)} + e^{-10s} \left( \frac{1}{(s+1)(s+a)} - \frac{1}{s(s+a)} \right) \right] \]

\[ = \frac{20}{R} \left[ \frac{1}{as} - \frac{1}{a(s+a)} + e^{-10s} \left( \frac{1}{(a-1)(s+1)} + \frac{1}{(1-a)(s+a)} - \frac{1}{as} + \frac{1}{a(s+a)} \right) \right], \]

where \( a = \frac{1}{RC} \). Thus, taking the inverse Laplace transform, we have

\[ q(t) = \frac{20}{R} \left[ \frac{1}{a} - \frac{1}{a}e^{-at} + u_{10}(t) \left( \frac{1}{a-1}(e^{-(t-10)} - e^{-(1-a)(t-10)}) - \frac{1}{a}(1-e^{-at}) \right) \right]. \]

Now, since \( i(t) = \frac{dq(t)}{dt} \) it follows that

\[ i(t) = \frac{20}{R} \left[ e^{-at} + u_{10}(t) \left( \frac{1}{a} - \frac{1}{a}e^{a(t-10)} - e^{-(t-10)} - e^{-at} \right) \right]. \]

Solutions to Section 8.8

1. TRUE. We have

\[ I = \int_{-\infty}^{\infty} F(t)dt, \]

where \( F(t) \) is the magnitude of the applied force at time \( t \).

3. FALSE. The correct formula is \( L[\delta(t-a)] = e^{-as} \).

5. FALSE. The initial conditions are unrelated to the instantaneous blow. They are the pre-blow position and velocity of the mass, and these are not related in any way to the instantaneous blow.

Problems:

1. Taking the Laplace transform of both sides of the differential equation \( y' - 2y = \delta(t - 2) \), we obtain

\[ L[y'] - 2L[y] = L[\delta(t - 2)] \implies sY(s) - y(0) - 2Y(s) = e^{-2s}. \]
Substituting the initial value, we have \( sY(s) - 1 - 2Y(s) = e^{-2s} \). Solving for \( Y(s) \), we have

\[
Y(s)(s - 2) = 1 + e^{-2s} \implies Y(s) = \frac{1}{s - 2} + \frac{e^{-2s}}{s - 2}.
\]
Taking the inverse Laplace transform, we conclude that \( y(t) = e^{2t} + u_2(t)e^{2(t-2)} \).

**3.** Taking the Laplace transform of both sides of the differential equation \( y' - 5y = 2e^{-t} + \delta(t - 3) \) and inserting the initial value, we obtain \( sY(s) - 5Y(s) = \frac{2}{s + 1} + e^{-3s} \). Solving for \( Y(s) \), we have

\[
Y(s) = \frac{2}{(s + 1)(s - 5)} + e^{-3s} \frac{1}{s - 5}
\]

\[
= \frac{1}{3(s - 5)} - \frac{1}{3(s + 1)} + e^{-3s} \frac{1}{s - 5}.
\]
Taking the inverse Laplace transform, we conclude that \( y(t) = \frac{1}{3}(e^{5t} - e^{-t}) + u_3(t)e^{5(t-3)} \).

**5.** Taking the Laplace transform of both sides of the differential equation \( y'' - 4y = \delta(t - 3) \) and inserting the initial values, we obtain \( s^2Y(s) - 0s - 1 - 4Y(s) = e^{-3s} \). Solving for \( Y(s) \), we have

\[
Y(s) = \frac{1}{s^2 - 4} + \frac{e^{-3s}}{s^2 - 4}
\]

\[
= \frac{1}{4(s - 2)} - \frac{1}{4(s + 2)} + e^{-3s} \left[ \frac{1}{4(s - 2)} - \frac{1}{4(s + 2)} \right].
\]
Taking the inverse Laplace transform, we conclude that

\[
y(t) = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}u_3(t)[e^{2(t-3)} - e^{-2(t-3)}]
\]

\[
= \frac{1}{2} \left[ e^{2t} - e^{-2t} \right] + u_3(t) \left( \frac{e^{2(t-3)} - e^{-2(t-3)}}{2} \right)
\]

\[
= \frac{1}{2} \left[ \sinh 2t + u_3(t) \sinh [2(t - 3)] \right].
\]

**7.** Taking the Laplace transform of both sides of the differential equation \( y'' - 4y' + 13y = \delta(t - \pi/4) \) and inserting the initial values, we have \( s^2Y(s) - 3s - 0 - 4(sY(s) - 3) + 13Y(s) = e^{-\pi s/4} \). Solving for \( Y(s) \), we obtain

\[
Y(s) = \frac{3s - 12}{(s - 2)^2 + 9} + \frac{e^{-\pi s/4}}{(s - 2)^2 + 9}
\]

\[
= 3 \left[ \frac{s - 2}{(s - 2)^2 + 9} - \frac{3}{(s - 2)^2 + 9} \right] + \frac{1}{3} e^{-\pi s/4} \left[ \frac{3}{(s - 2)^2 + 9} \right].
\]
Taking the inverse Laplace transform, we conclude that

\[
y(t) = 3e^{2t} \cos 3t - 2e^{2t} \sin 3t + \frac{1}{3}u_{\pi/4}(t)e^{2(t-\pi/4)} \sin [3(t - \pi/4)].
\]

**9.** Taking the Laplace transform of both sides of the differential equation \( y'' + 6y' + 13y = \delta(t - \pi/4) \) and inserting the initial values, we have \( s^2Y(s) - 5s - 5 + 6(sY(s) - 5) + 13Y(s) = e^{-\pi s/4} \). Solving for \( Y(s) \), we
have

\[ Y(s) = \frac{5s + 35}{s^2 + 6s + 13} + \frac{e^{-\pi s/4}}{s^2 + 6s + 13} \]
\[ = \frac{5(s + 3)}{(s + 3)^2 + 4} + \frac{20}{(s + 3)^2 + 4} + \frac{e^{-\pi s/4}}{(s + 3)^2 + 4}. \]

Taking the inverse Laplace transform, we conclude that

\[ y(t) = 5e^{-3t} \cos 2t + 10e^{-3t} \sin 2t + \frac{1}{2}e^{-3(t-\pi/4)}u_{\pi/4}(t) \sin [2(t - \pi/4)] \]
\[ = 5e^{-3t}(\cos 2t + 2 \sin 2t) - \frac{1}{2}e^{-3(t-\pi/4)}u_{\pi/4}(t) \cos 2t. \]

11. Taking the Laplace transform of both sides of the differential equation \( y'' + 16y = 4 \cos 3t + \delta(t - \pi/3) \)
and inserting the initial values, we have \( s^2Y(s) - 0s - 0 + 16Y(s) = \frac{4s}{s^2 + 9} + e^{-\pi s/3} \). Solving for \( Y(s) \), we find that

\[ Y(s) = \frac{4s}{(s^2 + 9)(s^2 + 16)} + \frac{e^{-\pi s/3}}{s^2 + 16} \]
\[ = \frac{4s}{7(s^2 + 9)} - \frac{4s}{7(s^2 + 16)} + \frac{e^{-\pi s/3}}{s^2 + 16}. \]

Taking the inverse Laplace transform, we conclude that

\[ y(t) = \frac{4}{7}(\cos 3t - \cos 4t) + \frac{1}{4} \sin [4(t - \pi/3)]u_{\pi/3}(t). \]

13. Taking the Laplace transform of both sides of the differential equation \( \frac{d^2y}{dt^2} + 4y = F_0 \cos 3t - 4\delta(t - 5) \)
and inserting the initial values, we have \( s^2Y(s) - 0s - 0 + 4Y(s) = \frac{F_0s}{s^2 + 9} - 4e^{-5s} \). Solving for \( Y(s) \), we obtain

\[ Y(s) = \frac{F_0s}{(s^2 + 9)(s^2 + 4)} - \frac{4e^{-5s}}{s^2 + 4} \]
\[ = \frac{F_0}{5} \left( \frac{s}{s^2 + 4} - \frac{s}{s^2 + 9} \right) - \frac{4e^{-5s}}{s^2 + 4}. \]

Taking the inverse Laplace transform, we have

\[ y(t) = \frac{F_0}{5} (\cos 2t - \cos 3t) - 2u_5(t) \sin [2(t - 5)]. \]

15. Taking the Laplace transform of both sides of the differential equation \( \frac{d^2y}{dt^2} + \omega_0^2y = F_0 \sin \omega t + A\delta(t-t_0) \),
we obtain \( s^2Y(s) - 0s - 0 + \omega_0^2Y(s) = \frac{F_0\omega}{s^2 + \omega^2} + Ae^{-t_0s} \). Solving for \( Y(s) \), we have

\[ Y(s) = \frac{F_0\omega}{(s^2 + \omega_0^2)(s^2 + \omega^2)} + \frac{Ae^{-t_0s}}{s^2 + \omega_0^2} \]
\[ = \frac{F_0\omega}{\omega_0^2 - \omega^2} (\frac{1}{s^2 + \omega^2} - \frac{1}{s^2 + \omega_0^2}) + \frac{Ae^{-t_0s}}{s^2 + \omega_0^2}. \]
Taking the inverse Laplace transform, we have

\[ y(t) = \frac{F_0}{(\omega_0^2 - \omega^2)\omega_0} (\omega_0 \sin \omega t - \omega \sin \omega_0 t) + \frac{A}{\omega_0} u_{t_0}(t) \sin (\omega_0 [t - t_0]) \]

\[ = \frac{F_0}{\omega_0 (\omega^2 - \omega_0^2)} (\omega \sin \omega_0 t - \omega_0 \sin \omega t) + \frac{A}{\omega_0} u_{t_0}(t) \sin (\omega_0 [t - t_0]). \]

**Solutions to Section 8.9**

**True-False Review:**

1. **TRUE.** We have \((f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau\) and \((g * f)(t) = \int_0^t g(t - \tau)f(\tau)d\tau\). The latter integral becomes the same as the first one via the \(u\)-substitution \(u = t - \tau\).

3. **FALSE.** The Convolution Theorem states that \(L[f * g] = L[f]L[g]\).

5. **FALSE.** For instance, if \(f\) is identically zero, then \(f * g = f * h = 0\) for all functions \(g\) and \(h\).

7. **TRUE.** This follows from the fact that constants can be factored out of integrals, and \(a(f * g), (af) * g,\) and \(f \ast (ag)\) can be expressed as integrals. In short, integration is linear.

**Problems:**

1. \((f * g)(t) = \int_0^t f(t - x)g(x)dx = \int_0^t (t - x)dx = \left[t x - \frac{x^2}{2}\right]_0^t = t^2 - \frac{t^2}{2} = \frac{t^2}{2}.\)

3. \((f * g)(t) = \int_0^t f(t - x)g(x)dx = \int_0^t e^{t-x}xdx = e^t \int_0^t e^{-x}dx = e^t [e^{-x}(-x - 1)]_0^t = e^t - 1 - t.\)

5. \((f * g)(t) = \int_0^t f(t - x)g(x)dx = \int_0^t e^{t-x} \sin xdx = e^t \int_0^t \sin xdx = [-e^t \cos x]_0^t = e^t(1 - \cos t).\)

7. We have

\[(f \ast (g \ast h))(t) = \int_0^t f(t - v)(g \ast h)(v)dv\]

\[= \int_0^t f(t - v) \left[ \int_0^v g(v - u)h(u)du \right] dv\]

\[= \int_0^t \int_0^v f(t - v)g(v - u)h(u)dudv.\]

The limits of integration are \(0 \leq u \leq v\) and \(0 \leq v \leq t\) so that the region of integration is that part of the \(v - u\) plane that lies just above the \(v\)-axis and below the line \(u = v\). Reversing the order of integration, the new limits are \(u \leq v \leq t\) and \(0 \leq u \leq t\). Thus the double integral above can be written

\[(f \ast (g \ast h))(t) = \int_0^t \int_u^t f(t - v)g(v - u)h(u)dvdu.\]

We now make a change of variable \(w = v - u\) in the first iterated integral. Then \(dw = dv\) and the new
$w$-limits are $0 \leq w \leq t - u$. Thus

\[
(f * (g * h))(t) = \int_0^t \int_0^{t-u} [f(t-(u+w))g(w)]dw\,h(u)\,du
\]

\[
= \int_0^t \int_0^{t-u} [f(t-u) - w)g(w)]dw\,h(u)\,du
\]

\[
= \int_0^t (f * g)(t-u)h(u)\,du
\]

\[
= (f * g) * h(t).
\]

Therefore, $(f * g) * h = f * (g * h)$.

9. $L[f * g] = L[f]L[g] = L[t]L[\sin t] = \frac{1}{s^2} \cdot \frac{1}{s^2 + 1} = \frac{1}{s^2(s^2 + 1)}$.

11. $L[f * g] = L[f]L[g] = L[\sin t]L[\cos 2t] = \frac{1}{s^2 + 1} \cdot \frac{s}{s^2 + 4} = \frac{s}{(s^2 + 1)(s^2 + 4)}$.

13. $L[f * g] = L[f]L[g] = L[t^2]L[e^{3t} \sin 2t] = \frac{2}{s^3} \cdot \frac{2}{(s - 3)^2 + 4} = \frac{4}{s^3[(s - 3)^2 + 4]}$.

15.

(a)

\[
L^{-1}[F(s)G(s)] = L^{-1}\left[\frac{1}{s + 1}, \frac{1}{s}\right]
\]

\[
= L^{-1}\left[\frac{1}{s + 1}\right] * L^{-1}\left[\frac{1}{s}\right]
\]

\[
= e^{-t} * 1
\]

\[
= \int_0^t e^{-(t-u)}du = e^{-t} \int_0^t e^u\,du = e^{-t}\left[e^u\right]_0^t = 1 - e^{-t}.
\]

(b)

\[
L^{-1}[F(s)G(s)] = L^{-1}\left[\frac{1}{s + 1}, \frac{1}{s}\right]
\]

\[
= L^{-1}\left[\frac{1}{s} - \frac{1}{s + 1}\right]
\]

\[
= L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s + 1}\right]
\]

\[
= 1 - e^{-t}.
\]

17.
(a) \[ L^{-1}[F(s)G(s)] = L^{-1} \left[ \frac{1}{s + 2} \cdot \frac{s + 2}{s^2 + 4s + 13} \right] \]

\[ = L^{-1} \left[ \frac{1}{s + 2} \right] * L^{-1} \left[ \frac{s + 2}{s^2 + 4s + 13} \right] \]

\[ = e^{-2t} * e^{-2t} \cos 3t \]

\[ = \int_0^t e^{-2(t-x)}e^{-2x} \cos 3x \, dx = e^{-2t} \int_0^t \cos 3x \, dx = \left[ e^{-2t} \frac{3}{3} \sin 3x \right]_0^t = \frac{e^{-2t}}{3} \sin 3t. \]

(b) \[ L^{-1}[F(s)G(s)] = L^{-1} \left[ \frac{1}{s + 2} \cdot \frac{s + 2}{s^2 + 4s + 13} \right] \]

\[ = L^{-1} \left[ \frac{1}{s^2 + 4s + 13} \right] \]

\[ = \frac{1}{3} L^{-1} \left[ \frac{3}{(s + 2)^2 + 9} \right] \]

\[ = e^{-2t} \sin 3t. \]

19. (a) \[ L^{-1}[F(s)G(s)] = L^{-1} \left[ \frac{1}{s^2} \frac{e^{-\pi s}}{s^2 + 1} \right] \]

\[ = L^{-1} \left[ \frac{1}{s^2} \right] * L^{-1} \left[ \frac{e^{-\pi s}}{s^2 + 1} \right] \]

\[ = t * u_{\pi}(t) \sin (t - \pi) \]

\[ = \int_0^t (t-x)u_{\pi}(x) \sin (x - \pi) \, dx \]

\[ = \int_0^t (x-t)u_{\pi}(x) \sin x \, dx = u_{\pi}(t)[-x \cos x + \sin x + t \cos x]_0^t = u_{\pi}(t)(\sin t - \pi + t). \]

(b) \[ L^{-1}[F(s)G(s)] = L^{-1} \left[ \frac{1}{s^2} \frac{e^{-\pi s}}{s^2 + 1} \right] \]

\[ = L^{-1} \left[ e^{-\pi s} \left( \frac{1}{s^2} - \frac{1}{s^2 + 1} \right) \right] \]

\[ = L^{-1} \left[ e^{-\pi s} \frac{1}{s^2} \right] - L^{-1} \left[ e^{-\pi s} \frac{1}{s^2 + 1} \right] \]

\[ = u_{\pi}(t)(t - \pi) - u_{\pi}(t) \sin (t - \pi) \]

\[ = u_{\pi}(t)(t - \pi + \sin t). \]
21. We have
\[
L^{-1}[F(s)G(s)] = L^{-1}\left[ \frac{s + 1}{s^2 + 2s + 2} \cdot \frac{1}{(s + 3)^2} \right] = L^{-1}\left[ \frac{s + 1}{s^2 + 2s + 2} \right] \ast L^{-1}\left[ \frac{1}{(s + 3)^2} \right] = (e^{-t} \cos t) \ast (e^{-3t}) = \int_0^t e^{-(t-x)} \cos (t-x) xe^{-3x} dx = \int_0^t e^{-(t+2x)} x \cos (t-x) dx.
\]

23. We have
\[
L^{-1}[F(s)G(s)] = L^{-1}\left[ \frac{s + 4}{s^2 + 8s + 25} \cdot \frac{se^{-\pi s/2}}{s^2 + 16} \right] = L^{-1}\left[ \frac{s + 4}{s^2 + 8s + 25} \right] \ast L^{-1}\left[ \frac{se^{-\pi s/2}}{s^2 + 16} \right] = (e^{-4t} \cos 3t) \ast u_{\pi/2}(t) \cos (4[t - \pi/2]) = (e^{-4t} \cos 3t) \ast (u_{\pi/2}(t) \cos 4t) = \int_{\pi/2}^t e^{-4(t-x)} \cos [3(t-x)] \cos 4x dx,
\]
where \( \pi/2 \leq t. \)

25. Applying the Laplace transform to both sides of the differential equation \( y'' + y = e^{-t} \) yields
\[
s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s + 1}.
\]
Substituting the initial values and solving for \( Y(s) \), we have
\[
Y(s) = \frac{1}{s^2 + 1} + \frac{1}{(s + 1)(s^2 + 1)}.
\]
Applying the inverse Laplace transform yields
\[
y(t) = L^{-1}[Y(s)] = L^{-1}\left[ \frac{1}{s^2 + 1} \right] + L^{-1}\left[ \frac{1}{(s + 1)(s^2 + 1)} \right] = \sin t + \sin t = \sin t + e^{-t} \ast \sin t = \sin t + \int_0^t \sin (t - \tau)e^{-\tau}d\tau.
\]

27. Applying the Laplace transform to both sides of the differential equation \( y'' + 16y = f(t) \), yields
\[
s^2Y(s) - sy(0) - y'(0) + 16Y(s) = F(s).
\]
Substituting the initial values and solving for \( Y(s) \) we find

\[
Y(s) = \frac{\alpha s + \beta}{s^2 + 16} + \frac{F(s)}{s^2 + 16}.
\]

Taking the inverse Laplace transform gives

\[
y(t) = L^{-1}[Y(s)]
\]

\[
= \alpha L^{-1} \left[ \frac{s}{s^2 + 16} \right] + \frac{\beta}{4} L^{-1} \left[ \frac{4}{s^2 + 16} \right] + L^{-1}[F(s)] * L^{-1} \left[ \frac{1}{s^2 + 16} \right]
\]

\[
= \alpha \cos 4t + \frac{\beta}{4} \sin 4t + \frac{1}{4} f(t) \sin 4t
\]

\[
= \frac{1}{4} (4\alpha \cos 4t + \beta \sin 4t) + \frac{1}{4} \int_0^t f(t - x) \sin 4x \, dx.
\]

**29.** Applying the Laplace transform to both sides of the differential equation \( y'' - a^2 y = f(t) \) and substituting the initial values yields

\[
s^2 Y(s) - \alpha s - \beta - a^2 Y(s) = F(s).
\]

Solving for \( Y(s) \) we have

\[
Y(s) = \frac{\alpha s + \beta}{2a} \left( \frac{1}{s - a} - \frac{1}{s + a} \right) + \frac{F(s)}{s^2 - a^2}.
\]

Taking the inverse Laplace transform gives

\[
y(t) = L^{-1}[Y(s)]
\]

\[
= \frac{1}{2a} L^{-1} \left[ \frac{\alpha s - \alpha}{s - a} + \frac{\beta - \beta}{s + a} \right] + \frac{1}{2a} L^{-1}[F(s)] * L^{-1} \left[ \frac{1}{s - a} - \frac{1}{s + a} \right]
\]

\[
= \frac{1}{2a} \left( \alpha a e^{at} + \beta e^{at} - a ae^{-at} - \beta e^{-at} \right) + \frac{1}{a} f(t) \sinh at
\]

\[
= \frac{\alpha a + \beta}{2a} e^{at} - \frac{\alpha a + \beta}{2a e^{at}} + \frac{1}{\alpha} \int_0^t f(t - x) \sinh (ax) \, dx
\]

\[
= \frac{\alpha a + \beta}{a} \sinh at + \frac{1}{\alpha} \int_0^t f(t - x) \sinh (ax) \, dx.
\]

**31.** Applying the Laplace transform to both sides of the differential equation \( y'' - 2ay' + (a^2 + b^2)y = f(t) \) and substituting the initial values yields

\[
s^2 Y(s) - \alpha s - \beta - 2a(sY(s) - \alpha) + (a^2 + b^2)Y(s) = F(s).
\]

Solving for \( Y(s) \) we have

\[
Y(s) = \frac{\alpha s + \beta - 2a\alpha}{(s - a)^2 + b^2} + \frac{F(s)}{(s - a)^2 + b^2}.
\]
Taking the inverse Laplace transform gives

\[ y(t) = L^{-1}[Y(s)] = \alpha L^{-1} \left[ \frac{s - a}{(s - a)^2 + b^2} \right] + \frac{\beta - a\alpha}{b} L^{-1} \left[ \frac{b}{(s - a)^2 + b^2} \right] + \frac{1}{b} L^{-1}[F(s)] * L^{-1} \left[ \frac{b}{(s - a)^2 + b^2} \right] \]

\[ = \alpha e^{at} \cos bt + \frac{\beta - a\alpha}{b} e^{at} \sin bt + \frac{1}{b} \int_{0}^{t} f(t - x)e^{ax} \sin bx \, dx. \]

33. Taking the Laplace transform of the given integral equation yields

\[ L[x(t)] = L \left[ 2e^{3t} - \int_{0}^{t} e^{2(t-\tau)}x(\tau)d\tau \right] \]

so that

\[ X(s) = \frac{2}{s - 3} - L[e^{2t} * x(t)]. \]

Applying the convolution theorem gives

\[ X(s) = \frac{2}{s - 3} - \frac{1}{s - 2} \cdot X(s). \]

Solving for \( X(s) \) gives

\[ X(s) = \frac{2(s - 2)}{(s - 1)(s - 3)} = \frac{1}{s - 3} + \frac{1}{s - 1}. \]

Consequently,

\[ x(t) = e^{3t} + e^t. \]

35. Taking the Laplace transform of the given integral equation yields

\[ L[x(t)] = L \left[ 1 + 2 \int_{0}^{t} \sin(t - \tau)x(\tau)d\tau \right] \]

so that

\[ X(s) = \frac{1}{s} + 2L[\sin t * x(t)]. \]

Applying the convolution theorem gives

\[ X(s) = \frac{1}{s} + \frac{2}{s^2 + 1} \cdot X(s). \]

Solving for \( X(s) \) gives

\[ X(s) = \frac{s^2 + 1}{s(s^2 - 1)} = \frac{1}{s - 1} + \frac{1}{s + 1} - \frac{1}{s}. \]

Consequently,

\[ x(t) = e^t + e^{-t} - 1 = 2 \cosh t - 1. \]
37. Taking the Laplace transform of the given integral equation yields

\[ L[x(t)] = L[2 + 2 \int_{0}^{t} \cos(2[t - \tau]) x(\tau) d\tau] \]

so that

\[ X(s) = \frac{2}{s} + 2L[\cos(2t) * x(t)]. \]

Applying the convolution theorem gives

\[ X(s) = \frac{2}{s} + \frac{2s}{s^2 + 4} \cdot X(s). \]

Solving for \( X(s) \) we find

\[ X(s) = \frac{2(s^2 + 4)}{s(s^2 - 2s + 4)} = \frac{2}{s} \cdot \frac{4}{s^2 - 2s + 4}. \]

Consequently,

\[ x(t) = 2 + \frac{4}{\sqrt{3}}e^t \sin(\sqrt{3}t). \]

39. THIS PROBLEM IS THE SAME AS #8 AND SHOULD BE REMOVED.

Solutions to Section 8.10

Problems:

1.

\[ F(s) = \int_{0}^{\infty} e^{-st}(3t - 4)dt \]

\[ = 3 \int_{0}^{\infty} te^{-st}dt - 4 \int_{0}^{\infty} e^{-st}dt \]

\[ = 3 \lim_{n \to \infty} \int_{0}^{n} te^{-st}dt - 4 \lim_{n \to \infty} \int_{0}^{n} e^{-st}dt \]

\[ = 3 \lim_{n \to \infty} \left[ \frac{-te^{-st}}{s} \right]_{0}^{n} + 3 \lim_{n \to \infty} \int_{0}^{n} \frac{1}{s} e^{-st}dt - 4 \lim_{n \to \infty} \int_{0}^{n} e^{-st}dt \]

\[ = 3 \lim_{n \to \infty} \left[ \frac{-ne^{-sn}}{s} \right] - \frac{3}{s^2} \lim_{n \to \infty} [e^{-st}]_{0}^{n} + \frac{4}{s} \lim_{n \to \infty} [e^{-st}]_{0}^{n} \]

\[ = 3 \lim_{n \to \infty} \left[ \frac{-ne^{-sn}}{s} \right] - \frac{3}{s^2} \lim_{n \to \infty} [e^{-sn} - 1] + \frac{4}{s} \lim_{n \to \infty} [e^{-sn} - 1] \]

\[ = \frac{3}{s^2} - \frac{4}{s}. \]
3.  

\[ F(s) = \int_0^\infty e^{-st}(4t^2)dt \]
\[ = 4 \lim_{n \to \infty} \int_0^n t^2 e^{-st}dt \]
\[ = 4 \lim_{n \to \infty} \left[ -\frac{t^2}{s} - \frac{2t}{s^2} - \frac{2}{s^3} \right]_0^n \]
\[ = 4 \lim_{n \to \infty} \left[ -\frac{2}{s^3} - e^{-sn} \left( \frac{n^2}{s} + \frac{2n}{s^2} + \frac{2}{s^3} \right) \right] \]
\[ = \frac{8}{s^3}. \]

5.  

\[ F(s) = \int_0^\infty e^{-st}(7te^{-t})dt \]
\[ = 7 \int_0^\infty t e^{-(s+1)t}dt \]
\[ = 7 \lim_{n \to \infty} \int_0^n t e^{-(s+1)t}dt \]
\[ = 7 \left[ \lim_{n \to \infty} \left( -\frac{t}{s+1} e^{-(s+1)t} \right)_0^n + \frac{1}{s+1} \int_0^n e^{-(s+1)t}dt \right] \]
\[ = 7 \left[ \lim_{n \to \infty} \left( -\frac{t}{s+1} e^{-(s+1)t} \right)_0^n - \left( \frac{1}{(s+1)^2} e^{-(s+1)t} \right)_0^n \right] \]
\[ = \frac{7}{(s+1)^2}. \]

7.  

\[ F(s) = \int_0^\infty e^{-st} \sin^2 at \, dt \]
\[ = \lim_{n \to \infty} \int_0^n e^{-st} \sin^2 at \, dt \]
\[ = \lim_{n \to \infty} \int_0^n e^{-st} \frac{1 - \cos 2at}{2} \, dt \]
\[ = \frac{1}{2} \left[ \lim_{n \to \infty} \int_0^n e^{-st} dt - \int_0^n e^{-st} \cos 2at \, dt \right] \]
\[ = \frac{1}{2s} - \frac{1}{2} \lim_{n \to \infty} \int_0^n e^{-st} \cos 2at \, dt \]
\[ = \frac{1}{2s} - \frac{1}{2} \lim_{n \to \infty} \left[ \frac{e^{-st}}{s^2 + 4a^2} \left[ -s \cdot \cos(2at) + 2a \sin(2at) \right] \right]_0^n \]
\[ = \frac{1}{2s} - \frac{1}{2(s^2 + 4a^2)}. \]
9. 

\[ F(s) = \int_0^\infty e^{-st} f(t)dt \]

\[ = \int_0^3 e^{-st}(t+1)dt + \int_3^\infty e^{-st}(t^2-1)dt \]

\[ = \int_0^3 e^{-st}(t+1)dt + \lim_{n \to \infty} \int_3^n e^{-st}(t^2-1)dt \]

\[ = \left[ \frac{t+1}{s} e^{-st} \right]_0^3 + \frac{1}{s} \int_0^3 e^{-st}dt + \lim_{n \to \infty} \left[ \left[ \frac{-t^2-1}{s} e^{-st} \right]_3^n + \frac{2}{s} \int_3^n te^{-st}dt \right] \]

\[ = \left[ \frac{t+1}{s} e^{-st} \right]_0^3 - \frac{1}{s^2} \left[ e^{-st} \right]_0^3 + \lim_{n \to \infty} \left[ \left[ \frac{-t^2-1}{s} e^{-st} \right]_3^n + \frac{2}{s} \left[ \frac{-t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_3^n \right] \]

\[ = \left[ \frac{-4}{s} e^{-3s} + \frac{1}{s} \right] + \left[ \frac{-1}{s^2} e^{-3s} + \frac{1}{s^2} \right] + \left[ \frac{8}{s} e^{-3s} + \frac{6}{s^2} e^{-3s} + \frac{2}{s^3} e^{-3s} \right] \]

\[ = \frac{e^{-3s} (4s^2 + 5s + 2)}{s^3} + \frac{s + 1}{s^2}. \]

11. We have

\[ L[f] = 5L[\cos 2t] - 7L[e^{-t}] - 3L[e^6] \]

\[ = \frac{5s}{s^2 + 4} - \frac{7}{s + 1} - \frac{3 \cdot 6!}{s^8}. \]

13. We have

\[ L[f] = \frac{s - 3}{(s - 3)^2 + 25} - \frac{2}{(s + 1)^2 + 4}. \]

15. We have

\[ L[f] = \sqrt{\frac{\pi}{s + 5}}. \]

17. We have

\[ L[f] = 2L[1] + 2L[e^{-t}u_1(t)] - 2L[u_1(t)]. \]

For the middle term, we wish to apply the Second Shifting Theorem. In order to do so, we note that

\[ e^{-t} = \frac{e^{-(t-1)}}{e}. \]

Thus, if we write \( f(t - 1) = \frac{e^{-(t-1)}}{e} \), then \( f(t) = \frac{e^{-t}}{e} \). Hence,

\[ L[u_1(t)e^{-t}] = L[u_1(t)f(t-1)] = e^{-s} \cdot \frac{1}{e} \cdot \frac{1}{s + 1} = \frac{e^{-s}}{e(s + 1)}. \]

Therefore,

\[ L[f] = 2L[1] + 2L[e^{-t}u_1(t)] - 2L[u_1(t)] \]

\[ = \frac{2}{s} + \frac{2e^{-s}}{e(s + 1)} - \frac{2e^{-s}}{s} \]

\[ = \frac{2}{s} (1 - e^{-s}) + \frac{2e^{-s}}{e(s + 1)} \]

\[ = \frac{2}{s} (1 - e^{-s}) + \frac{2e^{-(s+1)}}{s + 1}. \]
19. In this case, we observe that $f(t) = t^2 \ast e^t$. Therefore, by the convolution theorem, we have

$$L[f] = L[t^2]L[e^t] = \frac{2}{s^3} \cdot \frac{1}{s-1} = \frac{2}{s^3(s-1)}.$$ 

21. We have

$$L^{-1}[F(s)] = L^{-1}\left[\frac{4s}{s^2 + 9}\right] + L^{-1}\left[\frac{5}{s^2 + 9}\right]$$

$$= 4L^{-1}\left[\frac{s}{s^2 + 9}\right] + 5L^{-1}\left[\frac{1}{s^2 + 9}\right]$$

$$= 4 \cos 3t + \frac{5}{3} \sin 3t.$$ 

23. We recognize $F(s) = \frac{2}{s(s^2 + 16)}$ as a convolution product, with $g(t) = \frac{1}{2t}$ and $h(t) = \frac{4}{s^2 + 16}$. By the convolution theorem, we have

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{2s}\right] * L^{-1}\left[\frac{4}{s^2 + 16}\right]$$

$$= \frac{1}{2} \sin 4t$$

$$= \int_0^t \frac{1}{2} \sin 4w \, dw$$

$$= -\frac{1}{8} \cos 4w |^0_t$$

$$= \frac{1}{8} [1 - \cos 4t].$$

25. We have

$$L^{-1}[F(s)] = L^{-1}\left[\frac{2s + 5}{s(s^2 + 4s + 20)}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{4}{(s + 2)^2 + 16}\right] + \frac{5}{4} L^{-1}\left[\frac{4}{s((s + 2)^2 + 16)}\right]$$

The First Shifting Theorem can be used to evaluate the first inverse Laplace transform, while the second inverse Laplace transform can be evaluated by using the First Shifting Theorem along with the convolution theorem. Continuing, we have

$$L^{-1}[F(s)] = \frac{1}{2} L^{-1}\left[\frac{4}{(s + 2)^2 + 16}\right] + \frac{5}{4} \left[1 * e^{-2t} \sin 4t\right]$$

$$= \frac{1}{2} e^{-2t} \sin 4t + \frac{5}{4} \int_0^t e^{-2w} \sin 4w \, dw$$

$$= \frac{1}{2} e^{-2t} \sin 4t + \frac{5}{4} \left[\frac{-2}{20} e^{-2w} \sin 4w - \frac{2}{20} \cos 4w\right]^t_0$$

$$= \frac{1}{2} e^{-2t} \sin 4t + \frac{5}{4} \left[-\frac{1}{10} e^{-2t} (\sin 4t + 2 \cos 4t)\right]$$

$$= \frac{1}{4} + \frac{3}{8} e^{-2t} \sin 4t - \frac{1}{4} e^{-2t} \cos 4t.$$
27. We have
\[ f(t) = (2e^{-t} - 1)u_{\ln 2}(t) + 1, \]
so that
\[ L[f] = L[(2e^{-t} - 1)u_{\ln 2}(t)] + L[1] \]
\[ = 2L[e^{-t}u_{\ln 2}(t)] - L[u_{\ln 2}(t)] + L[1] \]
\[ = 2e^{-\ln 2s} \cdot \frac{1}{2(s+1)} - e^{-\ln 2s} \cdot \frac{1}{s} + \frac{1}{s} \]
\[ = \frac{1}{2s(s+1)} - \frac{1}{s^2} + \frac{1}{s} \]
\[ = \frac{1}{s} - \frac{1}{s(s+1)2s}. \]

![Figure for Exercise 27](image_url)

29. Making the substitution \( u = t - a \), we have
\[ L[f_a(t)] = \int_0^\infty e^{-st} f_a(t) \, dt \]
\[ = \int_0^\infty e^{-st} f(t-a) \, dt \]
\[ = \int_0^\infty e^{-s(u+a)} f(u) \, du \]
\[ = e^{-as} \int_0^\infty e^{-su} f(u) \, du \]
\[ = e^{-as} L[f] \]

31. (a) We have \( f'(x) = e^{ax} + a xe^{ax} \). Therefore, using the table of Laplace transforms, we immediately find that
\[ L[f'(x)] = L[e^{ax} + a xe^{ax}] = L[e^{ax}] + aL[xe^{ax}] = \frac{1}{s-a} + aL[f(x)]. \]
(b) We have \( L[f'] = sL[f] - f(0) \). Equating this with the result for \( L[f'] \) obtained in part (a), we have
\[ sL[f] - f(0) = aL[f] + \frac{1}{s-a}. \]
so that
\[(s - a)L[f] = \frac{1}{s - a} + f(0).\]

In this case, \(f(0) = 0\), so therefore,
\[L[f] = \frac{1}{(s - a)^2}.\]

(c) Part (b) establishes this result for \(n = 1\). Now assume that
\[L[x^n e^{ax}] = \frac{(n - 1)!}{(s - a)^n}.\]

To evaluate \(L[x^n e^{ax}]\), we first observe that \((x^n e^{ax})' = n x^{n-1} e^{ax} + a x^n e^{ax}\). Again using \(L[f'] = sL[f] - f(0)\) and the fact that \(f(0) = 0\) in this case, we have
\[s L[x^n e^{ax}] = L[(x^n e^{ax})'] = n L[x^{n-1} e^{ax}] + a L[x^n e^{ax}].\]

Rearranging and substituting the induction hypothesis, we obtain
\[(s - a)L[x^n e^{ax}] = n \cdot \frac{(n - 1)!}{(s - a)^n} = \frac{n!}{(s - a)^n},\]
so that
\[L[x^n e^{ax}] = \frac{n!}{(s - a)^{n+1}},\]
as required.

33. In Section 8.4, Problem 33, it is established that
\[L[f^{(n)}] = s^n L[f] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).\]

Therefore, taking the Laplace transform of both sides of
\[y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = f(t),\]
and using the given initial conditions yields
\[(s^n + a_1 s^{n-1} + \cdots + a_n) Y(s) = F(s).\]

Solving for \(Y(s)\) gives
\[Y(s) = \frac{1}{P(s)} \cdot F(s),\]
where
\[P(s) = s^n + a_1 s^{n-1} + \cdots + a_n.\]

Consequently,
\[y(t) = L^{-1} \left[ \frac{1}{P(s)} \cdot F(s) \right] = K(t) * f(t) = \int_0^t K(t - w) f(w) dw,\]
where \(K(t) = L^{-1} \left[ \frac{1}{P(s)} \right].\)
35. Taking the Laplace transform of both sides of the given differential equation we obtain

\[ [s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] - 8Y(s) = \frac{5}{s} \]

Substituting the initial values \( y(0) = 1 \) and \( y'(0) = 0 \) and solving for \( Y(s) \) yields

\[ Y(s) = \frac{s^2 - 2s + 5}{s(s^2 - 2s - 8)} = \frac{s^2 - 2s + 5}{s(s - 4)(s + 2)} \]

Next, we must determine a partial fractions decomposition of the right-hand side. We have

\[ \frac{s^2 - 2s + 5}{s(s - 4)(s + 2)} = \frac{A}{s} + \frac{B}{s - 4} + \frac{C}{s + 2} = \frac{A(s - 4)(s + 2) + Bs(s + 2) + Cs(s - 4)}{s(s - 4)(s + 2)} = \frac{(A + B + C)s^2 + (-2A + 2B - 4C)s - 8A}{s(s - 4)(s + 2)} \]

Equating numerators on both sides of the equation according to powers of \( s \), we obtain

\[ A + B + C = 1, \quad -2A + 2B - 4C = -2, \quad -8A = 5 \]

Solving this system, we find

\[ A = -\frac{5}{8}, \quad B = \frac{13}{24}, \quad C = \frac{13}{12} \]

Thus,

\[ \frac{s^2 - 2s + 5}{s(s - 4)(s + 2)} = -\frac{5}{8} \cdot \frac{1}{s} + \frac{13}{24} \cdot \frac{1}{s - 4} + \frac{13}{12} \cdot \frac{1}{s + 2} \]

We can solve

\[ Y(s) = -\frac{5}{8} \cdot \frac{1}{s} + \frac{13}{24} \cdot \frac{1}{s - 4} + \frac{13}{12} \cdot \frac{1}{s + 2} \]

for \( y(t) \) by taking the inverse Laplace transform of both sides:

\[ y(t) = -\frac{5}{8} + \frac{13}{24} e^{4t} + \frac{13}{12} e^{-2t} \]

37. Note that \( f(t) = 1 - u_{\pi/2}(t) \), so that we must solve

\[ y'' + y = 1 - u_{\pi/2}(t) \]

Taking the Laplace transform of both sides of the given differential equation we obtain

\[ s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1 - e^{-\pi s/2}}{s} \]

Substituting the initial values \( y(0) = 0 \) and \( y'(0) = 1 \) and solving for \( Y(s) \) yields

\[ Y(s) = \frac{s + 1 - e^{-\pi s/2}}{s(s^2 + 1)} = \frac{s + 1}{s(s^2 + 1)} - e^{-\pi s/2} \left[ \frac{1}{s(s^2 + 1)} \right] \].
Next, we determine a partial fractions decomposition for the first term on the right-hand side:

\[
\frac{s + 1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1} = \frac{A(s^2 + 1) + (Bs + C)s}{s(s^2 + 1)} = \frac{(A + B)s^2 + Cs + A}{s(s^2 + 1)}.
\]

Equating numerators on both sides according to powers of \(s\), we obtain

\[A + B = 0, \quad C = 1, \quad A = 1.\]

Therefore, \(A = 1, B = -1, \) and \(C = 1\). Hence,

\[
\frac{s + 1}{s(s^2 + 1)} = \frac{1}{s} + \frac{-s + 1}{s^2 + 1}.
\]

Thus,

\[
L^{-1}\left[\frac{s + 1}{s(s^2 + 1)}\right] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{-s + 1}{s^2 + 1}\right] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2 + 1}\right] - L^{-1}\left[\frac{s}{s^2 + 1}\right] = 1 + \sin t - \cos t.
\]

Next, we do a partial fractions decomposition to help us determine the inverse Laplace transform of the second term on the right-hand side above:

\[
\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1} = \frac{A(s^2 + 1) + (Bs + C)s}{s(s^2 + 1)} = \frac{(A + B)s^2 + Cs + A}{s(s^2 + 1)}.
\]

Equating numerators on both sides according to powers of \(s\), we obtain

\[A + B = 0, \quad C = 0, \quad A = 1.\]

Therefore, \(A = 1, B = -1, \) and \(C = 0\). Hence,

\[
\frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}.
\]

Hence,

\[
L^{-1}\left[e^{-\pi s/2} \left[\frac{1}{s(s^2 + 1)}\right]\right] = L^{-1}\left[e^{-\pi s/2} \left(\frac{1}{s} - \frac{s}{s^2 + 1}\right)\right] = L^{-1}\left[e^{-\pi s/2} \frac{1}{s}\right] - L^{-1}\left[se^{-\pi s/2} \frac{s}{s^2 + 1}\right] = u_{\pi/2}(t) - u_{\pi/2}(t) \cos(t - \pi/2) = u_{\pi/2}(t)[1 - \cos(t - \pi/2)].
\]
Putting this together with the inverse Laplace transform for the first term obtained above, we obtain the final answer:

\[ y(t) = 1 + \sin t - \cos t - u_{\pi/2}(t)[1 - \cos(t - \pi/2)]. \]

39. Taking the Laplace transform of both sides of the given differential equation we obtain

\[ s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + Y(s) = e^{-4s}. \]

Substituting the initial values \( y(0) = 0 \) and \( y'(0) = 0 \) and solving for \( Y(s) \) yields

\[ Y(s) = \frac{e^{-4s}}{(s + 1)^2}. \]

Now, since

\[ L^{-1}\left[\frac{1}{(s + 1)^2}\right] = te^{-t}, \]

we have

\[ y(t) = u_4(t)(t - 4)e^{-(t-4)}. \]

41. Taking the Laplace transform of each differential equation and using the given initial conditions yields:

\[ sX_1 - 1 = X_1 + 2X_2 \quad \text{and} \quad sX_2 = 2X_1 + X_2, \]

or equivalently,

\[ (s - 1)X_1 - 2X_2 = 1 \quad \text{and} \quad -2X_1 + (s - 1)X_2 = 0. \]

Solving this algebraic system of equations using Cramer’s rule we obtain:

\[
X_1(s) = \frac{1}{\begin{vmatrix} 1 & -2 \\ 0 & s - 1 \end{vmatrix}} = \frac{s - 1}{(s - 1)^2 - 4} = \frac{s - 1}{s^2 - 2s - 3} = \frac{s - 1}{(s - 3)(s + 1)}
\]

\[
= \frac{1}{2(s - 3)} + \frac{1}{2(s + 1)}.
\]

\[
X_2(s) = \frac{1}{\begin{vmatrix} 1 & 0 \\ s - 1 & -2 \end{vmatrix}} = \frac{2}{(s - 3)(s + 1)} = \frac{1}{2(s - 3)} - \frac{1}{2(s + 1)}.
\]

Taking the inverse Laplace transform of \( X_1(s) \) and \( X_2(s) \) yields, respectively,

\[ x_1(t) = \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \quad \text{and} \quad x_2(t) = \frac{1}{2}e^{3t} - \frac{1}{2}e^{-t}. \]

43. Taking the Laplace transform of the given system yields

\[ sX_1 - 1 = -2X_2 \quad \text{and} \quad sX_2 - 1 = 2X_1 + 4X_2, \]

or equivalently,

\[ sX_1 + 2X_2 = 1 \quad \text{and} \quad -2X_1 + (s - 4)X_2 = 1. \]
Solving this algebraic system of equations using Cramer’s rule we obtain:

\[
X_1(s) = \frac{\begin{vmatrix} 1 & 2 -4 \\ 1 & s -4 \end{vmatrix}}{\begin{vmatrix} s -4 \\ 2 s -4 \end{vmatrix}} = \frac{s -6}{s^2 -4s + 4} = \frac{s -6}{(s-2)^2}
\]

\[
= \frac{1}{s-2} - \frac{4}{(s-2)^2},
\]

\[
X_2(s) = \frac{1}{(s-2)^2} \cdot \begin{vmatrix} s & 1 \\ -2 & 1 \end{vmatrix} = \frac{s + 2}{(s-2)^2} = \frac{1}{s-2} + \frac{4}{(s-2)^2}
\]

Taking the inverse Laplace transform of \(X_1(s)\) and \(X_2(s)\) yields, respectively,

\[x_1(t) = e^{2t} - 4te^{2t}\quad\text{and}\quad x_2(t) = e^{2t} + 4te^{2t}.
\]

45. Taking the Laplace transform of the given integral equation and using the convolution theorem yields

\[
X(s) = \frac{2}{s^2} + \frac{1}{s^2 + 1}X(s).
\]

That is,

\[
X(s) \left(\frac{s^2}{s^2 + 1}\right) = \frac{2}{s^2},
\]

so that

\[
X(s) = \frac{2(s^2 + 1)}{s^4} = \frac{2}{s^2} + \frac{2}{s^4}.
\]

Taking the inverse Laplace transform yields

\[x(t) = 2t + \frac{1}{3}t^3.
\]

47. Taking the Laplace transform of the given integral equation and using the convolution theorem yields

\[
X(s) = \frac{4}{s^3} + \frac{2}{s^2 + 4}X(s).
\]

That is,

\[
X(s) \left(\frac{s^2 + 2}{s^2 + 4}\right) = \frac{4}{s^3},
\]

so that

\[
X(s) = \frac{4(s^2 + 4)}{s^3(s^2 + 2)}.
\]

Now we use a partial fractions decomposition:

\[
\frac{4(s^2 + 4)}{s^3(s^2 + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2 + 2}
\]

\[
= \frac{As^2(s^2 + 2) + Bs(s^2 + 2) + C(s^2 + 2) + (Ds + E)s^3}{s^3(s^2 + 2)}
\]

\[
= \frac{(A + D)s^4 + (B + E)s^3 + (2A + C)s^2 + 2Bs + 2C}{s^3(s^2 + 2)}.
\]
Equating numerators on both sides according to powers of $s$, we have

$$A + D = 0, \quad B + E = 0, \quad 2A + C = 4, \quad 2B = 0, \quad 2C = 16.$$ 

Solving this system, we obtain

$$A = -2, \quad B = 0, \quad C = 8, \quad D = 2, \quad E = 0.$$ 

Therefore,

$$\frac{4(s^2 + 4)}{s^3(s^2 + 2)} = -\frac{2}{s} + \frac{8}{s^3} + \frac{2s}{s^2 + 2}.$$ 

Hence,

$$x(t) = L^{-1}\left[-\frac{2}{s} + \frac{8}{s^3} + \frac{2s}{s^2 + 2}\right] = -2L^{-1}\left[\frac{1}{s}\right] + 8L^{-1}\left[\frac{1}{s^3}\right] + 2L^{-1}\left[\frac{s}{s^2 + 2}\right] = -2 + 4t^2 + 2\cos \sqrt{2}t.$$ 

**Solutions to Section 9.1**

**True-False Review:**

1. **TRUE.** Theorem 9.1.7 addresses this for a rational function $f(x) = \frac{p(x)}{q(x)}$. The radius of convergence of the power series representation of $f$ around the point $x_0$ is the distance from $x_0$ to the nearest root of $q(x)$.

3. **FALSE.** It could be the case, for example, that $b_n = -a_n$ for all $n = 0, 1, 2, \ldots$. Then

$$\sum_{n=0}^{\infty} (a_n + b_n)x^n = 0,$$

but the individual summations need not converge.

5. **TRUE.** This is part of the statement in Theorem 9.1.6.

7. **TRUE.** A polynomial is a rational function $p(x)/q(x)$ where $q(x) = 1$. Since $q(x) = 1$ has no roots, Theorem 9.1.7 guarantees that the power series representation of $p(x)/q(x)$ about any point $x_0$ has an infinite radius of convergence.

9. **FALSE.** According to the algebra of power series, the coefficient $c_n$ of $x^n$ is

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$ 

**Solutions to Section 9.2**

**True-False Review:**

1. **TRUE.** Since a polynomial is analytic at every $x_0$ in $\mathbb{R}$, it follows from Definition 9.2.1 that every point of $\mathbb{R}$ is an ordinary point.
3. **FALSE.** The nearest singularity to \( x = 2 \) occurs in \( p \) with root \( x = 1 \). Hence, \( R = 1 \) is the radius of convergence of \( p(x) = \frac{1}{x^2-1} \) about \( x = 2 \). Therefore, by Theorem 9.2.4, the radius of convergence of a power series solution to this differential equation is at least 1 (not at least 2).

5. **TRUE.** The radii of convergence of the power series expansions of \( p \) and \( q \) are both positive, and thus, Theorem 9.2.4 implies that the general solution to the differential equation can be represented as a power series with a positive radius of convergence.

7. **FALSE.** For example, the power series \( y_1(x) \) found in Example 9.2.6 is a solution to (9.2.10), but \( xy_1(x) \) is not a solution.

9. **TRUE.** The value of \( a_k \) depends on \( a_{k-1} \) and \( a_{k-3} \), so once \( a_0, a_1, \) and \( a_2 \) are specified, we can find \( a_3, a_4, a_5, \ldots \) uniquely from the recurrence relation.

**Solutions to Section 9.3**

**True-False Review:**

1. **FALSE.** Theorem 9.2.4 guarantees that \( R = 1 \) is a lower bound on the radius of convergence of the power series solutions to Legendre’s equation about \( x = 0 \). The radius of convergence could be greater than 1.

3. **TRUE.** If \( \alpha \) is a positive even integer or a negative odd integer or zero, then eventually the coefficients of (9.3.3) become zero, whereas if \( \alpha \) is a negative even integer or a positive odd integer, then eventually the coefficients of (9.3.4) become zero. Therefore, for any integer \( \alpha \), either (9.3.3) or (9.3.4) contains only finitely many nonzero terms.

**Solutions to Section 9.4**

**True-False Review:**

1. **FALSE.** It is required that \( (x-x_0)^2Q(x) \) be analytic at \( x = x_0 \), not that \( (x-x_0)Q(x) \) be analytic at \( x = x_0 \).

3. **TRUE.** This is well-illustrated by Examples 9.4.5 and 9.4.6. The values of \( a_0, a_1, a_2, \ldots \) in the Frobenius series solution are obtained by directly substituting it into both sides of the differential equation and matching up the coefficients of \( x \) resulting on each side of the differential equation.

**Solutions to Section 9.5**

**True-False Review:**

1. **FALSE.** The indicial equation (9.5.4) in this case reads \( r(r-1) - r + 1 = 0 \), or \( r^2 - 2r + 1 = 0 \), and the roots of this equation are \( r = 1, 1 \), so the roots are not distinct.

3. **FALSE.** The indicial equation (9.5.4) in this case reads \( r(r-1) + 9r + 25 = 0 \), or \( r^2 + 8r + 25 = 0 \). The quadratic formula quickly yields the roots \( r = -4 \pm 3i \), which do not differ by an integer.

5. **TRUE.** The indicial equation (9.5.4) in this case reads \( r(r-1) = 0 \), with roots \( r = 0 \) and \( r = 1 \). They differ by 1.

**Solutions to Section 9.6**
True-False Review:

1. **FALSE.** The requirement that guarantees the existence of two linearly independent Frobenius series solutions is that \(2p\) is not an integer, not that \(p\) is a positive noninteger.

3. **FALSE.** The gamma function is not defined for \(p = 0, -1, -2, \ldots\) integers.

5. **FALSE.** Although it is possible via Property 3 in Equation (9.6.27) to express \(J_p(x)\) in terms of \(J_{p-1}(x)\) and \(J_{p-2}(x)\) as
   \[
   J_p(x) = 2x^{-1}(p - 1)J_{p-1}(x) - J_{p-2}(x),
   \]
   the expression on the right-hand side is not a linear combination.

7. **FALSE.** From Equation (9.6.31), we see that \(J_p(\lambda_n x)\) and \(J_p(\lambda_m x)\) are orthogonal on (0,1) relative to the weight function \(w(x) = x\):
   \[
   \int_0^1 x J_p(\lambda_n x) J_p(\lambda_m x) dx = 0.
   \]